Københavns Universitet
MSc in Physics

ADVANCED CONDENSED MATTER THEORY

Problems on 1D Models and Bosonization

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N.B. two kinds of "text boxes" appear in this text:

A gray box — contains the problem text of the given question

A green box — contains the answer a given question, unless the problem involves diagrams

Note that we employ natural units $\hbar = k_B = 1$

II. Preliminaries: 1D Dirac Hamiltonian

II.5 The SSH Mode

To understand better why we can model the Fermi surface of Hamiltonian:

$$\hat{H}_0 = -t \sum_r \left[\hat{c}^\dagger_{r+a} \hat{c}_r + \text{H.c.} \right] + \mu \sum_r \hat{c}^\dagger_r \hat{c}_r \tag{1}$$

with a Dirac cone, consider the following model (Su-Schrieffer-Heeger model):

$$\hat{H}_{\text{SSH}} = -t_1 \sum_r \left[\hat{c}^\dagger_{2r+a} \hat{c}_{2r} + \text{H.c.} \right] - t_2 \sum_r \left[\hat{c}^\dagger_{2r+2a} \hat{c}_{2r+a} + \text{H.c.} \right] \tag{2}$$

Define a suitable unit cell, a suitable Brillouin zone, and **calculate** its spectrum. **What** happens for $t_1 = t_2$?

We start by noting that we will be assuming *periodic boundary conditions*. Choosing some arbitrary reference point, the position of the r'th site can be written as an integer multiple of the lattice constant a:

$$r = na \tag{3}$$

Using this, we can write the Hamiltonian in a more easily interpreted form:

$$\hat{H}_{\text{SSH}} = \sum_n \left[-t_1 \hat{c}^\dagger_{(2n+1)a} \hat{c}_{(2n)a} - t_2 \hat{c}^\dagger_{(2n+2)a} \hat{c}_{(2n+1)a} + \text{H.c.} \right] \tag{4}$$

We now see that the hopping is between odd and even numbered sites. The appropriate unit cell (see Fig. 1) must thus contain an even site (A) and an odd site (B), and thus we have a new lattice with lattice contant[1] $\tilde{a} \equiv 2a$. We can then write the Hamiltonian in terms of a sum over unit cells.

$$\hat{H}_{\text{SSH}} = \sum_n \left[-t_1 \hat{c}^\dagger_{n,B} \hat{c}_{n,A} - t_2 \hat{c}^\dagger_{n+1,A} \hat{c}_{n,B} + \text{H.c.} \right] \tag{5}$$

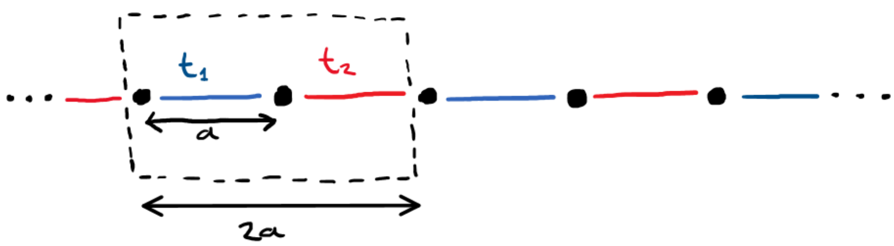


Figure 1: `The figure above shows a sketch SSH lattice. The lattice constant is` a, `but there are two different hopping elements` t_1 `and` t_2. `Our choice of unit cell (dashed square) has length` $2a$ `and has intra-cell hopping element` t_1, `while` t_2 `is the inter-cell hopping element.`

[1] This is important since the crystal momentum will be in terms of $\tilde{a}$, i.e. $k_n = \frac{2\pi}{N\tilde{a}} n$

Using the Fourier convention $\hat{c}_r = {}^1/\sqrt{N} \sum_k \mathrm{e}^{\mathrm{i}kr} \hat{c}_k$, we can also define the even and odd-site operators in the n'th cell as:

$$\hat{c}_{n,A} \equiv \hat{c}_{2na} = \frac{1}{\sqrt{N}} \sum_k \mathrm{e}^{\mathrm{i}k2na} \hat{c}_{k,A} \tag{6a}$$

$$\hat{c}_{n,B} \equiv \hat{c}_{(2n+1)a} = \frac{1}{\sqrt{N}} \sum_k \mathrm{e}^{\mathrm{i}k(2n+1)a} \hat{c}_{k,B} \tag{6b}$$

Since the lattice of unit cells is translation invariant, we expect the Hamiltonian to be simpler in momentum space. Based on this we now rewrite the Hamiltonian using Eq. (6a):

$$\begin{aligned}\hat{H}_{\mathrm{SSH}} = \Bigg[-t_1 \left(\frac{1}{\sqrt{N}} \sum_k \mathrm{e}^{-\mathrm{i}k(2n+1)a} \hat{c}^\dagger_{k,B}\right) \left(\frac{1}{\sqrt{N}} \sum_{k'} \mathrm{e}^{\mathrm{i}k2na} \hat{c}_{k'A}\right) - \\ t_2 \left(\frac{1}{\sqrt{N}} \sum_k \mathrm{e}^{-\mathrm{i}k2(n+1)a} \hat{c}^\dagger_{k,A}\right) \left(\frac{1}{\sqrt{N}} \sum_{k'} \mathrm{e}^{\mathrm{i}k(2n+1)a} \hat{c}_{k'B}\right) + \mathrm{H.c.}\Bigg] = \\ \sum_{k,k'} \Bigg[- t_1 \hat{c}^\dagger_{k,B} \hat{c}_{k',A} \mathrm{e}^{-\mathrm{i}ka} \underbrace{\frac{1}{N} \sum_n \mathrm{e}^{-\mathrm{i}2na(k-k')}}_{\delta_{k,k'}} - t_2 \hat{c}^\dagger_{k,A} \hat{c}_{k',B} \mathrm{e}^{-\mathrm{i}(2k-k')a} \underbrace{\frac{1}{N} \sum_n \mathrm{e}^{-\mathrm{i}2a(k-k')}}_{\delta_{k,k'}} + \mathrm{H.c.}\Bigg]\end{aligned} \tag{7}$$

Using the Kroenecker-delta's and writing the Hermitian conjugate terms explicitly we have[2]:

$$\hat{H}_{\mathrm{SSH}} = \sum_k \left[-t_1 \hat{c}^\dagger_{k,B} \hat{c}_{k,A} \mathrm{e}^{-ika} - t_1 \hat{c}^\dagger_{k,A} \hat{c}_{k,B} \mathrm{e}^{ika} - t_2 \hat{c}^\dagger_{k,A} \hat{c}_{k,B} \mathrm{e}^{-ika} - t_2 \hat{c}^\dagger_{k,B} \hat{c}_{k,A} \mathrm{e}^{ika}\right] \tag{8}$$

If we define the "spinor" $\hat{\psi}^\dagger_k \equiv \left(\hat{c}^\dagger_{k,A}, \hat{c}^\dagger_{k,B}\right)$, we can write the Hamiltonian in matrix form as:

$$\hat{H} = \sum_k \hat{\psi}^\dagger_k \cdot \boldsymbol{h}_k \cdot \hat{\psi}_k \equiv \sum_k \begin{pmatrix} \hat{c}^\dagger_{k,A} & \hat{c}^\dagger_{k,B} \end{pmatrix} \begin{bmatrix} 0 & -t_1 \mathrm{e}^{ika} - t_2 \mathrm{e}^{-ika} \\ -t_1 \mathrm{e}^{-ika} - t_2 \mathrm{e}^{ika} & 0 \end{bmatrix} \begin{pmatrix} \hat{c}_{k,A} \\ \hat{c}_{k,B} \end{pmatrix} \tag{9}$$

The matrix $\boldsymbol{h}_k$ is Hermitian, and thus unitarily diagonalisable. If denote the appropiate unitary $\boldsymbol{U}$, we have:

$$\hat{H} = \sum_k \hat{\psi}^\dagger_k \boldsymbol{U}\boldsymbol{U}^\dagger \boldsymbol{h}_k \boldsymbol{U}\boldsymbol{U}^\dagger \psi_k = \sum_k \underbrace{\left(\boldsymbol{U}^\dagger \hat{\psi}_k\right)^\dagger}_{\hat{\tilde{\psi}}^\dagger_k} \underbrace{\left[\boldsymbol{U}^\dagger \boldsymbol{h}_k \boldsymbol{U}\right]}_{\begin{bmatrix} E_+ & 0 \\ 0 & E_- \end{bmatrix}} \underbrace{\left(\boldsymbol{U}^\dagger \hat{\psi}_k\right)}_{\hat{\tilde{\psi}}_k} \equiv \sum_{k,n} \hat{\tilde{\psi}}^\dagger_{k,n} \hat{\tilde{\psi}}_{k,n} E_{k,n}$$

where we have defined a new set of operators, and introduced a band index $n = \pm$ which simply labels the entries in the diagonal matrix. The dispersion relation for the two bands can now be determined by finding the eigenvalues of $\boldsymbol{h}_k$. To do so we solve the characteristic equation

$$\begin{aligned}\det\left(\boldsymbol{h}_k - E\mathbb{1}\right) = 0 \Leftrightarrow \\ E^2 - \left(-t_1 \mathrm{e}^{ika} - t_2 \mathrm{e}^{-ika}\right)\left(-t_1 \mathrm{e}^{-ika} - t_2 \mathrm{e}^{ika}\right) = 0 \Leftrightarrow \\ E_\pm = \pm\sqrt{t_1^2 + t_2^2 + t_1 t_2 \mathrm{e}^{-2ika} + t_1 t_2 \mathrm{e}^{2ika}}\end{aligned} \tag{10}$$

We thus find that there are two bands with dispersion relations:

[2] For simplicity we take $t_1, t_2 \in \mathbb{R}$

$$\boxed{E_{k,\pm} = \pm\sqrt{t_1^2 + t_2^2 + 2t_1 t_2 \cos(\tilde{a}k)}} \tag{11}$$

where we have recognised the appearance of the lattice constant $\tilde{a} = 2a$. We note that having two different hopping elements introduce two bands, separated by non-zero band gap $\Delta E = 2\,|t_1 - t_2|$ (see Fig. 3a, Fig. 3b or Fig. 2 for a sketches of the dispersion in three different schemes!).

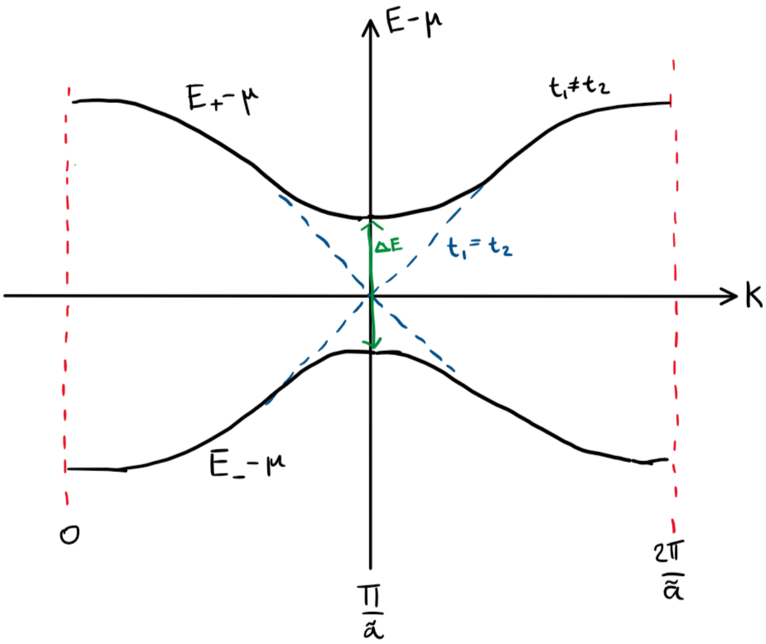


Figure 2: The figure above shows a sketch Sketch of the dispersion relation Eq. (11) in a single Brillouin zone (BZ). We see that for $t_1 = t_2$ (dashed blue) the dispersion is gapless and is linear near the Fermi surface, while for $t_1 \neq t_2$ (blasck solid) a gap ΔE is opened (green solid) in the spectrum

For $t_1 = t_2$ the result should reduce to the usual single-band dispersion relation:

$$E_k = -2\,|t|\cos(ka) \tag{12}$$

To see how Eq. (11) reduces to Eq. (12), we first set $t_1 = t_2$:

$$E_{k,\pm}|_{t_1=t_2=t} = \pm\sqrt{2t^2 + 2t^2\cos(\tilde{a}k)} = 2\,|t|\,\sqrt{\frac{1}{2} + \frac{1}{2}\cos(\tilde{a}k)} \tag{13}$$

Using the identity $2\cos^2\left(\frac{x}{2}\right) = 1 + \cos(x)$ we find:

$$E_{k,\pm}|_{t_1=t_2=t} = \pm 2\,|t|\left|\cos\left(\frac{k\tilde{a}}{2}\right)\right| \tag{14}$$

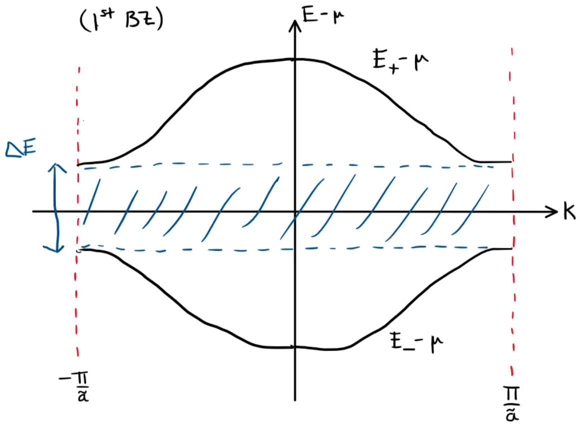
Let us now consider how this expression behaves in the extended zone scheme (see Fig. 3b for a sketch of the gapped dispersion in the extended zone scheme), i.e. where we consider the lower band in the first BZ and the upper band in the second BZ. Since the lattice constant is $\tilde{a}$, when we use our odd-even unit cell, we have:

$$\begin{cases} E_{k,-}|_{t_1=t_2=t} = 2\,|t|\cos\left(\frac{k\tilde{a}}{2}\right), & |k| \leq \frac{\pi}{\tilde{a}}\ (1^{\text{st}}\text{ BZ}) \\ E_{k,+}|_{t_1=t_2=t} = 2\,|t|\cos\left(\frac{k\tilde{a}}{2}\right), & \frac{\pi}{\tilde{a}} \leq |k| \leq \frac{2\pi}{\tilde{a}}\ (2^{\text{nd}}\text{ BZ}) \end{cases} \tag{15}$$

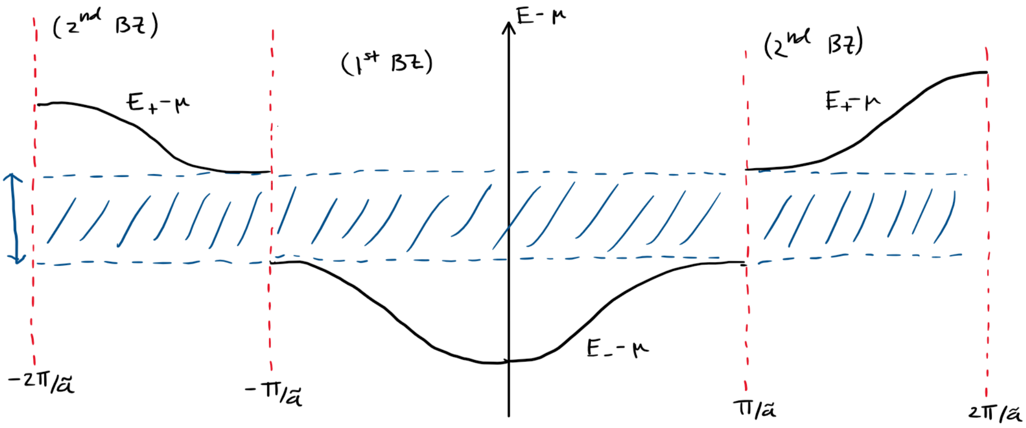
where we have used that the cosine is negative in the first BZ and positive in the second BZ. For a tight binding model with only one hopping element, the unit cell only needs to contain one site and has lattice constant a. In terms of $a = \tilde{a}/2$, we see that Eq. (15) is just a convoluted way of writing the expected dispersion relation in single BZ for a lattice with lattice constant a:

$$\boxed{E_{k,\pm}\big|_{t_1=t_2=t} = 2\,|t|\cos(ka), \quad |k| \leq \frac{\pi}{a}} \tag{16}$$

So we find that the dispersion relation reduces to the expected result, when we consider the appropriate lattice constant and the appropriate BZ.



(a) `SSH Dispersion in Reduced Zone Scheme`



(b) `SSH Dispersion in Extended Zone Scheme`

Figure 3: `Figure Fig. 3a Shows the dispersion relation Eq. (11) in the reduced zone scheme. Fig. 3b shows the same dispersion relation in the extended zone scheme.`

IV. Phenomenological Bosonization

IV.2 Bosonization of Bosons

There is a second brutal approximation, which is less rigorous (but it may still be justified by RG arguments): sometimes one can neglect the fluctuations of the density with respect to ρ_0: $\partial_x\theta(x) \ll \rho_0$ in the prefactor of ψ_B. In this case:

$$\psi_B^\dagger \approx \sqrt{\rho_0}\mathrm{e}^{-\mathrm{i}\varphi} \tag{17}$$

Consider the Hamiltonian:

$$H = \frac{1}{2m}\int \mathrm{d}x \left(\partial_x\psi_B^\dagger\right)(\partial_x\psi_B) + U\int \mathrm{d}x\,(\rho(x))^2 \tag{18}$$

IV.2.1 Bosonized Hamiltonian

By using this brutal approximation for ψ_B and:

$$\rho(x) = \rho_0 - \frac{\partial_x\theta(x)}{\pi} \tag{19}$$

for ρ, **express** H as a function of φ and θ.

Using the given approximation for the bosonized field the kinetic term becomes:

$$\begin{aligned} H_0[\phi,\theta] \approx \frac{1}{2m}\int \mathrm{d}x \left(\partial_x\left(\sqrt{\rho_0}\mathrm{e}^{\mathrm{i}\varphi}(x)\right)\right)\left(\partial_x\left(\sqrt{\rho_0}\mathrm{e}^{-\mathrm{i}\varphi(x)}\right)\right) = \\ \frac{\rho_0}{2m}\int \mathrm{d}x\left(\mathrm{i}\mathrm{e}^{\mathrm{i}\varphi(x)}\partial_x\varphi(x)\right)\left(-\mathrm{i}\mathrm{e}^{-\mathrm{i}\varphi(x)}\partial_x\varphi(x)\right) \Leftrightarrow \\ H_0[\phi,\theta] \approx \frac{\rho_0}{2m}\int \mathrm{d}x\,(\partial_x\varphi(x))^2 \end{aligned} \tag{20}$$

The density-density interaction becomes:

$$H_{\rho\text{-}\rho}[\varphi,\theta] \approx U\int \mathrm{d}x\left(\rho_0 - \frac{\partial_x\theta(x)}{\pi}\right)^2 = U\int \mathrm{d}x\left[\frac{1}{\pi^2}(\partial_x\theta(x))^2 - \frac{2\rho_0}{\pi}\partial_x\theta(x) + \rho_0^2\right] \tag{21}$$

We now drop the constant term and the term linear in $\partial_x\theta(x)$, since it can be integrated to a constant. We thus find that under the brutal approximation Eq. (17) for the field, and the approximation Eq. (19) for the density, the Hamiltonian becomes:

$$\boxed{H[\varphi,\theta] = \frac{1}{2\pi}\int \mathrm{d}x\left[\frac{\pi\rho_0}{m}(\partial_x\varphi)^2 + \frac{2U}{\pi}(\partial_x\theta)^2\right]} \tag{22}$$

We see that this is the Hamiltonian for an interacting Luttinger liquid:

$$H = \frac{1}{2\pi}\int \mathrm{d}x\left[uK(\partial_x\varphi)^2 + \frac{u}{K}(\partial_x\theta)^2\right] \tag{23}$$

where in this case:

$$K = \sqrt{(\tfrac{\pi\rho_0}{Um})/(\tfrac{2U}{\pi})} = \pi\sqrt{\frac{\rho_0}{2m}}, \quad u = \sqrt{\left(\frac{\pi\rho_0}{m}\right)\left(\frac{2U}{\pi}\right)} = \sqrt{\frac{2\rho_0 U}{m}}$$

That is, the Luttinger parameter K and the superfluid velocity u corresponding to Eq. (22) are:

$$K = \pi\sqrt{\frac{\rho_0}{2Um}} \tag{24a}$$

$$u = \sqrt{\frac{2\rho_0 U}{m}} \tag{24b}$$

IV.2.2 Lagrangian and Equations of Motion

By using that $\partial_x\theta/\pi$ and φ are canonically conjugate fields, **find** the Lagrangian for φ and its equations of motion.

Hint: The canonical conjugation is the relation:

$$\left[\frac{\partial_{x'}\theta(x')}{\pi}, \varphi(x)\right] = -\mathrm{i}\delta(x - x') \tag{25}$$

In this way you can obtain the canonical "momentum" operator Π_φ such that:

$$[\varphi(x), \Pi_\varphi(x')] = \mathrm{i}\delta(x - x') \tag{26}$$

This allows you in turn to get the relation between Π_φ and $\partial_t\phi$ needed in the Legendre transformation to obtain the Lagrangian.

Comparing the canonical conjugation relation Eq. (25) and the canonical commutation relation Eq. (26) we find the relation:

$$\Pi_\varphi = \frac{\partial_x\theta}{\pi} \tag{27}$$

We now recall the duality relation:

$$\partial_x\theta = \frac{K}{u}\partial_t\varphi \tag{28}$$

Using K and u from Eq. (24) the duality relation becomes:

$$\partial_x\theta = \frac{\pi}{2U}\partial_t\varphi \tag{29}$$

The canonical mometum conjugate to φ then is:

$$\Pi_\varphi = \frac{1}{2U}\partial_t\varphi \tag{30}$$

Using the duality relation Eq. (29) Hamiltonian Eq. (22) can be expressed in terms of only φ:

$$H = \int \mathrm{d}x \left[\frac{\rho_0}{2m}(\partial_x\varphi)^2 + \frac{1}{4U}(\partial_t\varphi)^2\right] \tag{31}$$

We can now obtain the Lagrangian by performing a Legendre tranformation:

$$L = \int \mathrm{d}x \Pi_\varphi \partial_t \varphi - H \tag{32}$$

Using the canonical momentum Eq. (30) and the Hamiltonian Eq. (31), we find that the Lagrangian is given by:

$$\boxed{L = \int \mathrm{d}x \left[\frac{1}{4U} \left(\partial_t \varphi\right)^2 - \frac{\rho_0}{2m} \left(\partial_x \varphi\right)^2 \right]} \tag{33}$$

To find the equations of motion, we first note that the Lagrangian density $\mathcal{L}$ corresponding to Eq. (33) is:

$$\mathcal{L} = \frac{1}{4U} \left(\partial_t \varphi\right)^2 - \frac{\rho_0}{2m} \left(\partial_x \varphi\right)^2 \tag{34}$$

We also recall that the EOM for the field φ can be obtained from the Euler-Lagrange equation:

$$\frac{\partial \mathcal{L}}{\partial \varphi} - \partial_x \frac{\partial \mathcal{L}}{\partial \left(\partial_x \varphi\right)} - \partial_t \frac{\partial \mathcal{L}}{\partial \left(\partial_t \varphi\right)} = 0 \tag{35}$$

We then have:

$$\begin{gathered} \partial_x \left[-\frac{\rho_0}{m} \partial_x \varphi \right] + \partial_t \left[\frac{1}{2U} \partial_t \varphi \right] = 0 \Leftrightarrow \\ \partial_x^2 \varphi - \frac{m}{2\rho_0 U} \partial_t^2 \varphi = 0 \end{gathered} \tag{36}$$

We now recognise that the coefficient in the second term is related to the superfluid velocity Eq. (24b), and so we find that the EOM can be written:

$$\boxed{\left(\partial_x - \frac{1}{u^2} \partial_t \right) \varphi(x,t) = 0} \tag{37}$$

We recognise this as the Klein-Gordon equation for a massless bosonic field.

IV.4 Bosonization of Fermions

Consider the definition of the density operator:

$$\rho(x) = \frac{k_F}{\pi} - \frac{\partial_x\theta(x)}{\pi} + \text{F.O.} \tag{38}$$

and its fast oscillating terms:

$$\text{F.O.} = \frac{\mathrm{i}}{2\pi a}\left[\mathrm{e}^{-2\mathrm{i}k_F x}\mathrm{e}^{2\mathrm{i}\theta(x)} - \mathrm{e}^{2\mathrm{i}k_F x}\mathrm{e}^{-2\mathrm{i}\theta(x)}\right] = \frac{1}{\pi a}\sin\left(2k_F x - 2\theta(x)\right) \tag{39}$$

IV.4.1 Bosonization of Density-Density Interaction

Calculate the operator $\rho(x)\rho(x')$ with a suitable Taylor expansion to order $(\partial_x\theta)^2$. Consider small distances $x - x' \approx a$, such that you can Taylor expand the fields ϕ and θ at first order, and the exponential at second order to get the required terms in $(\partial_x\theta)^2$. In doing so, separate fast and slow oscillating parts.

As a check: consider the density density interaction: $H_{\text{int}} = \tilde{U}\int \mathrm{d}x\rho(x)\rho(x-a)$; for $k_F \neq {}^{\pi}/_{2a}$, the slow oscillationg terms must yeild:

$$H_{\text{int,slow}} = \int \mathrm{d}x\frac{\tilde{U}}{\pi^2}\left(1 - \cos(2k_F a)\right)(\partial_x\theta)^2 + \frac{\tilde{U}}{a\pi^2}\sin(2k_F a)\partial_x\theta - \frac{2\tilde{U}k_F}{\pi^2}\partial_x\theta \tag{40}$$

The second and third terms can be integrated out and give only a constant contribution (which is zero for systems with a conserved number of particles!). $\tilde{U}$ has units of energy times distance ($\tilde{U} = Ua$)

Using the given approximation for the bosonized density operator, we have:

$$\begin{gathered}\rho(x)\rho(x') = \left(\rho_0 - \frac{\partial_x\theta(x)}{\pi} + \text{F.O.}(x)\right)\left(\rho_0 - \frac{\partial_{x'}\theta(x')}{\pi} + \text{F.O.}(x')\right) = \\ \rho_0^2 - \rho_0\frac{\partial_x\theta(x)}{\pi} - \rho_0\frac{\partial_{x'}\theta(x')}{\pi} + \frac{\partial_x\theta(x)}{\pi}\frac{\partial_{x'}\theta(x')}{\pi} + \\ \rho_0\text{F.O.}(x') + \rho_0\text{F.O.}(x) - \frac{\partial_x\theta(x)}{\pi}\text{F.O}(x') - \text{F.O.}(x)\frac{\partial_x'\theta(x')}{\pi} + \text{F.O.}(x)\text{F.O.}(x')\end{gathered}$$

Where we have used that for Fermions $\rho_0 = {}^{k_F}/_{\pi}$. The only potentially slowly oscillating terms in the above expressions are the ones with no factors of F.O. and the term with a product of fast oscillating terms. Let us first consider simplest slowly oscillating terms. Since we assume that the field $\theta(x)$, and $\phi(x)$ as well, wary slowly, and that a is a very small length, we make the approximation:

$$\rho_0^2 - \rho_0\frac{\partial_x\theta(x)}{\pi} - \rho_0\frac{\partial_{x'}\theta(x')}{\pi} + \frac{\partial_x\theta(x)}{\pi}\frac{\partial_{x'}\theta(x')}{\pi} \approx \rho_0^2 - 2\rho_0\frac{\partial_x\theta(x)}{\pi} + \left(\frac{\partial_x\theta(x)}{\pi}\right)^2 \tag{41}$$

Likewise we can approximate[3]:

$$\rho_0\text{F.O.}(x') + \rho_0\text{F.O.}(x) - \frac{\partial_x\theta(x)}{\pi}\text{F.O}(x') - \text{F.O.}(x)\frac{\partial_{x'}\theta(x')}{\pi} \approx \frac{2}{\pi a}\left(\rho_0 - \frac{\partial_x\theta(x)}{\pi}\right)\sin\left(2k_F x - 2\theta(x)\right), \tag{42}$$

[3]Here we truly just use the leading order approximation $x \approx x'$. The reasoning is that not only is this term fast oscillating at general fillings, but also at half filling, so it will not be relevant in any of the cases we consider, even if we include the $-2k_F a$ phase shift stemming from $x' \approx x - a$

where we have used that $[\partial_x\theta(x), \theta(x')] = 0$ to collect the terms with the derivatives. Finally we turn our attention to the final term:

$$\begin{aligned}\text{F.O.}(x)\text{F.O.}(x') = \left(\frac{\mathrm{i}}{2\pi a}\right)^2 \left(\mathrm{e}^{-2\mathrm{i}k_F x}\mathrm{e}^{2\mathrm{i}\theta(x)} - \mathrm{e}^{2\mathrm{i}k_F x}\mathrm{e}^{-2\mathrm{i}\theta(x)}\right)\left(\mathrm{e}^{-2\mathrm{i}k_F x'}\mathrm{e}^{2\mathrm{i}\theta(x')} - \mathrm{e}^{2\mathrm{i}k_F x'}\mathrm{e}^{-2\mathrm{i}\theta(x')}\right) = \\ -\frac{1}{(2\pi a)^2}\left(\mathrm{e}^{-2\mathrm{i}k_F x}\mathrm{e}^{2\mathrm{i}\theta(x)}\mathrm{e}^{-2\mathrm{i}k_F x'}\mathrm{e}^{2\mathrm{i}\theta(x')} + \mathrm{e}^{2\mathrm{i}k_F x}\mathrm{e}^{-2\mathrm{i}\theta(x)}\mathrm{e}^{2\mathrm{i}k_F x'}\mathrm{e}^{-2\mathrm{i}\theta(x')}\right) + \\ \frac{1}{(2\pi a)^2}\left(\mathrm{e}^{-2\mathrm{i}k_F x}\mathrm{e}^{2\mathrm{i}\theta(x)}\mathrm{e}^{2\mathrm{i}k_F x'}\mathrm{e}^{-2\mathrm{i}\theta(x')} + \mathrm{e}^{2\mathrm{i}k_F x}\mathrm{e}^{-2\mathrm{i}\theta(x)}\mathrm{e}^{-2\mathrm{i}k_F x'}\mathrm{e}^{2\mathrm{i}\theta(x')}\right)\end{aligned}$$

Using that the field commute with themselves: $[\theta(x), \theta(x')] = 0$, we can freely combine the exponentials:

$$\begin{aligned}\text{F.O.}(x)\text{F.O.}(x') = -\frac{1}{(2\pi a)^2}\left(\mathrm{e}^{-2\mathrm{i}(k_F(x+x')-\theta(x)-\theta(x'))} + \mathrm{e}^{2\mathrm{i}(k_F(x+x')-\theta(x)-\theta(x'))}\right) + \\ \frac{1}{(2\pi a)^2}\left(\mathrm{e}^{-2\mathrm{i}k_F(x-x')}\mathrm{e}^{2\mathrm{i}(\theta(x)-\theta(x'))} + \mathrm{e}^{2\mathrm{i}k_F(x-x')}\mathrm{e}^{-2\mathrm{i}(\theta(x)-\theta(x'))}\right)\end{aligned}$$

The terms in the first parentheses are fast oscillating. Using once again that the fields are assumed slowly oscillating, i.e. $\theta(x') \approx \theta(x)$, and using[4] $x + x' = 2x + a$ the fast oscillating term becomes:

$$-\frac{1}{(2\pi a)^2}\left(\mathrm{e}^{-2\mathrm{i}(k_F(x+x')-\theta(x)-\theta(x'))} + \mathrm{e}^{2\mathrm{i}(k_F(x+x')-\theta(x)-\theta(x'))}\right) \approx -\frac{2}{(2\pi a)^2}\cos\left(4k_F x - 4\theta(x) + 2k_F a\right) \tag{43}$$

Using the slow oscillation of the fields to make the first order expansion: $\theta(x) - \theta(x') \approx a\partial_x\theta(x)$, we can then write the slowly oscillating terms as:

$$\begin{aligned}\frac{1}{(2\pi a)^2}\left(\mathrm{e}^{-2\mathrm{i}k_F(x-x')}\mathrm{e}^{2\mathrm{i}(\theta(x)-\theta(x'))} + \mathrm{e}^{2\mathrm{i}k_F(x-x')}\mathrm{e}^{-2\mathrm{i}(\theta(x)-\theta(x'))}\right) \approx \\ \frac{1}{(2\pi a)^2}\left(\mathrm{e}^{-2\mathrm{i}k_F(x-x')}\mathrm{e}^{2\mathrm{i}a\partial_x\theta(x)} + \mathrm{e}^{2k_F\mathrm{i}(x-x')}\mathrm{e}^{-2\mathrm{i}a\partial_x\theta(x)}\right)\end{aligned} \tag{44}$$

Since the fields wary slowly, the derivatives are small, and we can Taylor expands the exponentials:

$$\mathrm{e}^{2\mathrm{i}a\partial_x\theta(x)} \approx 1 + 2a\mathrm{i}\partial_x\theta(x) - 2a^2\left(\partial_x\theta(x)\right)^2 \tag{45}$$

$$\mathrm{e}^{-2\mathrm{i}a\partial_x\theta(x)} \approx 1 - 2a\mathrm{i}\partial_x\theta(x) - 2a^2\left(\partial_x\theta(x)\right)^2 \tag{46}$$

The slowly oscillating term then is:

$$\begin{aligned}\frac{1}{(2\pi a)^2}\left(\mathrm{e}^{-2\mathrm{i}k_F(x-x')}\mathrm{e}^{2\mathrm{i}a\partial_x\theta(x)} + \mathrm{e}^{2k_F\mathrm{i}(x-x')}\mathrm{e}^{-2\mathrm{i}a\partial_x\theta(x)}\right) \approx \\ \frac{1}{(2\pi a)^2}\left(2\cos(2k_F a) + 4a\sin(2k_F a)\partial_x\theta(x) - 4a^2\cos(2k_F a)\left(\partial_x\theta(x)\right)^2\right)\end{aligned} \tag{47}$$

Combining all of the above we find that we get three types of contributions: First there are constant terms which can be removed by shifting the reference energy. Next there are slowly oscillating terms

[4] Note, unlike the fast oscillating term Eq. (39), we now need to use $x + x' = 2x + a$. If we didn't include the a in the density-density interaction, we would have a term like $\cos(4k_F x - 4\theta(x))$, which at half filling results in a different sign due to the lack $2k_F a$ phase shift!

(S.O.). Finally there are the fast oscillation (F.O.) terms:

$$\rho(x)\rho(x-a) \approx \underbrace{\rho_0^2 + \frac{1}{2(\pi a)^2}\cos(2k_F a) +}_{\text{cont.}} \tag{48a}$$

$$\underbrace{\frac{1}{a\pi^2}\sin(2k_F a)\partial_x\theta(x) - 2\rho_0\frac{\partial_x\theta(x)}{\pi} + \frac{1}{\pi^2}\left(1-\cos(2k_F a)\right)\left(\partial_x\theta(x)\right)^2 +}_{\text{S.O.}} \tag{48b}$$

$$\underbrace{\frac{2}{\pi a}\left(\rho_0 - \frac{\partial_x\theta(x)}{\pi}\right)\sin\left(2k_F x - 2\theta(x)\right) - \frac{2}{(2\pi a)^2}\cos\left(4k_F x - 4\theta(x) + 2k_F a\right)}_{\text{F.O.}} \tag{48c}$$

Dropping the constant terms since they just correspond to a shift in energy, we find that the density-density product, including the fast oscillating terms, is given by:

$$\boxed{\begin{aligned}\rho(x)\rho(x')\big|_{x-x'\approx a} \approx \left(\frac{1}{a\pi^2}\sin(2k_F a) - \frac{2k_F}{\pi^2}\right)\partial_x\theta(x) + \frac{1}{\pi^2}\left(1-\cos(2k_F a)\right)\left(\partial_x\theta(x)\right)^2 + \\ \frac{2}{\pi a}\left(\rho_0 - \frac{\partial_x\theta(x)}{\pi}\right)\sin\left(2k_F x - 2\theta(x)\right) - \frac{2}{(2\pi a)^2}\cos\left(4k_F x - 4\theta(x) + 2k_F a\right) \qquad (49)\end{aligned}}$$

Using this result, we find that that the slowly oscillating part of a density-density interaction:

$$H_{\text{int}} = \tilde{U}\int \mathrm{d}x\rho(x)\rho(x-a) \tag{50}$$

can be described by the bosonized Hamiltonian:

$$H_{\text{int,slow}} = \int \mathrm{d}x\frac{\tilde{U}}{\pi^2}\left(1-\cos(2k_F a)\right)\left(\partial_x\theta\right)^2 + \tilde{U}\left(\frac{1}{a\pi^2}\sin(2k_F a)\partial_x\theta - \frac{2k_F}{\pi^2}\right)\partial_x\theta \tag{51}$$

which is what we should expect.

IV.4.2 Interaction at Half-Filling

What happens for $k_F = \pi/2a$?

At certain fillings, fast oscillating terms can become slowly oscillating. To understand this we must first recall that we start from a discrete lattice model where the position is an integer multiple of the lattice constant[5], i.e. $x = ja, j \in \mathbb{Z}$. this means that at half filling, where $k_F = \pi/2a$ we have:

$$k_F x = j\frac{\pi}{2}, \quad \text{at half filling} \tag{52}$$

Consequently, if a terms is oscillating at a frequency, $4j'k_F, j' \in \mathbb{Z}$, we will have $k_F x = n2\pi, n \in \mathbb{Z}$. This means that any complex exponential depending on $4k_F x$ will simply be equal to unity at half

[5]This point is technically subtle. In the continuum model x is of course not an integer, and it is therefore not clear why the fast oscillation should simply give unity. It is possible to account for the special behaviour at half filling by including an additional term in the continuum Hamiltonian involving an interaction of the density and a periodic potential, but that is beyond the scope of this solution.

filling, and therefore doesn't oscillate fast! This is exactly the case for the last of the fast oscillating terms in the density-density interaction Eq. (49)

$$-\frac{2}{(2\pi a)^2}\cos\left(4k_F x - 4\theta(x) + 2k_F a\right)\Big|_{k_F=\frac{\pi}{2a}} = \frac{2}{(2\pi a)^2}\cos(4\theta(x)) \tag{53}$$

This term oscillated slowly at half filling, and will not simply average out any more if we integrate over position, as is the case in the Hamiltonian. We will return to the physical interpetation of this term in Problem V.3.3.

V. Field-Theoretical (Axiomatic) Bosonization

V.3 Details on the bosonization of fermions

Consider a one dimensional chain of fermions with generic $k_F \neq {}^{\pi}/{}_{2a}$ and a nearest-neighbor interaction as in Hamiltonian:

$$\hat{H} = \hat{H}_0 + \hat{H}_{\text{int}} \equiv \hat{H}_0 + U \sum_r \hat{c}^\dagger_{r+a} \hat{c}_{r+a} \hat{c}^\dagger_r \hat{c}_r \tag{54}$$

V.3.1 Bosonization of Nearest Neighbor Chain of Fermions

Based on the standard bosonization prescription:

$$\psi^\dagger(x) = \frac{1}{\sqrt{2\pi a}} \left[\mathrm{e}^{\mathrm{i}k_F x} \mathrm{e}^{-\mathrm{i}(\varphi(x)+\theta(x))} + \mathrm{e}^{-\mathrm{i}k_F x} \mathrm{e}^{-\mathrm{i}(\varphi(x)-\theta(x))} \right], \tag{55a}$$

$$\psi(x) = \frac{1}{\sqrt{2\pi a}} \left[\mathrm{e}^{-\mathrm{i}k_F x} \mathrm{e}^{\mathrm{i}(\varphi(x)+\theta(x))} + \mathrm{e}^{\mathrm{i}k_F x} \mathrm{e}^{\mathrm{i}(\varphi(x)-\theta(x))} \right] \tag{55b}$$

and the result Eq. (40), **derive** the slow-oscillating part of the Hamiltonian as a function of θ and φ, starting from the free Hamiltonian $\hat{H}_0$ and adding the interaction. Use a second order Taylor expansion considering the lattice spacing a as a small parameter. **verify** that you get:

$$H = \int \mathrm{d}x \, \frac{2ta \sin(k_F a)}{2\pi} \left[(\partial_x \varphi)^2 + (\partial_x \theta)^2 \right] + \int \mathrm{d}x \, \frac{U}{\pi^2} \left(1 - \cos(2k_F a)\right) (\partial_x \theta)^2 \tag{56}$$

First we take the continuum limit of Eq. (1) by following the pescription[6]:

$$\sum_r \to \int \frac{\mathrm{d}x}{a} \tag{57a}$$

$$c_r \to \sqrt{a}\psi(x) \tag{57b}$$

$$n_r = c_r^\dagger c_r \to a\rho(x) \tag{57c}$$

The continuum description of H_0 then becomes:

$$H_0 = -t \int \mathrm{d}x \left[\psi^\dagger(x+a)\psi(x) + \text{H.c.} \right] + \mu \int \mathrm{d}x \rho(x) \tag{58}$$

The chemical potential term, when bozonized according to Eq. (38), only gives a constant term and a term linear in $\partial_x \theta(x)$, so we will simply drop it[7]. To make the connection to our earlier results clearer we assume periodic boundary conditions, so that we may freely shift the variables and consider:

$$H_0 = -t \int \mathrm{d}x \left[\psi^\dagger(x)\psi(x-a) + \text{H.c.} \right] \tag{59}$$

[6]Note that in our convention, due to the $a^{-1/2}$ prefactor in Eq. (55), the factors of a^{-1} needed to convert the sum to an integral are already included in the fields.

[7]We could also say that for simplicity we we consider a system with a fixed number of particles, i.e. which can't exchange particles with a reservoir, and therefore we drop the chemical potential term.

We now want to bosonize the hopping term, which using $x' = x - a$, requires that we find:

$$\psi^\dagger(x)\psi(x') = \frac{1}{2\pi a}\left[\mathrm{e}^{\mathrm{i}k_F x}\mathrm{e}^{-\mathrm{i}(\varphi(x)+\theta(x))} + \mathrm{e}^{-\mathrm{i}k_F x}\mathrm{e}^{-\mathrm{i}(\varphi(x)-\theta(x))}\right]\left[\mathrm{e}^{-\mathrm{i}k_F x'}\mathrm{e}^{\mathrm{i}(\varphi(x')+\theta(x'))} + \mathrm{e}^{\mathrm{i}k_F x'}\mathrm{e}^{\mathrm{i}(\varphi(x')-\theta(x'))}\right] =$$
$$\frac{1}{2\pi a}\left[\mathrm{e}^{\mathrm{i}k_F(x-x')}\mathrm{e}^{-\mathrm{i}(\varphi(x)+\theta(x))}\mathrm{e}^{\mathrm{i}(\varphi(x')+\theta(x'))} + \mathrm{e}^{-\mathrm{i}k_F(x-x')}\mathrm{e}^{-\mathrm{i}(\varphi(x)-\theta(x))}\mathrm{e}^{\mathrm{i}(\varphi(x')-\theta(x'))} + \right.$$
$$\left.\mathrm{e}^{\mathrm{i}k_F(x+x')}\mathrm{e}^{-\mathrm{i}(\varphi(x)+\theta(x))}\mathrm{e}^{\mathrm{i}(\varphi(x')-\theta(x'))} + \mathrm{e}^{-\mathrm{i}k_F(x+x')}\mathrm{e}^{-\mathrm{i}(\varphi(x)-\theta(x))}\mathrm{e}^{\mathrm{i}(\varphi(x')+\theta(x'))}\right]$$

To further reduce this expression we must combine the exponentials. However, the fields θ and ϕ don't commute, but satisfy the commutation relation:

$$\left[\theta(x), \varphi(x')\right] = -\mathrm{i}\pi\Theta(x - x') \tag{60}$$

Since the commutator of θ and φ is number, we can use the following version of the BCH lemma:

$$\mathrm{e}^A\mathrm{e}^B = \mathrm{e}^{A+B+\frac{1}{2}[A,B]} \tag{61}$$

First let us consider the terms proportional to $\mathrm{e}^{\pm\mathrm{i}k_F(x-x')}$, i.e. the slowly oscillating ones:

$$\begin{aligned}\frac{1}{2\pi a}&\left[\mathrm{e}^{\mathrm{i}k_F(x-x')}\mathrm{e}^{-\mathrm{i}(\varphi(x)+\theta(x))}\mathrm{e}^{\mathrm{i}(\varphi(x')+\theta(x'))} + \mathrm{e}^{-\mathrm{i}k_F(x-x')}\mathrm{e}^{-\mathrm{i}(\varphi(x)-\theta(x))}\mathrm{e}^{\mathrm{i}(\varphi(x')-\theta(x'))}\right] = \\ &\frac{1}{2\pi a}\left[\mathrm{e}^{\mathrm{i}k_F a}\mathrm{e}^{-\mathrm{i}(\phi(x)-\phi(x')+\theta(x)-\theta(x'))-\mathrm{i}\frac{\pi}{2}} + \mathrm{e}^{-\mathrm{i}k_F a}\mathrm{e}^{-\mathrm{i}(\varphi(x)-\varphi(x')+\theta(x')-\theta(x))+\mathrm{i}\frac{\pi}{2}}\right]\end{aligned} \tag{62}$$

where we have used the commutators:

$$\left[-\mathrm{i}\left(\varphi(x)+\theta(x)\right), \mathrm{i}\left(\varphi(x-a)+\theta(x-a)\right)\right] = \underbrace{\left[\varphi(x), \theta(x-a)\right]}_{0} + \underbrace{\left[\theta(x), \varphi(x-a)\right]}_{-\mathrm{i}\pi} = -\mathrm{i}\pi \tag{63}$$

$$\left[-\mathrm{i}\left(\varphi(x)-\theta(x)\right), \mathrm{i}\left(\varphi(x-a)-\theta(x-a)\right)\right] = -\underbrace{\left[\varphi(x), \theta(x-a)\right]}_{0} - \underbrace{\left[\theta(x), \varphi(x-a)\right]}_{-\mathrm{i}\pi} = \mathrm{i}\pi \tag{64}$$

Next we expand the fields, assuming that they are slowly oscillating and a is small:

$$\theta(x) - \theta(x-a) \approx a\partial_x\theta(x) \tag{65a}$$
$$\varphi(x) - \varphi(x-a) \approx a\partial_x\varphi(x) \tag{65b}$$

Using the Taylor expansions of the fields, and rewriting the phases $\mathrm{e}^{\pm\mathrm{i}\frac{\pi}{2}} = \pm\mathrm{i}$ we get:

$$\begin{aligned}\frac{1}{2\pi a}&\left[\mathrm{e}^{\mathrm{i}k_F a}\mathrm{e}^{-\mathrm{i}(\phi(x)-\phi(x')+\theta(x)-\theta(x'))-\mathrm{i}\frac{\pi}{2}} + \mathrm{e}^{-\mathrm{i}k_F a}\mathrm{e}^{-\mathrm{i}(\varphi(x)-\varphi(x')+\theta(x')-\theta(x))+\mathrm{i}\frac{\pi}{2}}\right] \approx \\ &\frac{1}{2\pi a}\left[-\mathrm{i}\mathrm{e}^{\mathrm{i}k_F a}\mathrm{e}^{-\mathrm{i}a(\partial_x\theta(x)+\partial_x\varphi(x))} + \mathrm{i}\mathrm{e}^{-\mathrm{i}k_F a}\mathrm{e}^{\mathrm{i}a(\partial_x\theta(x)-\partial_x\varphi(x))}\right] \approx \\ &\frac{1}{2\pi a}\left[\mathrm{i}\mathrm{e}^{-\mathrm{i}k_F a}\left(1 + \mathrm{i}a\partial_x\theta(x) - \mathrm{i}a\partial_x\varphi(x) - \frac{a^2}{2}\left(\partial_x\theta(x) - \partial_x\varphi(x)\right)^2\right)\right. \\ &\left.-\mathrm{i}\mathrm{e}^{\mathrm{i}k_F a}\left(1 - \mathrm{i}a\partial_x\theta(x) - \mathrm{i}\partial_x\varphi(x) - \frac{a^2}{2}\left(\partial_x\theta(x) + \partial_x\varphi(x)\right)^2\right)\right]\end{aligned} \tag{66}$$

Where we have used that the derivatives are small, to also expand the exponentials, to second order. We thus find:

$$\begin{aligned}\psi^\dagger(x)\psi(x-a) \approx -\frac{a\sin(k_F a)}{2\pi}\left(\left(\partial_x\theta(x)\right)^2 + \left(\partial_x\varphi(x)\right)^2\right) - \frac{1}{\pi}\left(\cos(k_F a)\partial_x\theta - \mathrm{i}\sin(k_F a)\partial_x\varphi\right) + \\ \mathrm{i}\frac{a}{2\pi}\cos(k_F a)\{\partial_x\theta, \partial_x\varphi\} + \frac{1}{\pi a}\sin(k_F a) + \text{F.O.}\end{aligned} \tag{67}$$

where $\{A, B\} \equiv AB + BA$ is the anti-commutator and F.O. is the fast oscillating term. Even though we will neglect the fast oscillating terms $\sim e^{\pm i(x+x')}$, we now find them for the sake of completeness. Using the same approach as for the slowly oscillating term we have:

$$\text{F.O.} = \frac{1}{2\pi a}\left[e^{ik_F(x+x')}e^{-i(\varphi(x)+\theta(x))}e^{i(\varphi(x')-\theta(x'))} + e^{-ik_F(x+x')}e^{-i(\varphi(x)-\theta(x))}e^{i(\varphi(x')+\theta(x'))}\right] = \frac{1}{2\pi a}\left[e^{-ik_F(x+x')}e^{2i\theta(x)+i\frac{\pi}{2}} + e^{ik_F(x+x')}e^{-2i\theta(x)-i\frac{\pi}{2}}\right] \tag{68}$$

where we have used $\theta(x) + \theta(x-a) \approx 2\theta(x)$ and have neglected $\varphi(x) - \varphi(x-a) \approx a\partial_x\varphi(x)$, since we assume that the fields are large compared to the derivatives, which are small due to the slowly varying fields. Finally we approximate[8] $x + x' = 2x + a \approx 2x$, and find:

$$\text{F.O.} = \frac{1}{2\pi a}\left[ie^{-i2k_F x}e^{2i\theta(x)+i\frac{\pi}{2}} - ie^{i2k_F x}e^{-2i\theta(x)}\right] \tag{69}$$

We can now include the hermitian conjugate of Eq. (67), and after dropping the fast oscillating terms we get:

$$\psi^\dagger(x)\psi(x-a) + \text{H.c.} \approx -\frac{a}{\pi}\sin(k_F a)\left((\partial_x\theta)^2 + (\partial_x\varphi)^2\right) - \frac{2}{\pi}\cos(k_F a)\partial_x\theta + \frac{2}{\pi a}\sin(k_F a) \tag{70}$$

We thus find that the bosonized hopping Hamiltonian is given by:

$$H_0 = \int dx \frac{at\sin(k_F a)}{\pi}\left[(\partial_x\theta)^2 + (\partial_x\varphi)^2\right] \tag{71}$$

where usual we drop the term linear in $\partial_x\theta(x)$ and the constant term since they are unimportant. Finally we turn our attention to the interaction term:

$$H_{\text{int}} = U\sum_r c^\dagger_{r+a}c_{r+a}c^\dagger_r c_r = U\sum_r n^\dagger_{r+a}n_r \to U\int dx\ a\rho(x+a)\rho(x) \tag{72}$$

Shifting the variable of the integral by a, we can directly use the result from Problem IV.4.1. Doing so we find that the bosonized Hamiltonian, when dropping constant and boundary terms, is given by:

$$\boxed{H = \int dx\ \frac{at\sin(k_F a)}{\pi}\left[(\partial_x\varphi)^2 + (\partial_x\theta)^2\right] + \int dx\ \frac{Ua}{\pi^2}\left(1 - \cos(2k_F a)\right)(\partial_x\theta)^2} \tag{73}$$

Note that in this case U actually has dimension of energy.

[8] Note this is consistent with Eq. (39)

V.3.2 Luttinger Parameter and Fermi Velocity

From Eq. (56) **calculate** the value of K and u as a function of U and v_F, where:

$$v_F = 2ta\sin(k_F a) \tag{74}$$

is the Fermi velocity of the non-interacting model H_0.

Using the trigonometric identity $1 - \cos(x) = 2\sin^2\left(\frac{x}{2}\right)$, and factoring out a $1/2\pi$, we can rewrite the Hamiltonian as:

$$H = \frac{1}{2\pi}\int \mathrm{d}x\ 2ta\sin(k_F a)\left[(\partial_x\varphi)^2 + (\partial_x\theta)^2\right] + \frac{1}{2\pi}\int \mathrm{d}x\ \frac{4Ua}{\pi}\sin^2(k_F a)\,(\partial_x\theta)^2 \tag{75}$$

Using the Fermi velocity $v_F = 2ta\sin(k_F a)$ for the non-interacting electrons, we can rewrite this as:

$$H = \frac{1}{2\pi}\int \mathrm{d}x\left[v_F\,(\partial_x\varphi)^2 + \left(v_F + \frac{Uv_F^2}{\pi a t^2}\right)(\partial_x\theta)^2\right] \tag{76}$$

To find the Luttinger parameter and Fermi velocity, recall that the Hamiltonian for the Luttinger liquid is:

$$H = \frac{1}{2\pi}\int \mathrm{d}x\left(uK\,(\partial_x\varphi)^2 + \frac{u}{K}\,(\partial_x\theta)^2\right) \tag{77}$$

Comparing Eq. (76) to the Luttinger liquid Eq. (77), we find:

$$K = \sqrt{\frac{A_\varphi}{B_\theta}}, \quad u = \sqrt{A_\varphi B_\theta} \tag{78}$$

where A_φ is the coefficient of $(\partial_x\varphi)^2$, and B_θ the coefficient of $(\partial_x\theta)^2$. Using these relations we find that the luttinger parameter is given by:

$$K = \sqrt{\frac{v_F}{\left(v_F + \frac{Uv_F^2}{\pi a t^2}\right)}} = \frac{1}{\sqrt{1 + \frac{Uv_F}{\pi a t^2}}} \tag{79}$$

We note that in the non-interacting limit:

$$\lim_{U\to 0} K = 1 \tag{80}$$

If we consider a *genuine* continuum model, we can also take the strongly interacting limit[9]

$$\lim_{U\to\infty} K = 0 \tag{81}$$

[9] Note that in general what constitutes the strongly interacting limit is a subtle matter. In essence, it depends on what model we actually consider, i.e. did we start from a genuine continuum model or did we take the continuum limit of a discrete model. If we start from a lattice model, K will be bound from below, with the bound being set by the interaction range. Moral of the story: we can always trust the expression for K in the weak interaction limit, while in the strong interaction limit we must think more carefully to avoid systematic errors which arise when going from the lattice to the continuum.

as we should expect for Fermions. The Fermi velocity is

$$u = \sqrt{v_F\left(v_F + \frac{Uv_F^2}{\pi a t^2}\right)} = v_F\sqrt{1 + \frac{Uv_F}{\pi a t^2}} \tag{82}$$

In the non-interaction limit this reduces to the Fermi-velocity for the tight-binding model:

$$\lim_{U\to 0} u = v_F \tag{83}$$

while in the strongly interacting limit the Fermi velocity diverges:

$$\lim_{U\to\infty} u = \infty \tag{84}$$

We have thus found that the Luttinger parameter and Fermi-velocity for interacting Fermions are:

$$\boxed{K = \frac{1}{\sqrt{1 + \frac{Uv_F}{\pi a t^2}}}, \quad u = v_F\sqrt{1 + \frac{Uv_F}{\pi a t^2}}} \tag{85}$$

V.3.3 Interaction at Half Filling

What is the additional (slow-oscillating) interaction term that appears for $k_F = \pi/2a$

As we dicussed in Problem IV.4.2 fast oscillating terms become relevant at half filling, i.e. $k_F = \pi/2a$ if they oscillate like $\mathrm{e}^{\pm \mathrm{i}4k_F x}$. Such terms arise from the density-density interaction, since 4 factors of ψ or $\psi^\dagger$, or combinations thereof, are needed to get the required phase. The only fast oscillating term, which arises when bosonizing Eq. (54), which become relevant at half filling, is again the contribution to $\rho(x)\rho(x')$ in Eq. (53). This results in a sine-Gordon interaction term given by:

$$H_{\text{Umklapp}} = \frac{1}{2\pi}\int \mathrm{d}x \frac{U}{a\pi}\cos(4\theta(x)) \tag{86}$$

This term is physically due to Umklapp scattering. To understand why this is the case, recall that the fermionic field can be expressed in terms of right and left moving fields:

$$\psi^\dagger(x) = \mathrm{e}^{\mathrm{i}k_F x}\psi_L^\dagger + \mathrm{e}^{-\mathrm{i}k_F x}\psi_R^\dagger, \tag{87a}$$

$$\psi(x) = \mathrm{e}^{-\mathrm{i}k_F x}\psi_L + \mathrm{e}^{\mathrm{i}k_F x}\psi_R \tag{87b}$$

Since the density-density interaction is given by product $\rho\rho = \psi^\dagger\psi\psi^\dagger\psi$, we see that fast oscillating terms which become relevant at half-filling are:

$$\rho(x)\rho(x') = \mathrm{e}^{\mathrm{i}4k_F(x+x')}\psi_L^\dagger\psi_R\psi_L^\dagger\psi_R + \mathrm{e}^{-4\mathrm{i}k_F(x+x')}\psi_R^\dagger\psi_L\psi_R^\dagger\psi_L + \dots \tag{88}$$

The term $\sim \psi_L^\dagger\psi_L^\dagger\psi_R\psi_R$ describes two right moving electrons scattering into to left moving, while the term $\sim \psi_R^\dagger\psi_R^\dagger\psi_L\psi_L$ describes the opposite process. These are indeed both Umklapp processes, but why are these processes important at half filling? The explanation lies in periodicity of the BZ. At half filling the scattering process which changes the momentum $k = k_F \to k' = k_F + 2k_F$ and $k = k_F \to k' = -k_F$ are equivalent, due to the periodicity of the BZ. Consequently, exactly at half filling the processes with momentum exchange $2k_F$ can scatter two left movers into two right movers and vice versa, since they scatter from one half of the BZ and into the other, and thus obtain the opposite momentum.

We now understand that the additional sine-gordon like interaction term Eq. (86), which becomes relevant[10] at half filling, i.e. $k_F = \frac{\pi}{2a}$, is due to the fact that Umklapp scattering becomes important at half filling[11].

V.4 Bosonization of 1D p-wave SC

Let us now consider a superconducting system, to get a feeling about how to use all this machinary. Take the previous chain (i.e. from Exercise IV.4 for $k_F \neq \pi/2a$. Let us suppose that our chain is put in proximity with a p-wave superconductor. Therefore we include an additional term in the tight-binding model:

$$H_{\mathrm{SC}} = -\Delta \sum_r \left(c_r^\dagger c_{r+a}^\dagger + c_{r+a} c_r \right), \tag{89}$$

with real $\Delta > 0$.

V.4.1 Bosonization of Superconducting Term

Find the bosonized description of the previous term (as usual, consider the lattice spacing a to be a small parameter).

We start by taking the continuum limit of Eq. (89), following the usual prescription Eq. (57):

$$H_{\mathrm{SC}} = -\Delta \int \mathrm{d}x \left[\psi^\dagger(x) \psi^\dagger(x+a) + \mathrm{H.c.} \right] \tag{90}$$

In terms of the bosonized fields Eq. (55), the pair creation operator is:

$$\begin{aligned}
\psi^\dagger(x)\psi^\dagger(x+a) = \frac{1}{2\pi a} & \left[\mathrm{e}^{\mathrm{i}k_F x} \mathrm{e}^{-\mathrm{i}(\varphi(x)+\theta(x))} + \mathrm{e}^{-\mathrm{i}k_F x} \mathrm{e}^{-\mathrm{i}(\varphi(x)-\theta(x))} \right] \times \\
& \left[\mathrm{e}^{\mathrm{i}k_F (x+a)} \mathrm{e}^{-\mathrm{i}(\varphi(x+a)+\theta(x+a))} + \mathrm{e}^{-\mathrm{i}k_F (x+a)} \mathrm{e}^{-\mathrm{i}(\varphi(x+a)-\theta(x+a))} \right] = \\
\frac{1}{2\pi a} & \left[\mathrm{e}^{\mathrm{i}k_F a} \mathrm{e}^{-\mathrm{i}(\varphi(x)-\theta(x))} \mathrm{e}^{-\mathrm{i}(\varphi(x+a)+\theta(x+a))} + \mathrm{e}^{-\mathrm{i}k_F a} \mathrm{e}^{-\mathrm{i}(\varphi(x)+\theta(x))} \mathrm{e}^{-\mathrm{i}(\varphi(x+a)-\theta(x+a))} \right. \\
& \left. \mathrm{e}^{\mathrm{i}k_F (2x+a)} \mathrm{e}^{-\mathrm{i}(\varphi(x)+\theta(x))} \mathrm{e}^{-\mathrm{i}(\varphi(x+a)+\theta(x+a))} \mathrm{e}^{-\mathrm{i}k_F (2x+a)} \mathrm{e}^{-\mathrm{i}(\varphi(x)-\theta(x))} \mathrm{e}^{-\mathrm{i}(\varphi(x+a)-\theta(x+a))} \right]
\end{aligned} \tag{91}$$

The terms oscillating like $\mathrm{e}^{\pm \mathrm{i}2k_F x}$ are fast oscillating, so we will not consider them in detail. Next we use the BCH lemma Eq. (61) to combine the exponentials in the slowly oscillating term:

$$\begin{aligned}
& \frac{1}{2\pi a} \left[\mathrm{e}^{\mathrm{i}k_F a} \mathrm{e}^{-\mathrm{i}(\varphi(x)-\theta(x))} \mathrm{e}^{-\mathrm{i}(\varphi(x+a)+\theta(x+a))} + \mathrm{e}^{-\mathrm{i}k_F a} \mathrm{e}^{-\mathrm{i}(\varphi(x)+\theta(x))} \mathrm{e}^{-\mathrm{i}(\varphi(x+a)-\theta(x+a))} \right] = \\
& \frac{1}{2\pi a} \left[\mathrm{e}^{\mathrm{i}k_F a} \mathrm{e}^{-\mathrm{i}(\varphi(x+a)+\varphi(x)+\theta(x+a)-\theta(x)) - \mathrm{i}\frac{\pi}{2}} + \mathrm{e}^{-\mathrm{i}k_F a} \mathrm{e}^{-\mathrm{i}(\varphi(x)+\varphi(x+a)+\theta(x)-\theta(x+a)) + \mathrm{i}\frac{\pi}{2}} \right]
\end{aligned} \tag{92}$$

[10] Or at least slow oscillating

[11] We will later see that when the interaction strength U is sufficiently strong, the Umklapp scattering is relevant (in the RG sence) and leads to a phase transition to a Mott-like insulator

where we have used:

$$[-\mathrm{i}\left(\varphi(x)-\theta(x)\right), -\mathrm{i}\left(\varphi(x+a)+\theta(x+a)\right)] = -\underbrace{[\varphi(x),\theta(x+a)]}_{\mathrm{i}\pi} + \underbrace{[\theta(x),\varphi(x,+a)]}_{0} = -\mathrm{i}\pi \tag{93}$$

$$[-\mathrm{i}\left(\varphi(x)+\theta(x)\right), -\mathrm{i}\left(\varphi(x+a)-\theta(x+a)\right)] = \underbrace{[\varphi(x),\theta(x+a)]}_{\mathrm{i}\pi} - \underbrace{[\theta(x),\varphi(x,+a)]}_{0} = \mathrm{i}\pi \tag{94}$$

Based on the assumption that the fields vary slowly over the small distance a, we can make the approximations:

$$\varphi(x+a)+\varphi(x) \approx 2\varphi(x) \tag{95}$$

$$\theta(x+a)-\theta(x) \approx a\partial_x\theta(x) \tag{96}$$

We then have:

$$\begin{aligned}\frac{1}{2\pi a}&\left[\mathrm{e}^{\mathrm{i}k_Fa}\mathrm{e}^{-\mathrm{i}(\varphi(x+a)+\varphi(x)+\theta(x+a)-\theta(x))-\mathrm{i}\frac{\pi}{2}} + \mathrm{e}^{-\mathrm{i}k_Fa}\mathrm{e}^{-\mathrm{i}(\varphi(x)+\varphi(x+a)+\theta(x)-\theta(x+a))+\mathrm{i}\frac{\pi}{2}}\right] \approx \\ &\frac{1}{2\pi a}\left[-\mathrm{i}\mathrm{e}^{\mathrm{i}k_Fa}\mathrm{e}^{-\mathrm{i}(2\varphi(x)+a\partial_x\theta(x))} + \mathrm{i}\mathrm{e}^{-\mathrm{i}k_Fa}\mathrm{e}^{-\mathrm{i}(2\varphi(x)-a\partial_x\theta(x))}\right] \approx \\ &\frac{1}{2\pi a}\left[-\mathrm{i}\mathrm{e}^{\mathrm{i}k_Fa}\mathrm{e}^{-\mathrm{i}2\varphi(x)} + \mathrm{i}\mathrm{e}^{-\mathrm{i}k_Fa}\mathrm{e}^{-\mathrm{i}2\varphi(x)}\right]\end{aligned} \tag{97}$$

where we have assumed that $a\partial_x\theta(x)$ is negligible compared to φ, again based on the assumption that the fields vary slowly, but not that the fields are small! The bosonized pair-creation operator then is:

$$\psi^\dagger(x)\psi^\dagger(x+a) = \frac{1}{2\pi a}\left[\mathrm{i}\mathrm{e}^{-\mathrm{i}(k_Fa+2\varphi(x)} - \mathrm{i}\mathrm{e}^{\mathrm{i}(k_Fa-2\varphi(x))}\right] + \text{F.O.} \tag{98}$$

Including the hermitian conjugate, and dropping the fast oscillating terms, we get:

$$\psi^\dagger(x)\psi^\dagger(x+a) + \text{H.c.} = \frac{1}{2\pi a}\left[\mathrm{i}\mathrm{e}^{-\mathrm{i}(k_Fa+2\varphi(x))} - \mathrm{i}\mathrm{e}^{\mathrm{i}(k_Fa-2\varphi(x))} - \mathrm{i}\mathrm{e}^{\mathrm{i}(k_Fa+2\varphi(x))} + \mathrm{i}\mathrm{e}^{-\mathrm{i}(k_Fa-2\varphi(x))}\right] \Leftrightarrow \tag{99}$$

$$\psi^\dagger(x)\psi^\dagger(x+a) + \text{H.c.} = \frac{1}{\pi a}\left[\sin\left(k_Fa+2\varphi(x)\right) + \sin\left(k_Fa-2\varphi(x)\right)\right] \tag{100}$$

Using the trigonometric addition formula $\sin(x)+\sin(y) = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$, we can finally write the Hamiltonian Eq. (90) in its bosonized form:

$$\boxed{H_{\mathrm{SC}} = -\frac{2\Delta}{\pi a}\sin(k_Fa)\int \mathrm{d}x\cos(2\varphi(x))} \tag{101}$$

Alternatively we can write the Superconducting Hamiltonian in terms of vertex operators:

$$H_{\mathrm{SC}} = -g_\Delta\int \mathrm{d}x\left[V_\Delta(x) + V_\Delta^\dagger(x)\right], \quad V_\Delta(x) \equiv \mathrm{e}^{\mathrm{i}2\varphi(x)}, \quad g_\Delta \equiv \frac{\Delta}{\pi a}\sin(k_Fa) \tag{102}$$

which is more convenient when doing any RG analysis.

V.4.2 RG Analysis of Superconducting Perturbation

Based on $\langle\varphi(x)\varphi(0)\rangle = -\frac{1}{2K}\ln|x|$, **what** is the scaling dimension of the operator you found in H_{SC} as a function of K? **When** is it relevant?

We now wish to find the scaling dimension of the superconducting perturbation. To do so we examine the scaling dimension of the vertex operators. To do so, we first find the correlation function of the vertex operators, taken with respect to the *unperturbed* Luttinger liquid action:

$$\left\langle V_\Delta^\dagger(x)V_\Delta(0)\right\rangle = \left\langle \mathrm{e}^{-\mathrm{i}2\varphi(x)}\mathrm{e}^{2\mathrm{i}\varphi(0)}\right\rangle \tag{103}$$

Next we use $[\varphi(x), \varphi(0)]$ to combine the exponentials. Furthermore, since the unperturbed Luttinger liquid has a quadratic action, we can rewrite the correlation function exactly using a second order cumulant expansion[12]:

$$\left\langle V_\Delta^\dagger(x)V_\Delta(0)\right\rangle = \left\langle \mathrm{e}^{-2\mathrm{i}(\varphi(x)-\varphi(0))}\right\rangle = \mathrm{e}^{-2\mathrm{i}\langle\varphi(x)\rangle+2\mathrm{i}\langle\varphi(0)\rangle-2\left[\left\langle(\varphi(x)-\varphi(0))^2\right\rangle-\langle(\varphi(x)-\varphi(0))\rangle^2\right]} \tag{104}$$

Since the Luttinger liquid is transitionally invariant, the mean value is position independent:

$$\langle\varphi(x)\rangle = \langle\varphi(0)\rangle \tag{105}$$

So the correlation function simplifies to:

$$\left\langle V_\Delta^\dagger(x)V_\Delta(0)\right\rangle = \underbrace{\mathrm{e}^{-2\left(\left\langle\varphi^2(x)\right\rangle+\left\langle\varphi^2(0)\right\rangle\right)}}_{C}\mathrm{e}^{4\langle\varphi(x)\varphi(0)\rangle} = C\mathrm{e}^{-\frac{2}{K}\ln|x|} = \frac{C}{|x|^{2/K}} \tag{106}$$

We thus find that the correlation function for the Vertex operators, is:

$$\left\langle V_\Delta^\dagger(x)V_\Delta(0)\right\rangle = \frac{C}{|x|^{2/K}} \tag{107}$$

where C is a non-universal constant, i.e. it depends on the momentum cutoff, which is system dependent. We now examine how the correlation function behaves under a rescaling:

$$x = bx', \quad b = 1 + \mathrm{d}\ell \tag{108}$$

Rescaling Eq. (107), we have:

$$\begin{aligned}\left\langle V_\Delta^\dagger(x')V_\Delta(0)\right\rangle = C\mathrm{e}^{-\frac{2}{K}\ln|x|} = \frac{C}{|bx'|^{2/K}} \Leftrightarrow \\ \left\langle V_\Delta^\dagger(x')V_\Delta(0)\right\rangle = b^{-\frac{2}{K}}\frac{C}{|x'|^{2/K}}\end{aligned} \tag{109}$$

Since the correlation function decays as a power law, the scaling dimension D_Δ can be found from:

$$\left\langle V_\Delta^\dagger(x')V_\Delta(0)\right\rangle = b^{-2D_\Delta}\left\langle V_\Delta'^\dagger(x')V_\Delta'(0)\right\rangle \tag{110}$$

Comparing Eq. (109) and Eq. (110) we find that the scaling dimension of the superconductor vertex operator is:

$$\boxed{V_\Delta(x) = b^{-D_\Delta}V_\Delta'(x'), \quad D_\Delta = \frac{1}{K}} \tag{111}$$

[12]Recall that the second order cumulant expansion is: $\langle \mathrm{e}^x\rangle_0 = \mathrm{e}^{\langle x\rangle_0+\frac{1}{2}\left(\langle x^2\rangle-\langle x\rangle^2\right)}$

To determine when the Superconducting perturbation is relevant, we examine how the coupling constant g_Δ behaves under the RG flow. This requires that we first find the RG equation, which we can obtain as follows. First we note that the euclidean time action corresponding to the perturbation Eq. (101) is:

$$\mathcal{S}_{\mathrm{SC}} = -g_\Delta \int \mathrm{d}x\mathrm{d}\tau \left[V_\Delta(x) + V_\Delta^\dagger(x)\right] \tag{112}$$

Rescaling this action using Eq. (108) for x, and an analogous scaling for τ we have:

$$\mathcal{S}'_{\mathrm{SC}} = -_\Delta \int \mathrm{d}x'\mathrm{d}\tau'\ g_\Delta b^2 \left[b^{-D_\Delta} V'_\Delta(x') + b^{-D_\Delta} V'^\dagger_\Delta(x')\right] = -\int \mathrm{d}x'\mathrm{d}\tau'\ g_\Delta b^{2-D_\Delta} \left[V'_\Delta(x') + V'^\dagger_\Delta(x')\right] \tag{113}$$

Comparing this to:

$$\mathcal{S}'_{\mathrm{SC}} = -\int \mathrm{d}x'\mathrm{d}\tau'\ g'_\Delta \left[V'_\Delta(x') + V'^\dagger_\Delta(x')\right] \tag{114}$$

We find:

$$g'_\Delta = b^{2-D_\Delta} g_\Delta \tag{115}$$

For an infinitesimal rescaling $b = 1 + \mathrm{d}\ell$ we get:

$$\begin{aligned} g'_\Delta = (1+\mathrm{d}\ell)^{2-D_\Delta}\, g_\Delta &= (1 + (2-D_\Delta)\mathrm{d}\ell)\, g_\Delta \Leftrightarrow \\ \frac{g'_\Delta - g_\Delta}{\mathrm{d}\ell} &= (2-D_\Delta) g_\Delta \Rightarrow \\ \frac{\mathrm{d}g_\Delta(\ell)}{\mathrm{d}\ell} &= (2-D_\Delta) g_\Delta \end{aligned} \tag{116}$$

We thus have the RG equation:

$$g_\Delta(\ell) = g_\Delta(0)\mathrm{e}^{(2-D_\Delta)\ell} \tag{117}$$

The perturbation is relevant when the exponential is growing, i.e. for $D_\Delta < 2$. We thus find that the superconducting term Eq. (101) is:

$$\boxed{H_{\mathrm{SC}} : \begin{cases} \text{relevant}, & K > \frac{1}{2} \\ \text{irrelevant}, & K < \frac{1}{2} \end{cases}} \tag{118}$$

V.4.3 Strong Δ limit

Consider a situation in which H_{SC} is relevant. Based on the bosonized description, **what** happens when Δ becomes large compared to the kinetic energy and interaction term (on a "semiclassical" level)? In particular, **what** happens to the field φ? **How many** semiclassical minima of H_{SC} as a function of φ are there?

We now consider the full Hamiltonian $H = H_0 + H_{\mathrm{int}} + H_{\mathrm{SC}}$, which we can write:

$$H = \frac{1}{2\pi}\int \mathrm{d}x \left(v_F K \left(\partial_x\varphi\right)^2 + \frac{v_F}{K}\left(\partial_x\theta\right)^2\right) - 2g_\Delta \int \mathrm{d}x \cos(2\varphi(x)) \tag{119}$$

where K and v_F are given in Eq. (85). In the limit $\Delta \gg U, t$ where the superconducting term will dominate. If it is sufficiently large we can make a semi-classical approximation which will pin $\varphi(x)$ to the field configuration $\varphi^{\text{s-c}}(x)$ which minimises the energy, i.e. the Hamiltonian H_{SC}. How we minimise the energy depends on the sign of the Hamiltonian, which is determined by $-g_\Delta \propto -\Delta \sin(k_F a)$. Since we assume $\Delta > 0$, and since $\sin(k_F a)$ always is positive, as $k_F \in (0; \frac{\pi}{a}]$, we have $-g_\Delta < 0$. Since $-g_\Delta < 0$, we must maximise the cosine, to minimise the energy from the superconducting term, which implies:

$$
\begin{aligned}
\cos(2\varphi(x)) &= 1 \Rightarrow \\
2\varphi(x) &= 2n\pi, \quad n \in \mathbb{Z} \Leftrightarrow \\
\varphi(x) &= \pi n, \quad n \in \mathbb{Z}
\end{aligned}
\tag{120}
$$

However, the integer n is actually limited to two values. To see this, we note that the bosonized fields Eq. (55), are invariant under $\varphi(x) \to \varphi(x) + 2\pi$, so the field $\varphi(x)$ is only uniquely defined for $\varphi(x) \in [0; 2\pi)$. We thus find that when the superconducting term is much larger than the kinetic and interaction term, the field φ is semi-classically pinned to a constant value:

$$
\boxed{\varphi^{\text{s-c}}(x) = \pi n, \quad n = 0, 1}
\tag{121}
$$

which minimises the energy of the superconducting term. Furthermore this means that all fluctuations in φ are killed off, and due to the canonical relation Eq. (25) between φ and $\partial_x \theta$, the θ field will become wildly fluctuating when φ is pinned; just as would be the case for e.g. momentum if the position is measured! We see that there are two distinct values φ can be pinned to, but they are *degenerate*; both the pinned field configurations lead to the same semi-classical minimum of the Hamiltonian:

$$
H^{\text{s-c}} = -\frac{2L\Delta}{\pi a} \sin(k_F a)
\tag{122}
$$

where L is the length of the system.

V.4.4 Correlation function

What do you expect from correlations of the kind $\left\langle \psi_R^\dagger(x)\psi_L^\dagger(x+a)\psi_R(y)\psi_L(y+a)\right\rangle$ (consider only the dominant term in the limit $|x-y| \gg a$?

First recall that the bosonized left and right moving fields are given by:

$$
\psi_L(x) = \frac{1}{\sqrt{2\pi a}} \mathrm{e}^{\mathrm{i}(\varphi(x)+\theta(x))}
\tag{123a}
$$

$$
\psi_R(x) = \frac{1}{\sqrt{2\pi a}} \mathrm{e}^{\mathrm{i}(\varphi(x)-\theta(x))}
\tag{123b}
$$

We now consider the pair-correlation function. Using Eq. (123a) we have:

$$
\begin{aligned}
&\left\langle \psi_R^\dagger(x)\psi_L^\dagger(x+a)\psi_R(y)\psi_L(y+a)\right\rangle = \\
&\frac{1}{(2\pi a)^2}\left\langle \mathrm{e}^{-\mathrm{i}(\varphi(x)-\theta(x))}\mathrm{e}^{-\mathrm{i}(\varphi(x+a)+\theta(x+a))}\mathrm{e}^{\mathrm{i}(\varphi(y)-\theta(y))}\mathrm{e}^{\mathrm{i}(\varphi(y+a)+\theta(y+a))}\right\rangle = \\
&\frac{1}{(2\pi a)^2}\left\langle \mathrm{e}^{-\mathrm{i}(\varphi(x)+\varphi(x+a)-\theta(x)+\theta(x+a))-\mathrm{i}\frac{\pi}{2}}\mathrm{e}^{\mathrm{i}(\varphi(y)+\varphi(y+a)+\theta(y+a)-\theta(y))-\mathrm{i}\frac{\pi}{2}}\right\rangle = \\
&-\frac{1}{(2\pi a)^2}\left\langle \mathrm{e}^{-\mathrm{i}(\theta(x+a)-\theta(x)+\varphi(x)+\varphi(x+a))}\mathrm{e}^{\mathrm{i}(\varphi(y)+\varphi(y+a)+\theta(y+a)-\theta(y))}\right\rangle
\end{aligned}
\tag{124}
$$

where we have combined the exponential using the BCH lemma and the commutators:

$$[-\mathrm{i}(\varphi(x)-\theta(x)), -\mathrm{i}(\varphi(x+a)+\theta(x+a))] = -\underbrace{[\varphi(x),\theta(x+a)]}_{\mathrm{i}\pi} + \underbrace{[\theta(x),\varphi(x+a)]}_{0} = -\mathrm{i}\pi \tag{125}$$

$$[\mathrm{i}(\varphi(y)-\theta(y)), \mathrm{i}(\varphi(y+a)+\theta(y+a))] = -\underbrace{[\varphi(y),\theta(y+a)]}_{\mathrm{i}\pi} + \underbrace{[\theta(y),\varphi(y+a)]}_{0} = -\mathrm{i}\pi \tag{126}$$

Next we wish to split the θ and φ fields, by rearranging the BCH Lemma:

$$\mathrm{e}^{A+B} = \mathrm{e}^{A}\mathrm{e}^{B}\mathrm{e}^{-\frac{1}{2}[A,B]} \tag{127}$$

We then have:

$$-\frac{1}{(2\pi a)^2}\left\langle \mathrm{e}^{-\mathrm{i}(\theta(x+a)-\theta(x)+\varphi(x)+\varphi(x+a))}\mathrm{e}^{\mathrm{i}(\varphi(y)+\varphi(y+a)+\theta(y+a)-\theta(y))}\right\rangle =$$
$$-\frac{1}{(2\pi a)^2}\left\langle \mathrm{e}^{-\mathrm{i}(\theta(x+a)-\theta(x))}\mathrm{e}^{-\mathrm{i}(\varphi(x)+\varphi(x+a))-\mathrm{i}\frac{\pi}{2}}\mathrm{e}^{\mathrm{i}(\varphi(y)+\varphi(y+a))+\mathrm{i}\frac{\pi}{2}}\mathrm{e}^{\mathrm{i}(\theta(y+a)-\theta(y))}\right\rangle \tag{128}$$

Since we consider the semi-classical limit where the field is pinned to a constant, i.e. $\phi(x) \to \varphi^{\text{s-c}}$, we have:

$$\varphi(x) = \varphi(x+a) = \varphi(y) = \varphi(y+a) = \varphi^{\text{s-c}} \tag{129}$$

Consequently the middle factors in the correlation function are simply unity in the semi-classical approximation:

$$\mathrm{e}^{-\mathrm{i}(\varphi(x)+\varphi(x+a))-\mathrm{i}\frac{\pi}{2}}\mathrm{e}^{\mathrm{i}(\varphi(y)+\varphi(y+a))+\mathrm{i}\frac{\pi}{2}} \approx \mathrm{e}^{-2\mathrm{i}\varphi^{\text{s-c}}-\mathrm{i}\frac{\pi}{2}}\mathrm{e}^{2\mathrm{i}\varphi^{\text{s-c}}+\mathrm{i}\frac{\pi}{2}} = 1 \tag{130}$$

Thus, when the φ field is semi-classically pinned the correlation function reduces to:

$$\left\langle \psi_R^\dagger(x)\psi_L^\dagger(x+a)\psi_R(y)\psi_L(y+a)\right\rangle^{\text{s-c}} = -\frac{1}{(2\pi a)^2}\left\langle \mathrm{e}^{-\mathrm{i}(\theta(x+a)-\theta(x))}\mathrm{e}^{\mathrm{i}(\theta(y+a)-\theta(y))}\right\rangle \tag{131}$$

Since the average is with respect to the full action including the super conducting term, we can't evaluate the correlation exactly. Instead we do as follows. First we Taylor expanded the fields, by assuming that they are slowly varying, and then we Taylor expand the exponentials in the resulting derivatives of the fields, which are small:

$$-\frac{1}{(2\pi a)^2}\left\langle \mathrm{e}^{-\mathrm{i}(\theta(x+a)-\theta(x))}\mathrm{e}^{\mathrm{i}(\theta(y+a)-\theta(y))}\right\rangle \approx -\frac{1}{(2\pi a)^2}\left\langle \mathrm{e}^{-\mathrm{i}a\partial_x\theta(x)}\mathrm{e}^{\mathrm{i}a\partial_y\theta(y)}\right\rangle \approx$$
$$-\frac{1}{(2\pi a)^2}\left\langle (1-\mathrm{i}a\partial_x\theta(x))\,(1+\mathrm{i}a\partial_y\theta(y))\right\rangle =$$
$$-\frac{1}{(2\pi a)^2}\Big(1-\mathrm{i}a\partial_x\left\langle\theta(x)\right\rangle+\mathrm{i}a\partial_y\left\langle\theta(y)\right\rangle+a^2\partial_x\partial_y\left\langle\theta(x)\theta(y)\right\rangle\Big) \tag{132}$$

Since the perturbation H_{SC} opens a gap in the spectrum of the Fermions, we expect that at large separations $|x-y| \gg a$, any correlations function should decay $\sim \mathrm{e}^{-\Delta|x-y|}$, and consequently only the leading order term is relevant. We thus find that when φ is semi-classically pinned, the correlation function approaches a constant:

$$\boxed{\left\langle \psi_R^\dagger(x)\psi_L^\dagger(x+a)\psi_R(y)\psi_L(y+a)\right\rangle \approx -\frac{1}{(2\pi a)^2}} \tag{133}$$

To understand why this is reasonable, we note that we have almost calculated the pair-correlation function, i.e. the correlation function of the order parameter for the superconductor[13]:

$$\left\langle \psi_R^\dagger \psi_L^\dagger \psi_L \psi_R \right\rangle \sim \left\langle \bar{\Delta}\Delta \right\rangle , \tag{134}$$

which indeed should approach a constant at large distances, when we are in the superconducting phase which displays *long range order*[14].

V.4.5 Bosonization of SC with Position Dependent Phase

Suppose now that the SC order parameter acquires a position dependent phase:

$$H_{\text{SC}} = -\sum_r \left(\Delta \mathrm{e}^{\mathrm{i}\phi r} c_r^\dagger c_{r+a}^\dagger + \Delta \mathrm{e}^{-\mathrm{i}\phi r} c_{r+a} c_r \right) , \tag{135}$$

with $\Delta > 0$ and $\phi \ll {}^1\!/\!{}_a$.

What is the bosonized description of this interaction?

Following the same steps as Problem V.4.1, we first go to the continuum limit:

$$H_{\text{SC}} = -\Delta \int \mathrm{d}x \left(\mathrm{e}^{\mathrm{i}\phi x} \psi^\dagger(x) \psi^\dagger(x+a) + \text{H.C.} \right) \tag{136}$$

Using Eq. (98), and dropping the fast oscillating terms, the bosonized Hamiltonian becomes:

$$H_{\text{SC}} = -\frac{\Delta}{2\pi a} \int \mathrm{d}x \left[\mathrm{i}\mathrm{e}^{-\mathrm{i}(k_F a + 2\varphi(x) - \phi x)} - \mathrm{i}\mathrm{e}^{\mathrm{i}(k_F a - 2\varphi(x)) + \phi x} - \mathrm{i}\mathrm{e}^{\mathrm{i}(k_F a + 2\varphi(x) - \phi x)} + \mathrm{i}\mathrm{e}^{-\mathrm{i}(k_F a - 2\varphi(x) + \phi x)} \right] \tag{137}$$

Writing this in terms of sines , and using the same trigonometric identity as before we find that the bosonized form of the Hamiltonian, when the order parameter has a position dependent phase, is given by:

$$\boxed{H_{\text{SC}} = -\frac{2\Delta}{\pi a} \sin(k_F a) \int \mathrm{d}x \cos(2\varphi(x) - \phi x)} \tag{138}$$

V.4.6 Strong Δ limit revisited

If Δ is strong, **what** happens semiclassically to φ? Remember that the current is proportional to $j \propto \partial_x \varphi$. **What** do you conclude?

[13]Note the slightly different ordering of the fields compared to Eq. (133), which is there ensure that the cooper-pair density $\langle \bar{\Delta}\Delta \rangle \to n_0$ is positive. That is, the minus is Eq. (133) is cancelled by the minus which follows from anti-commuting two of the fields.

[14]Here there is a small subtlety, namely should we expect ODLRO in 1+1D? Naively, we should not, since the Mermin–Wagner theorem states that it is impossible for continuous symmetry breaking to occur and result in order in less than $2 + 1D$. However we can "circumvent" the Mermin-Wagner theorem in this case, since the Hamiltonian we are considering isn't truly the full one, but rather an effective one which arises from some interaction in a truly 3+1D system. For example the pairing term can arise by surrounding a 1D wire with a thick cylindrical 3D layer of superconducting material, in which case the pairing term will arise due to the superconducting proximity effect.

We once again consider the limit where $\Delta \gg t, U$. Just as before, we make a semi-classical approximation and minimise the pairing Hamiltonian with respect to φ. This implies:

$$\begin{aligned} \cos(2\varphi(x) - \phi x) = 1 \Leftrightarrow \\ 2\varphi(x) - \phi x = 2n\pi, \quad n \in \mathbb{Z} \end{aligned} \tag{139}$$

We again note that due to the invariance of the bosonized fields under a 2π shift of φ, there are only allowed values for n, resulting in only two distinct semi-classical minima. We thus find that when Δ is sufficiently strong, the filed gets pinned to the semiclassical value:

$$\boxed{\varphi^{\text{s-c}}(x) = \frac{\phi}{2}x + n\pi, \qquad n = 0, 1} \tag{140}$$

Importantly, the position dependent phase of the order parameter results in a semi-classical field configuration which also depends on position. If we now recall that the current is given by:

$$j = -K v_F \partial_x \varphi(x) \tag{141}$$

Using Eq. (140), we find that:

$$j^{\text{s-c}} = -K v_F \partial_x \left(\frac{\phi}{2}x + n\pi \right) = -\frac{K v_F}{2}\phi \tag{142}$$

We thus see that when φ is pinned to the semiclassical field Eq. (140), the system develops a non-zero current, if the order parameter has a spatially dependent phase[15]:

$$\boxed{j^{\text{s-c}} = -\frac{K v_F}{2}\phi} \tag{143}$$

V.4.7 Increasing ϕ

Imagine to progressively increase ϕ: **what** happens to the kinetic energy? **What** happens to the current?

From Eq. (143) we see that the current grows linearly in ϕ, and thus progressively increasing ϕ will likewise increase the current.

Next we note that the full Hamiltonian can be written purely in terms of φ by using the duality relation:

$$\partial_x \theta = \frac{K}{v_F}\partial_t \varphi \tag{144}$$

Using this relation to rewrite the kinetic part Eq. (77) of the Hamiltonian, we have:

$$H = \frac{1}{2\pi}\int \mathrm{d}x \left(v_F K \left(\partial_x \varphi\right)^2 + \frac{K}{v_F}\left(\partial_t \varphi\right)^2 \right) - \frac{2\Delta}{\pi a}\sin(k_F a)\int \mathrm{d}x \cos(2\varphi(x) - \phi x) \tag{145}$$

Using the semiclassical field Eq. (140) the Hamiltonian becomes:

$$H = \frac{1}{2\pi}\int \mathrm{d}x v_F K \frac{\phi^2}{4} - \int \mathrm{d}x \frac{2\Delta}{\pi a}\sin(k_F a) \tag{146}$$

[15]We could also write the current $j^{\text{s-c}} = -\frac{u}{2}\phi$. where we have used Eq. (85), to rewrite $K v_F = u$

Denoting the system length L we thus find that the semiclassical Hamiltonian is:

$$H^{\text{s-c}} = \underbrace{\frac{Kv_F L}{8\pi}\phi^2}_{H_0} - \underbrace{\frac{2L\Delta}{\pi a}\sin(k_F a)}_{H_{\text{SC}}} \tag{147}$$

We see the kinetic energy grows quadratically, i.e. $\sim \phi^2$, so increasing ϕ also increases the kinetic energy. We have thus found that both the current $j^{\text{s-c}}$ and the kinetic energy $H_0^{\text{s-c}}$ increases with increasing ϕ:

$$\boxed{j^{\text{s-c}} \sim \phi, \quad H_0^{\text{s-c}} \sim \phi^2} \tag{148}$$

V.4.8 Large ϕ limit

Imagine that ϕ increases a lot, such that $v_F\phi^2 \gtrsim \Delta/a$. **What** happens to the system? **What** do you expect to see in the current?

If ϕ is increases such that the kinetic energy becomes much larger than the superconducting term, i.e. $v_F\phi^2 \gg \frac{\Delta}{a}$. In this limit, the kinetic term dominates over the superconducting pairing term, and thus in our semiclassical approximation, we should minimise the kinetic term instead. From the Hamiltonian Eq. (145), we see that all the kinetic terms depend on squared of derivatives, and thus the kinetic energy is minimised by pinning $\varphi(x)$ to a constant value, such that the derivatives vanish:

$$\varphi^{\text{s-c}} = \varphi_0 \tag{149}$$

However, when the field is pinned to a constant, the current Eq. (141) *vanishes*:

$$j^{\text{s-c}} \propto \partial_x \varphi_0 = 0 \tag{150}$$

We thus find that if ϕ is increased sufficiently we transition from superconducting phase to a normal phase which does not carry supercurrent. The transition occurs when $v_F\phi^2 \sim \Delta/a$, which results in both the superconducting part and the kinetic part being equally relevant. Since all terms in the Hamiltonian are relevant at the phase transition, it is non-trivial to analyse and beyond the scope of this solution.

V.4.9 Critical Field

If you consider that ϕ is proportional to a magnetic field $\phi \propto B$, can you **relate** the previous observations with a known SC effect (even though we are only in 1D, and we do not truly have long-range superconducting order)?

We can now rephrase all of our above observations in terms of the magnetic field B. Firstly we see that applying an external field breaks the spatial homogeneity of the superconducting order parameter, and the resulting gradient in the phase of the order parameter result in a supercurrent. As the strength of the external magnetic field B is increased the current increases in strength, until the field hits a *critical value* B_c where the system makes a phase transition from a superconducting to a normal phase, resulting in the supercurrent vanishing. All of the above is just what we expect from our previous encounters with super-conductivity.

V.5 Double sine-Gordon model

Let us consider the Hamiltonian:

$$H = \frac{v}{2\pi}\int \mathrm{d}x \left(K\left(\partial_x\varphi\right)^2 + \frac{\left(\partial_x\theta\right)^2}{K}\right) + \frac{1}{2\pi}\int \mathrm{d}x \left(A\cos(\alpha\theta) + B\cos(\beta\varphi)\right) \tag{151}$$

V.5.1 RG Analysis of Interaction Terms

Determine for which values of the Luttinger parameter K the A and B terms are relevant/irrelevant in the RG sense.

Hint: split the cosines into exponentials to evaluate their scaling dimension through their correlation functions.

Writing the Hamiltonian in terms of vertex operators we have:

$$H = \frac{v}{2\pi}\int \mathrm{d}x \left(K\left(\partial_x\varphi\right)^2 + \frac{\left(\partial_x\theta\right)^2}{K}\right) + \int \mathrm{d}x \left(\frac{A}{2}V_\alpha + \frac{B}{2}V_\beta + \mathrm{H.c.}\right), \quad V_\alpha = \mathrm{e}^{\mathrm{i}\alpha\theta},\ V_\beta = \mathrm{e}^{\mathrm{i}\beta\varphi} \tag{152}$$

To determine when the perturbations are relevant or irrelevant, we first find their scaling dimension. To do so we consider the correlation functions. We start by recalling that the correlation functions for the fields, with respect to the unperturbed Luttinger liquid action, are given by:

$$\langle\varphi(x)\varphi(0)\rangle = -\frac{1}{2K}\ln|x| \tag{153a}$$

$$\langle\theta(x)\theta(0)\rangle = -\frac{K}{2}\ln|x| \tag{153b}$$

and that $\langle\varphi^2(x)\rangle = \text{const.}$ and $\langle\theta^2(x)\rangle = \text{const.}$ where the constants are different and non-universal. The calculation of the vertex operator correlation function is now completely analogous to the one we did in Problem V.4.2:

$$\left\langle V_\alpha^\dagger(x)V_\alpha(0)\right\rangle = \left\langle \mathrm{e}^{-\mathrm{i}\alpha\theta(x)}\mathrm{e}^{\mathrm{i}\alpha\theta(0)}\right\rangle = \underbrace{\mathrm{e}^{-\frac{\alpha^2}{2}\left(\left\langle\theta^2(x)\right\rangle+\left\langle\theta^2(0)\right\rangle\right)}}_{C_\alpha}\mathrm{e}^{\alpha^2\langle\theta(x)\theta(0)\rangle} = \frac{C_\alpha}{|x|^{\frac{K\alpha^2}{2}}} \tag{154}$$

$$\left\langle V_\beta^\dagger(x)V_\beta(0)\right\rangle = \left\langle \mathrm{e}^{-\mathrm{i}\beta\varphi(x)}\mathrm{e}^{\mathrm{i}\beta\varphi(0)}\right\rangle = \underbrace{\mathrm{e}^{-\frac{\beta^2}{2}\left(\left\langle\varphi^2(x)\right\rangle+\left\langle\varphi^2(0)\right\rangle\right)}}_{C_\beta}\mathrm{e}^{\beta^2\langle\varphi(x)\varphi(0)\rangle} = \frac{C_\beta}{|x|^{\frac{\beta^2}{2K}}} \tag{155}$$

We thus have the long range correlation functions:

$$\left\langle V_\alpha^\dagger(x)V_\alpha(0)\right\rangle = \frac{C_\alpha}{|x|^{\frac{K\alpha^2}{2}}} \tag{156a}$$

$$\left\langle V_\beta^\dagger(x)V_\beta(0)\right\rangle = \frac{C_\beta}{|x|^{\frac{\beta^2}{2K}}} \tag{156b}$$

Since the correlations decay as power laws, we can simply read off the scaling dimensions D_α and D_β by using:

$$\left\langle V_{\alpha/\beta}^\dagger(x)V_{\alpha/\beta}(0)\right\rangle = \frac{C_{\alpha/\beta}}{|x|^{2D_{\alpha/\beta}}} \tag{157}$$

We thus find the scaling dimensions:

$$D_\alpha = \frac{K\alpha^2}{4} \tag{158a}$$

$$D_\beta = \frac{\beta^2}{4K} \tag{158b}$$

Now that we now the scaling dimensions, we want to find the RG equations for the coupling constants A and B. To do so we note that the actions, in euclidean time, corresponding to the two perturbations are:

$$\mathcal{S}_{I,A} = \frac{1}{2\pi}\int \mathrm{d}x\mathrm{d}\tau A\left(V_\alpha + V_\alpha^\dagger\right) \tag{159a}$$

$$\mathcal{S}_{I,B} = \frac{1}{2\pi}\int \mathrm{d}x\mathrm{d}\tau A\left(V_\beta + V_\beta^\dagger\right) \tag{159b}$$

Rescaling the action we have:

$$\mathcal{S}'_{I,A} = \frac{1}{2\pi}\int \mathrm{d}x'\mathrm{d}\tau' b^2 A\left(b^{-D_\alpha}V'_\alpha + b^{-D_\alpha}V'^\dagger_\alpha\right) = \frac{1}{2\pi}\int \mathrm{d}x'\mathrm{d}\tau' \underbrace{Ab^{2-D_\alpha}}_{A'}\left(V'_\alpha + V'^\dagger_\alpha\right) \tag{160}$$

$$\mathcal{S}'_{I,B} = \frac{1}{2\pi}\int \mathrm{d}x'\mathrm{d}\tau' b^2 B\left(b^{-D_\beta}V'_\beta + b^{-D_\beta}V'^\dagger_\beta\right) = \frac{1}{2\pi}\int \mathrm{d}x'\mathrm{d}\tau' \underbrace{Bb^{2-D_\beta}}_{B'}\left(V'_\beta + V'^\dagger_\beta\right) \tag{161}$$

$$A' = b^{2-D_\alpha}A \tag{162}$$

$$B' = b^{2-D_\beta}B \tag{163}$$

These equations have exatly the same form as Eq. (115), so following the same steps as before we find that the RG flow of A and B is given by:

$$A(\ell) = A(0)\mathrm{e}^{(2-D_\alpha)\ell} \tag{164a}$$

$$B(\ell) = B(0)\mathrm{e}^{(2-D_\beta)\ell} \tag{164b}$$

Since e.g. A is relevant if it increases under the RG flow, we find that A is relevant if the exponent is positive, ie. $2 > D_\alpha$, and likewise for B. Using the scaling dimensions Eq. (158) we thus find that the relevance of the two sine-Gordon terms depend on the Luttinger parameter K as follows:

$$\boxed{A : \begin{cases}\text{Relevant}, & K < \frac{8}{\alpha^2} \\ \text{Irrelevant}, & K > \frac{8}{\alpha^2}\end{cases} \quad , \quad B : \begin{cases}\text{Relevant}, & K > \frac{\beta^2}{8} \\ \text{Irrelevant}, & K < \frac{\beta^2}{8}\end{cases}} \tag{165}$$

The cases $K = \frac{8}{\alpha^2}$ and $K = \frac{\beta^2}{8}$ are marginal, and would require further analysis which is beyond the scope of this solution.

V.5.2 RG Analysis for Model With $\mathbb{Z}_p$ Symmetry

This part is inspired by models with $\mathbb{Z}_p$ symmetry. Consider $\alpha = \beta = \sqrt{2p}$ with $p \in \mathbb{N}$. For **which** values of p is it possible that there exist a K such that both the terms are irrelevant?

For $\alpha = \beta = \sqrt{2p}$, the relevance criteria Eq. (165) become:

$$A : \begin{cases} \text{Relevant}, & K < \frac{4}{p} \\ \text{Irrelevant}, & K > \frac{4}{p} \end{cases} \quad , \quad B : \begin{cases} \text{Relevant}, & K > \frac{p}{4} \\ \text{Irrelevant}, & K < \frac{p}{4} \end{cases} \tag{166}$$

For both the A and B term to be irrelevant we must have:

$$\begin{aligned} \frac{4}{p} < K < \frac{p}{4} \\ \Rightarrow \frac{4}{p} < \frac{p}{4} \Leftrightarrow \\ 4 < p \end{aligned} \tag{167}$$

We thus find that both terms can be relevant is

$$\boxed{p > 4 \Rightarrow A \text{ and } B \textit{ can} \text{ be simultaneously irrelevant}} \tag{168}$$

V.5.3 Phases Transition

What are, in your opinion, the implications on the phase diagram of such a system as a function of p and K? **Which** gapped and gapless phases could you expect?

Based on Eq. (166) we now wish to analyse which phases of we can expect. If both A and B are irrelevant, the system is just a Luttinger liquid and thus in a gapless phase. If the A term is irrelevant, and the B term is relevant, the spectrum is gapped by the A-term perturbation. Likewise if the B term is irrelevant, and the A term is relevant, the spectrum is gapped by the B-term perturbation. If neither the A or B term are irrelevant, the most relevant term will gap the spectrum. Let us now examine in more detail how the actual phase diagram will look. First let us consider the regions where only one term or none are relevant:

- For $\frac{4}{p} < K < \frac{p}{4}$, both perturbations are irrelevant and the system is gapless
- For $K < \frac{4}{p} \wedge K < \frac{p}{4}$, the A terms is relevant and the system is gapped.
- For $K > \frac{4}{p} \wedge K > \frac{p}{4}$, the B terms is relevant and the system is gapped

Finally let us consider the region $\frac{p}{4} < K < \frac{4}{p}$, where both terms increase with the RG flow. from Eq. (164) we see that the term with the smallest scaling dimension will grow the fastest, and thus be relevant. since $D_\alpha \propto K$ and $D_\beta \propto K^{-1}$, we see that:

- For $\frac{p}{4} < K < \frac{4}{p} \wedge K < 1$ the A term is most relevant and the system is gapped
- $\frac{p}{4} < K < \frac{4}{p} \wedge K > 1$ the B term is most relevant and the system is gapped

Combining all of these consideration we get the phase diagram in Fig. 4

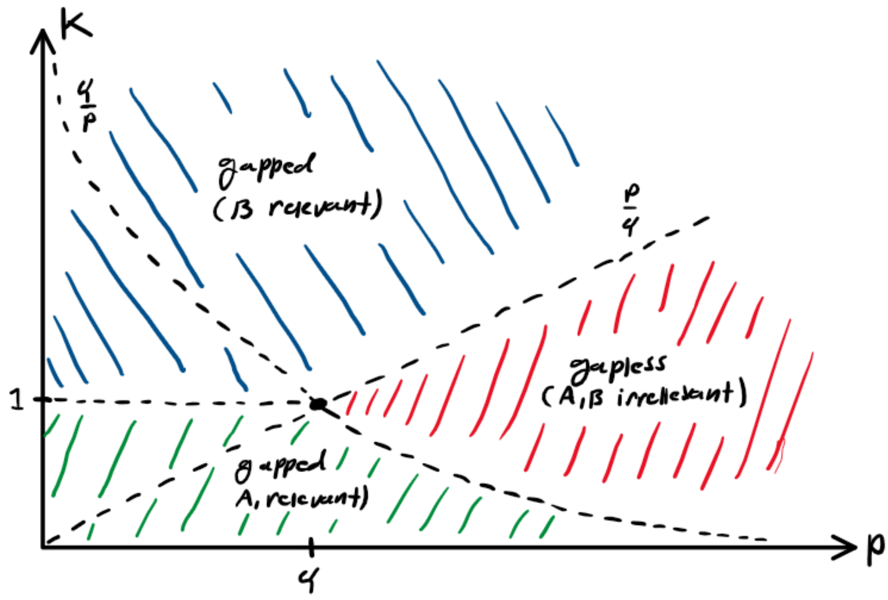


Figure 4: The figure above shows the K-p phase diagram for the double sine-Gordon model when $\alpha = \beta = \sqrt{2p}$. The system has three phases: a gapless phase (red), a phase gapped by the A term (green) and a phase gapped by the B term (blue). Note that p is an integer, so the contonius lines $\frac{p}{4}$ and $\frac{4}{p}$ are there for visual clarity.

V.6 Luttinger parameter for bosons

Calculate the Luttinger parameter K for the bosonic system in Exercise IV.2. For spinless bosonic systems in the non-interacting limit, **how** does K behave?

Since we could we already calculated the Luttinger parameter back in Problem IV.2.1 and found that it was given by Eq. (24a), which we for convinience restate here:

$$K = \pi\sqrt{\frac{\rho_0}{2Um}} \tag{169}$$

In the non-interacting limit $U \to 0$ we find that the Luttinger parameter for spinless bosons diverges:

$$\boxed{\lim_{U\to 0} K = \infty} \tag{170}$$

We can now make the comparison between spinless Fermions and spinless Bosons[16]:

$$\text{Bosons}: \begin{cases} \text{Weak interaction}, & K \to \infty \\ \text{Strong interaction}, & K \to 0 \end{cases}, \quad \text{Fermions}: \begin{cases} \text{Weak interaction}, & K \to 1 \\ \text{Strong interaction}, & K \to 0 \end{cases} \tag{171}$$

[16] Here we should stress that the strongly interacting limit has a subtlety: We are actually considering hardcore Bosons, and these are equivalent to Fermions through the Jordan-Wigner transformation. This means that what we refer to as the strongly interacting limit here actually corresponds not only to a large U, but also to having sufficiently long range interactions!

V.7 Bose Hubbard model

Several one-dimensional systems can be modelled as locally interacting bosons. These systems include ultracold bosonic atoms in 1D optical lattices or 1D arrays of superconducting islands connected by Josephson junctions, in which the Cooper pairs can be thought as bosons hopping on a discrete chain. To describe these systems, we may consider the following model for bosons on a chain (Bose-Hubbard model):

$$H = -t\sum_r \left[b^\dagger_{r+a} b_r + \text{H.c.}\right] + U\sum_r n_r^2 \tag{172}$$

Here the operators b and $b^\dagger$ are standard bosonic operators and $n_r = b^\dagger_r b_r$ measures the number of particles in the site r. $U > 0$ represents a local repulsive interaction. In the following, let us assume that we can vary the density of the system $\rho_0 = N/L = \sum_r n_r/L$ as we wish.

V.7.1 Bosonization of the Bose-Hubbard Model

Concerning the kinetic term, bosonize it through the basic approximation $b_r \to \psi = \sqrt{\rho_0}\mathrm{e}^{\mathrm{i}\varphi}$ as in Exercise IV.2. Concerning the interaction, consider $n_r = a\rho$ and express the operator ρ via:

$$\rho(x) = \left(\rho_0 - \frac{\partial_x\theta(x)}{\pi}\right)\sum_p \mathrm{e}^{\mathrm{i}2p(\pi\rho_0 x - \theta(x))} \tag{173}$$

by taking into account the harmonics $p = -1, 0, 1$ only.

> **Write** the bosonized Hamiltonian, including the fast-oscillating terms that you obtain in this way. You are supposed to find additional terms with respect to what you found in Exercise IV.2

We start by taking the continuum limit of Eq. (172), which yields:

$$H = -t\int \mathrm{d}x \left(\psi^\dagger(x+a)\psi(x) + \text{H.c.}\right) + U\int \mathrm{d}x a\left(\rho(x)\right)^2 \tag{174}$$

First let us consider the Kinetic term. Assuming as usual that the fields are slowly varying, we get:

$$\psi^\dagger(x+a)\psi(x) = \sqrt{\rho_0}\mathrm{e}^{-\mathrm{i}\varphi(x+a)}\sqrt{\rho_0}\mathrm{e}^{\mathrm{i}\varphi(x)} = \rho_0\mathrm{e}^{-\mathrm{i}(\varphi(x+a)-\varphi(x))} \approx$$

$$\rho_0\mathrm{e}^{-\mathrm{i}a\partial_x\varphi(x)} \approx \rho_0\left(1 - \mathrm{i}a\partial_x\varphi(x) - \frac{a^2}{2}\left(\partial_x\varphi\right)^2\right) \Leftrightarrow$$

$$\psi^\dagger(x+a)\psi(x) = \rho_0\left(1 - \mathrm{i}a\partial_x\varphi(x) - \frac{a^2}{2}\left(\partial_x\varphi\right)^2\right) \tag{175}$$

So the Bosonized from of the kinetic term is:

$$-t\int \mathrm{d}x\left(\psi^\dagger(x+a)\psi(x) + \text{H.c.}\right) \approx \frac{1}{2\pi}\int \mathrm{d}x\left(2\pi\rho_0 t a^2\left(\partial_x\varphi\right)^2 - 4\pi t\rho_0\right) \tag{176}$$

The last term is just a constant and therefore unimportant. Next we consider the interaction term. By including the $p = 0, \pm 1$ harmonics we get:

$$\rho^2(x) = \left(\rho_0 - \frac{\partial_x\theta}{\pi}\right)\left[1 + \mathrm{e}^{2\mathrm{i}(\pi\rho_0 x - \theta(x))} + \mathrm{e}^{-2\mathrm{i}(\pi\rho_0 x - \theta(x))}\right]\left(\rho_0 - \frac{\partial_x\theta}{\pi}\right)\left[1 + \mathrm{e}^{2\mathrm{i}(\pi\rho_0 x - \theta(x))} + \mathrm{e}^{-2\mathrm{i}(\pi\rho_0 x - \theta(x))}\right] \tag{177}$$

Since $[\partial_x\theta(x),\theta(x')]=0$, and $[\theta(x),\theta(x')]=0$, we can freely commute all the terms and get:

$$\rho^2(x)=\left(\rho_0-\frac{\partial_x\theta}{\pi}\right)^2\left[3+\mathrm{e}^{2\mathrm{i}(\pi\rho_0 x-\theta(x))}+\mathrm{e}^{-2\mathrm{i}(\pi\rho_0 x-\theta(x))}+\mathrm{e}^{4\mathrm{i}(\pi\rho_0 x-\theta(x))}+\mathrm{e}^{-4\mathrm{i}(\pi\rho_0 x-\theta(x))}\right]=$$
$$\left(\rho_0-\frac{\partial_x\theta}{\pi}\right)^2[3+2\cos(2\pi\rho_0 x-2\theta(x))+2\cos(4\pi\rho_0 x-4\theta(x))] \tag{178}$$

where the three comes from $1+\mathrm{e}^{2\mathrm{i}(\pi\rho_0 x-\theta(x))}\mathrm{e}^{-2\mathrm{i}(\pi\rho_0 x-\theta(x))}+\mathrm{e}^{-2\mathrm{i}(\pi\rho_0 x-\theta(x))}\mathrm{e}^{2\mathrm{i}(\pi\rho_0 x-\theta(x))}=3$. Multiplying everything out we find:

$$(\rho(x))^2=3\rho_0^2-\frac{3}{\pi}\partial_x\theta+\frac{3}{\pi^2}\left(\partial_x\theta\right)^2+$$
$$2\rho_0^2\cos(2\pi\rho_0 x-2\theta(x))+2\rho_0^2\cos(4\pi\rho_0 x-4\theta(x))-\frac{2}{\pi}\cos(2\pi\rho_0 x-2\theta(x))\partial_x\theta-$$
$$\frac{2}{\pi}\cos(4\pi\rho_0 x-4\theta(x))\partial_x\theta+\frac{2}{\pi}\cos(2\pi\rho_0 x-2\theta(x))(\partial_x\theta)^2+\frac{2}{\pi}\cos(4\pi\rho_0 x-4\theta(x))(\partial_x\theta)^2 \tag{179}$$

We note that the first term is constant, and thus unimportant. Likewise the linear term vanishes when integrated, for a constant number of particles. The third term contributes to the kinetic part. In general the remaining terms, those with cosines, are fast oscillating and thus average out, except when the density and x are commensurate, which we will detail further in the next question. Dropping the constant terms, we thus find the the bosonized Bose-Hubbard is given by:

$$H=\frac{1}{2\pi}\int\mathrm{d}x\left(2\pi\rho_0 ta^2\left(\partial_x\varphi\right)^2+\frac{6Ua}{\pi}\left(\partial_x\theta\right)^2\right)+$$
$$Ua\int\mathrm{d}x\left[\rho_0^2\cos(2\pi\rho_0 x-2\theta(x))+2\rho_0^2\cos(4\pi\rho_0 x-4\theta(x))-\frac{2}{\pi}\cos(2\pi\rho_0 x-2\theta(x))\partial_x\theta-\right.$$
$$\left.\frac{2}{\pi}\cos(4\pi\rho_0 x-4\theta(x))\partial_x\theta+\frac{2}{\pi}\cos(2\pi\rho_0 x-2\theta(x))(\partial_x\theta)^2+\frac{2}{\pi}\cos(4\pi\rho_0 x-4\theta(x))(\partial_x\theta)^2\right] \tag{180}$$

For the interacting Luttinger liquid part of the Hamiltonian we have:

$$K=\sqrt{(2\pi\rho_0 ta^2)/(\tfrac{6Ua}{\pi})}=\pi\sqrt{\frac{\rho_0 at}{3U}},\quad u=\sqrt{(2\pi\rho_0 ta^2)\left(\frac{6Ua}{\pi}\right)}=\sqrt{12\rho_0 Uta^3}$$

That is, the Luttinger parameter K and the superfluid velocity u are:

$$K=\pi\sqrt{\frac{\rho_0 at}{3U}} \tag{181a}$$
$$u=\sqrt{12\rho_0 Uta^3} \tag{181b}$$

V.7.2 RG Analysis

Analyse the behaviour of these additional interactions as a function of ρ_0 and U. Focus on the ones potentially more relevant in the RG sense. Based on the result of Ex. V.6, do you think there are regimes in which these additional operators become relevant? For which values of ρ_0 and U do you expect they may give rise to phase transitions?

If you arrived here, the description you found is a quite elegant way of studying the Mott - superfluid phase transitions in these systems.

As we noted before, all of the interaction terms are fast oscillating at general fillings, and thus average out. However at commensurate filling the terms become slowly oscillating. We see that as follows. Consider a system where there are an integer number n_0 number of particles per site, e.g. 1 particle. In this case the density is:

$$\rho_0 = \frac{N}{L} = \frac{\#\text{ sites} \cdot n_0}{\#\text{ sites} \cdot a} = \frac{n_0}{a}, \quad n_0 \in \mathbb{N} \tag{182}$$

Recall again that the model we started by considering was discrete and thus the position is given by:

$$x = ja, \quad j \in \mathbb{Z} \tag{183}$$

Furthermore, note that there are two types of fast oscillating terms: those that oscillate $\sim \mathrm{e}^{\pm \mathrm{i}2\pi\rho_0 x}$ and those which oscillate like $\sim \mathrm{e}^{\pm \mathrm{i}4\pi\rho_0 x}$. Importantly, when there is an integer number of particles per site, both types do in fact not oscillate, but are unity:

$$\mathrm{e}^{\pm \mathrm{i}2\pi\rho_0 x}\Big|_{\rho_0 = n_0/a} = \mathrm{e}^{\pm \mathrm{i}2\pi n_0 j} = 1 \tag{184}$$

$$\mathrm{e}^{\pm \mathrm{i}4\pi\rho_0 x}\Big|_{\rho_0 = n_0/a} = \mathrm{e}^{\pm \mathrm{i}4\pi n_0 j} = 1 \tag{185}$$

We thus refer to integer filling as *commensurate* filling since $\rho_0 x$ is an integer in this case. Importantly we find that at commensurate filling the interactions are no longer fast oscillating:

$$H\big|_{\rho_0 = n_0/a} = \frac{1}{2\pi}\int \mathrm{d}x \left(K v_F \left(\partial_x \varphi\right)^2 + \frac{v_F}{K}\left(\partial_x \theta\right)\right) + Ua\int \mathrm{d}x \Big[\rho_0^2 \cos(2\theta(x)) + 2\rho_0^2\cos(4\theta(x))$$
$$-\frac{2}{\pi}\cos(2\theta(x))\partial_x\theta - \frac{2}{\pi}\cos(4\theta(x))\partial_x\theta + \frac{2}{\pi}\cos(2\theta(x))(\partial_x\theta)^2 + \frac{2}{\pi}\cos(4\theta(x))(\partial_x\theta)^2\Big]$$

We now need to figure out which interaction is most relevant. To do so recall that the cosine terms can be written in terms of vertex operators $V_\gamma = \mathrm{e}^{\mathrm{i}\gamma\theta}$, and that in general the scaling dimension D_γ is found from:

$$\left\langle V_\gamma^\dagger(x) V_\gamma(0)\right\rangle = \left\langle \mathrm{e}^{-\mathrm{i}\gamma\theta(x)}\mathrm{e}^{\mathrm{i}\gamma\theta(0)}\right\rangle = \underbrace{\mathrm{e}^{-\frac{\gamma^2}{2}\left(\langle\theta^2(x)\rangle + \langle\theta^2(0)\rangle\right)}}_{C_\gamma}\mathrm{e}^{\gamma^2\langle\theta(x)\theta(0)\rangle} = \frac{C_\gamma}{|x|^{\frac{K\gamma^2}{2}}} \Rightarrow \tag{186}$$

$$D_\gamma = \frac{K\gamma^2}{4} \tag{187}$$

However some the the terms in the interaction Hamiltonian also depends on derrivatives which scale like:

$$\partial_x = b^{-1}\partial_{x'} \tag{188}$$

and the bosonic fields which have trivial scaling in $1+1D$. Taking into account the contribution to the scaling from both the vertex operators and the derivatives, we find that the different Terms have scaling dimensions:

$$H_{\text{int}} = Ua\int \mathrm{d}x \Bigg[\rho_0^2 \underbrace{\cos(2\theta(x))}_{D=K} + 2\rho_0^2 \underbrace{\cos(4\theta(x))}_{D=4K}$$
$$-\frac{2}{\pi}\underbrace{\cos(2\theta(x))\partial_x\theta}_{D=K+1} - \frac{2}{\pi}\underbrace{\cos(4\theta(x))\partial_x\theta}_{D=4K+1} + \frac{2}{\pi}\underbrace{\cos(2\theta(x))(\partial_x\theta)^2}_{D=K+2} + \frac{2}{\pi}\underbrace{\cos(4\theta(x))(\partial_x\theta)^2}_{4K+1}\Bigg] \tag{189}$$

All of the terms have slightly different coupling constants but what matters is that they all have an exponential RG flow $g(\ell) = g(0)\mathrm{e}^{(2-D)\ell}$, and thus the most relevant term is the one with the smallest scaling dimension, which in this case is the one with scaling dimension $D = K$. Based on these considerations we consider the Hamiltonian:

$$H|_{\rho_0 = n_0/a} = \frac{1}{2\pi}\int \mathrm{d}x \left(Kv_F\,(\partial_x\varphi)^2 + \frac{v_F}{K}\,(\partial_x\theta)\right) + \frac{1}{2\pi}\int \mathrm{d}x\; g\cos(2\theta(x)), \quad g \equiv 2\pi U\rho_0^2 a \tag{190}$$

The relevance of the remaining interaction term is given by:

$$g : \begin{cases} \text{Relevant}, & K < 2 \\ \text{Irrelevant}, & K > 2 \end{cases} \tag{191}$$

We thus find that at commensurate filling, the system is in a gapless phase for $K > 2$ and a gapped phase for $K < 2$. To interpret the phases recall that $K \propto U^{-\frac{1}{2}}$, and therefore K becomes smaller for a stronger on-site interactions. This means that the gapped phase corresponds to the case where U is large. Furthermore, since the Luttinger parameter K depends on the filling ρ_0, we also expect that the critical interaction strength U_c which separates the phases, must also depend on the filling. Physically we can now understand that for sufficiently *strong* interactions (large U) and at commensurate filling ($a\rho_0$ integer), the spectrum of the Bosons becomes gapped, and the system is in a Mott insulating phase; for weak interactions (small U) the system is a gapless superfluid. Moreover, there are multiple Mott-like insulating phases which depend on how many particles n_0 are per site.

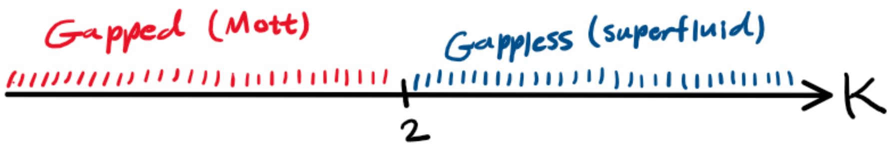


Figure 5: The figure above shows the K phase diagram for the Bose Hubbard model at commensurate filling. At incommensurate filling the spectrum is always gapless. K.