Sketches and hints for the solution of some of the exercises

A. Exercise II.5: SSH Model

The model is characterized by a staggered tunneling amplitude, therefore its unit cell must include two sites. We may define the even and odd sites with A and B degrees of freedom. To derive the Hamiltonian in momentum space we assume, for simplicity, periodic boundary conditions with L unit cells ($2L$ sites). The Hamiltonian in momentum space is a 2×2 Hamiltonian with the following form:

$$H = \sum_k \left(c_A^\dagger(k), c_B^\dagger(k)\right) \begin{pmatrix} 0 & -t_1 - t_2 e^{-ik2a} \\ -t_1 - t_2 e^{ik2a} & 0 \end{pmatrix} \begin{pmatrix} c_A(k) \\ c_B(k) \end{pmatrix} . \tag{1}$$

The unit cell has two sites, thus its size is $2a$, and the Brillouin zone can be taken with $k \in \left[0, \frac{\pi}{a}\right)$ with momentum intervals $\delta k = 2\pi/L$. To derive the previous Hamiltonian we applied a standard Fourier transform of the kind:

$$c_{2r} = c_A(r) = \frac{1}{\sqrt{L}} \sum_{k \in \mathrm{B.Z.}} e^{ik2r} c_A(k) , \quad c_{2r+a} = c_B(r) = \frac{1}{\sqrt{L}} \sum_{k \in \mathrm{B.Z.}} e^{ik2r} c_B(k) . \tag{2}$$

The spectrum of the system results:

$$E(k) = \pm\sqrt{(t_1 + t_2 \cos 2ka)^2 + t_2^2 \sin^2 2ka} = \pm\sqrt{t_1^2 + t_2^2 + 2t_1 t_2 \cos 2ka} . \tag{3}$$

For $|t_1| \neq |t_2|$ this corresponds to two energy bands separated by a gap. If $t_1 = t_2$ the energy becomes:

$$E_{t_1=t_2}(k) = \pm t_1 \sqrt{2 + 2\cos 2ka} = \pm 2t_1 \cos ka . \tag{4}$$

The two bands now touch in $k = \pi/2a$, which behaves as a Dirac point with linear dispersion (if you Taylor expand around that point). Physically, however, $t_1 = t_2$ corresponds to a translational invariant chain. Therefore, for this specific case, we may obtain two equivalent descriptions: either a single band with a cosine dispersion defined in a BZ with $k \in \left[0, \frac{2\pi}{a}\right)$, or two bands, touching in a Dirac point, defined in a folded BZ with $k \in \left[0, \frac{\pi}{a}\right)$. This shows in a clear way that our description of the translational invariant system in terms of a Dirac point with left and right movers is correct (up to translations in momentum space: the Dirac point in Eq. (4) was centered in $\pi/2a$, but a different (arbitrary) choice for the phases in the Fourier transform can translate it in momentum space).

B. Exercise IV.2

I consider the approximation for the bosonic field $\psi_B^\dagger \approx \sqrt{\rho_0} e^{-i\varphi}$. The Hamiltonian density results:

$$\mathcal{H} = \frac{\hbar^2}{2m} \left(\partial_x \psi_B^\dagger\right) \left(\partial_x \psi_B\right) + U \left(\rho(x)\right)^2 = \frac{\hbar^2 \rho_0}{2m} \left(\partial_x \varphi\right)^2 + U \left(\rho_0 - \frac{\partial_x \theta}{\pi}\right)^2 ; \tag{5}$$

$\mathcal{H}$ has the form of a Luttinger liquid plus a term linear in $\partial_x \theta$ that can be integrated away. From the commutation relation:

$$\left[\frac{\partial_x \theta(x')}{\pi}, \varphi(x)\right] = -i\delta(x - x') , \tag{6}$$

we obtain that the conjugate momentum of the field φ is:

$$\Pi_\varphi = \hbar \frac{\partial_x \theta}{\pi} . \tag{7}$$

The related Hamiltonian equation is:

$$\partial_t \varphi = \frac{\partial \mathcal{H}}{\partial \Pi_\varphi} = \frac{2U}{\hbar} \left(\frac{\partial_x \theta}{\pi} - \rho_0\right) . \tag{8}$$

Thus:

$$\Pi_\varphi = \hbar\frac{\partial_x\theta}{\pi} = \hbar^2\frac{\partial_t\varphi}{2U} + \hbar\rho_0\,. \tag{9}$$

The corresponding Lagrangian density results:

$$\mathcal{L} = \Pi_\varphi\partial_t\varphi - \mathcal{H} = \left(\hbar^2\frac{\partial_t\varphi}{2U} + \hbar\rho_0\right)\partial_t\varphi - \frac{\hbar^2\rho_0}{2m}(\partial_x\varphi)^2 - \frac{\hbar^2\left(\partial_t\varphi\right)^2}{4U} = \frac{\hbar^2\left(\partial_t\varphi\right)^2}{4U} - \frac{\hbar^2\rho_0}{2m}(\partial_x\varphi)^2 + \hbar\rho_0\partial_t\varphi\,. \tag{10}$$

The corresponding Lagrange equation is:

$$\partial_t\frac{d\mathcal{L}}{d(\partial_t\varphi)} + \partial_x\frac{d\mathcal{L}}{d(\partial_x\varphi)} = 0 \quad\Longrightarrow\quad \frac{\hbar^2\partial_t^2\varphi}{2U} - \frac{\hbar^2\rho_0}{m}\partial_x^2\varphi = 0\,, \tag{11}$$

which is consistent with a velocity $u = \sqrt{2\rho_0 U/m}$ (and $K = \pi\hbar\sqrt{\rho_0/(2Um)}$).

C. Exercise IV.3

The fast oscillating terms read:

$$\begin{aligned}\rho_f(x) = \lim_{x\to y} N^2\left[e^{ik_F(x+y)}e^{-i(\varphi(x)+\theta(x))}e^{i(\varphi(y)-\theta(y))} + e^{-ik_F(x+y)}e^{-i(\varphi(x)-\theta(x))}e^{i(\varphi(y)+\theta(y))}\right] = \\ = \lim_{x\to y} N^2\left[e^{ik_F(x+y)}e^{-i(\varphi(x)+\theta(x))+i(\varphi(y)-\theta(y))+\frac{1}{2}[\varphi(x),-\theta(y)]+\frac{1}{2}[\theta(x),\varphi(y)]}\right. \\ \left. + e^{-ik_F(x+y)}e^{-i(\varphi(x)-\theta(x))+i(\varphi(y)+\theta(y))+\frac{1}{2}[\varphi(x),\theta(y)]-\frac{1}{2}[\theta(x),\varphi(y)]}\right] = \\ = N^2\left[-ie^{2ik_Fx}e^{-2i\theta(x)} + ie^{-2ik_Fx}e^{2i\theta(x)}\right]\end{aligned} \tag{12}$$

D. Exercise IV.4

1. Here I consider the term stemming from the product of fast-oscillating terms with fast-oscillating terms, since they are the most interesting (the product between slow and fast terms are always fast-oscillating, the product of the slow terms only is trivial). They result:

$$\begin{aligned}\rho_f(x)\rho_f(x-a) = N^4\Big[-e^{2i\theta(x)+2i\theta(x-a)-4ik_Fx+2ik_Fa} - e^{-2i\theta(x)-2i\theta(x-a)+4ik_Fx-2ik_Fa} \\ + e^{-2ik_Fa}e^{2i(\theta(x)-\theta(x-a))} + e^{2ik_Fa}e^{-2i(\theta(x)-\theta(x-a))}\Big]\,.\end{aligned} \tag{13}$$

   The first two terms are fast-oscillating for generic k_F, the other two terms are slow oscillating. When considering the lattice spacing a as a small parameter we get:

$$\begin{aligned}\rho_f(x)\rho_f(x-a) \approx N^4\Big[-e^{4i\theta(x)-4ik_Fx+2ik_Fa} - e^{-4i\theta(x)+4ik_Fx-2ik_Fa} \\ + e^{-2ik_Fa}\left(1+2ia\partial_x\theta(x) - 2a^2(\partial_x\theta(x))^2\right) + e^{2ik_Fa}\left(1-2ia\partial_x\theta(x) - 2a^2(\partial_x\theta(x))^2\right)\Big]\,.\end{aligned} \tag{14}$$

   Here we are considering only the dominant terms in the Taylor expansion with small a. Concerning the second line, the constant term is an energy shift and the term linear in $\partial_x\theta$ can be integrated away. We consider only the quadratic term and we get:

$$\rho_f(x)\rho_f(x-a) \approx \frac{1}{4\pi^2a^2}\left[-e^{4i\theta(x)-4ik_Fx+2ik_Fa} - e^{-4i\theta(x)+4ik_Fx-2ik_Fa} - 4a^2\cos(2k_Fa)(\partial_x\theta)^2\right]\,. \tag{15}$$

   The last term provides a correction to the quadratic Hamiltonian consistent with the result in the text. This contribution of the fast-oscillating term must be added to the standard slow-oscillating part $(\partial_x\theta)^2/\pi^2$.

2. For $k_F = \pi/2a$ the first term of the previous equation becomes slow oscillating and the Hamiltonian acquires an additional umklapp scattering term of the kind:

$$H_{\rm umklapp} = \int dx\,\frac{\tilde{U}}{2\pi^2a^2}\cos 4\theta \tag{16}$$

E. Exercise V.3

I use the standard bosonized form of the fermionic field to get the kinetic energy. First of all we map the Hamiltonian from the lattice to the field formulation:

$$H_0 = -t \sum_r \left(c^\dagger_{r+a} c_r + \text{H.c.} \right) \to -t \int dx \, \left(\psi^\dagger(x+a)\psi(x) + \text{H.c.} \right). \tag{17}$$

Then I apply bosonization:

$$\psi^\dagger(x+a)\psi(x) = \\ \frac{1}{2\pi a}\left[e^{ik_F(x+a)} e^{-i(\varphi(x+a)+\theta(x+a))} + e^{-ik_F(x+a)} e^{-i(\varphi(x+a)-\theta(x+a))} \right] \left[e^{-ik_F x} e^{i(\varphi(x)+\theta(x))} + e^{ik_F x} e^{i(\varphi(x)-\theta(x))} \right]. \tag{18}$$

As usual I focus on the slow-oscillating terms and here I neglect the terms proportional to $e^{\pm i 2k_F x}$:

$$\begin{aligned}
&\psi^\dagger(x+a)\psi(x) + \text{H.c.} \approx \\
&\frac{1}{2\pi a}\left[e^{ik_F a} e^{-i(\varphi(x+a)+\theta(x+a))} e^{i(\varphi(x)+\theta(x))} + e^{-ik_F a} e^{-i(\varphi(x+a)-\theta(x+a))} e^{i(\varphi(x)-\theta(x))} \right] + \text{H.c.} = \\
&\frac{1}{2\pi a}\left[-ie^{ik_F a} e^{-i(\varphi(x+a)+\theta(x+a)-\varphi(x)-\theta(x))} + ie^{-ik_F a} e^{-i(\varphi(x+a)-\theta(x+a)-\varphi(x)+\theta(x))} \right] + \text{H.c.} \approx \\
&\frac{1}{2\pi a}\left[-ie^{ik_F a} e^{-ia(\partial_x\varphi(x)+\partial_x\theta(x)) - i\frac{a^2}{2}\partial_x^2 \dots} + ie^{-ik_F a} e^{-ia(\partial_x\varphi(x)-\partial_x\theta(x)) - i\frac{a^2}{2}\partial_x^2 \dots} \right] + \text{H.c.} \approx \\
&\frac{1}{2\pi a}\Bigg[-ie^{ik_F a}\left(1 - ia\left(\partial_x\varphi(x)+\partial_x\theta(x)\right) - i\frac{a^2}{2}\partial_x^2 \dots - \frac{a^2}{2}\left(\partial_x\varphi(x)+\partial_x\theta(x)\right)^2 \right) \\
&+ie^{-ik_F a}\left(1 - ia\left(\partial_x\varphi(x)-\partial_x\theta(x)\right) - i\frac{a^2}{2}\partial_x^2 \dots - \frac{a^2}{2}\left(\partial_x\varphi(x)-\partial_x\theta(x)\right)^2 \right) \Bigg] + \text{H.c.} = \\
&\text{const} + \frac{1}{2\pi a}\left[-4a\cos(k_F a)\partial_x\theta(x) - 2a^2\cos(k_F a)\partial_x^2\theta \right] + \frac{1}{2\pi a}\left[-4\frac{a^2}{2}\sin k_F a \left((\partial_x\varphi)^2 + (\partial_x\theta)^2 \right) \right]
\end{aligned} \tag{19}$$

where I applied the CBH formula and I got the phases $\pm i$ from the usual commutation relation $[\theta(x+a), \varphi(x)] = -i\pi$, I Taylor expanded the operators at second order and we can neglect terms that are just total derivatives (thus boundary contributions after the integration): for example, the terms in $a^2\partial_x^2\theta$. The terms linear in $\partial_x\theta$ are boundary terms as well, which correspond to the fluctuation of the total number of particles, which is zero for systems with particle-number conservation. For practical purposes, therefore, we consider only the last quadratic term and we get:

$$-t\sum_r \left(c^\dagger_{r+a} c_r + \text{H.c.} \right) \to \int dx \, \frac{2ta\sin k_F a}{2\pi}\left[(\partial_x\varphi)^2 + (\partial_x\theta)^2 \right]. \tag{20}$$

The interaction part has been calculated in the previous exercises.
Setting $v_F = 2ta\sin k_F a$, the Luttinger parameter and velocity read:

$$K = \sqrt{\frac{\pi v_F}{\pi v_F + 2U\left(1-\cos 2k_F a\right)}}, \quad u = \sqrt{v_F\left(v_F + \frac{2U}{\hbar\pi}\left(1-\cos 2k_F a\right) \right)}. \tag{21}$$

You may observe that u has the same form of the effective velocity we derived for the Luttinger model (see Eq. 41). Repulsive interactions increase the velocity. About the umklapp term, we have already derived it before.

F. Exercise V.4

1. The SC-term reads:

$$\begin{aligned}
-\Delta c^\dagger_x c^\dagger_{x+a} + \text{H.c.} \approx -\frac{\Delta}{2\pi a}&\left[e^{ik_F x} e^{-i(\varphi(x)+\theta(x))} + e^{-ik_F x} e^{-i(\varphi(x)-\theta(x))} \right] \cdot \\
&\cdot \left[e^{ik_F(x+a)} e^{-i(\varphi(x+a)+\theta(x+a))} + e^{-ik_F(x+a)} e^{-i(\varphi(x+a)-\theta(x+a))} \right] + \text{H.c.}
\end{aligned} \tag{22}$$

Taking the slow-oscillating term we get:

$$\begin{aligned}
&-\Delta c_x^\dagger c_{x+a}^\dagger + \text{H.c.} \approx \\
&-\frac{\Delta}{2\pi a}\left[e^{-ik_F a}e^{-i(\varphi(x)+\theta(x)+\varphi(x+a)-\theta(x+a))+\frac{1}{2}[\varphi(x),\theta(x+a)]} + e^{ik_F a}e^{-i(\varphi(x)-\theta(x)+\varphi(x+a)+\theta(x+a))-\frac{1}{2}[\varphi(x),\theta(x+a)]}\right] + \text{H.c.} \\
&= -\frac{\Delta}{2\pi a}\left[ie^{-ik_F a}e^{-i(\varphi(x)+\theta(x)+\varphi(x+a)-\theta(x+a))} - ie^{ik_F a}e^{-i(\varphi(x)-\theta(x)+\varphi(x+a)+\theta(x+a))}\right] + \text{H.c.} \\
&\approx -\frac{2\Delta}{\pi a}\sin(k_F a)\cos(2\varphi(x))
\end{aligned} \tag{23}$$

where, in the last step, I just consider the dominating term for small a, thus I take $\theta(x) \approx \theta(x+a)$.

2. The scaling dimension D of $\cos 2\varphi$ can be obtained from the correlation functions of $e^{i2\varphi}$ using, as usual, the cumulant expansion. We get that $D = 1/K$, thus the operator is relevant for $K > 1/2$.

3. When we want semiclassically to minimize the SC term, we must consider the field φ pinned around one of its minima. Therefore $\varphi(x) = 0, \pi$ (assuming $\sin k_F a > 0$). The distinct values are 2 because $\varphi = \varphi + 2\pi$ since the fermionic operator is invariant under this translation by 2π.

4. The dominating term of that correlation is given by:

$$\left\langle e^{2i\varphi(x)}e^{-2i\varphi(y)}\right\rangle . \tag{24}$$

If the field φ is pinned, this expression is essentially constant and the correlation does not decay. It is a kind of superconducting long-range order. One could be concerned that this result violates Mermin-Wagner theorem. The point is that we are assuming, since the beginning, to have a well defined Δ parameter with the same phase everywhere. This effectively breaks explicitly (and not spontaneously) the U(1) symmetry, thus Mermin-Wagner, strictly speaking, does not apply to our toy model. However, you may wonder what is the origin of this superconducting term. To get such an ordered superconducting term, it must be inherited by a background superconductor that lives in three dimensions.

5. By redoing the calculation before you may get:

$$H_{\text{sc}} \approx -\int dx \frac{2\Delta}{\pi a}\sin(k_F a)\cos(2\varphi(x) - \phi x) \tag{25}$$

6. In the regime in which Δ dominates, the two semiclassical solutions are $\varphi = \phi x/2 - \pi/2 \pm \pi/2$. In this case the current $j = -u\partial_x\varphi/\pi$ becomes linear in the parameter ϕ.

7. This implies that the kinetic energy term proportional to $(\partial_x\varphi)^2 \propto \phi^2$. We expect that, as long as Δ can be considered dominating, the kinetic energy grows quadratically with ϕ.

8. At a certain point the kinetic energy grows too much: if $v_F\phi^2$ becomes comparable with $\frac{2\Delta}{\pi a}\sin(k_F a)$ then it is more energetically convenient, for the system, to minimize the kinetic energy rather than the SC energy. The system undergoes a phase transition from a Δ dominated regime with φ pinned, j growing linearly in ϕ and long range SC correlations, back to a Luttinger liquid with a vanishing expectation value of the current. This is because the Δ term, for ϕ sufficiently large, becomes fast oscillating. This transition given by a relevant sine-Gordon term becoming fast-oscillating is usually called commensurate-incommensurate phase transition and it appears in many physical situations (the most notable: superfluid-Mott insulator transition for bosonic 1D gases).

9. Since ϕ is proportional to the magnetic field B (ϕ plays the role of a 1D vector potential A), we have a current growing linearly with B like in a standard superconductor. This is similar to the Meissner effect. At a certain point we reach a critical magnetic field and the superconducting order is broken.

G. Exercise V.5

1. As usual, we can use the correlation functions of the operators to obtain their scaling dimension. The scaling dimension of the A-term is $D_A = \alpha^2 K/4$, the one of the B-term is $D_B = \beta^2/4K$. By imposing $2 - D_{A/B} > 0$ you get the values of K for which these operators are relevant.

2. By replacing $\alpha = \beta = \sqrt{2p}$ one gets that the two sine-Gordon potentials can be both irrelevant around $K = 1$ for $p \geq 5$.

3. The previous observation implies that, for $p \leq 4$ there are two different gapped phases, respectively with φ and θ ordered; and, for $K = 1$, there is a phase transition between the two phases (it is a second order phase transition, but this is difficult to show). For $p \geq 5$, instead, a gapless phase opens between the two phases. In this situation, by increasing K we find three phases: at low K a gapped phase with θ ordered, at intermediate K a Luttinger liquid, at large K a phase with φ ordered. These phases are separated by two different BKT transitions of the kind we studied.