TOPOLOGICAL PHASES of MATTER

- They present phase transitions without local order parameters
- They display features that are robust against a suitable class of local perturbations

	SYMMETRY - PROTECTED	TOPOLOGICAL ORDER
GAPPED	· Topological features are robust as long as some symm. are fulfilled · Edge modes	→ True T.O. in 2D [FQH, Toric code] · Anyons · Degenerate GSs · Long-range entanglement · String symmetries
	IQH, Majorana	
GAPLESS	· Topological semimetals	· Critical anyonic models

Symmetry breaking	· Degeneracy without symmetries
Symmetry → Noether → Conservation	· Conservation laws without sym.

We will discuss about topological order

Intro on Anyons

- Spin Statistics holds in 3+1 D with Lorentz
- In 2D new kinds of particle may emerge.

  Particle: localization + mass
  New kinds: neither bosons nor fermions } Anyons

- A necessary condition to have anyons is topological order.

  0 Majoranas

→ 1 Definition of T.O.

2 TORIC CODE

3 Theory of Anyons

4 Topological quantum computation

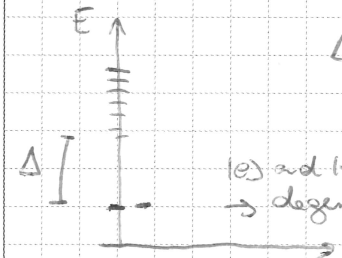
5 Majorana modes as anyons [T.O. in 1D!]

TOPOLOGICAL ORDER

To begin defining what is topological order one strategy is to define "what it is good for".

Therefore let us assume we want to build a quantum memory.

Spectrum of a many-body state



$\Delta \gg T$, noise, pert....

$|0\rangle$ and $|1\rangle$
→ degenerate states ⟺ logical states

a) Δ suppresses processes that excite the state to non-logical states [furthermore: gap ⟺ localization $\langle C(x) C(x+r)\rangle \sim e^{-r/\xi}$

b) Degeneracy of $|0\rangle$ and $|1\rangle$ avoids unwanted dynamical phases, and decay from one to another ...

c) Degenerate states? How to get them?

→ SSB ferromagnet | The Hamiltonian has a symmetry
The GSs do not!

→ I may exploit SSB to store logical qubits in the degenerate states of a ferromagnet

→ If however I perturb the system by violating the symmetry [magnetic field]
I remove the degeneracy → we have some problem

→ The problems are given also by decoherence.

$|\psi_S\rangle_{sys} |\psi_E\rangle_{env.} \leftrightarrow O_S \otimes O_E$ is a perturb.

Let us assume that, in general

a) $\langle \psi_{g_1} | O_S \psi_{g_2} \rangle \neq 0$ → "spin flips"

b) $\langle \psi_{g_1} | O_S \psi_{g_1} \rangle \neq \langle \psi_{g_2} | O_S \psi_{g_2} \rangle$ → dynamical phases "unpredictable"

→ Both processes are dangerous.

What is in general O_S?

→ Locality O_S is a (sum of) local operators

for example $H_{pert} = \sum_x \vec{h}(x,t) \cdot \vec{\sigma}(x)$

CONCLUSION: if I encode logical states in the GSs of a many-body system, very often local noise kills my qubits.

We want qubits stable against any <u>local</u> noise!!

[and not based on symmetries and SSB]

KITAEV CHAIN - FERMIONIC DESCRIPTION

$$H = \sum_x \mu c^\dagger_x c_x - \frac{t}{2} \sum_x (c^\dagger_x c_{x+1} + c^\dagger_{x+1} c_x)$$
$$- \sum_x \left(\frac{\tilde{\Delta}}{2} c^\dagger_x c^\dagger_{x+1} + \frac{\tilde{\Delta}^*}{2} c_{x+1} c_x \right)$$

$U(1) \Rightarrow \mathbb{Z}_2$ $\quad c_x \to -c_x$ instead of $c_x \to e^{i\alpha} c_x$

If $\tilde{\Delta} = \Delta e^{i\varphi} \to$ gauge tr : $c^\dagger_x \to e^{-i\varphi/2} c^\dagger_x$

$$\Rightarrow H = \sum_x \mu c^\dagger_x c_x - \frac{t}{2} \sum_x (c^\dagger_x c_{x+1} + c^\dagger_{x+1} c_x) - \frac{\Delta}{2} \sum_x (c^\dagger_x c^\dagger_{x+1} + c_{x+1} c_x)$$

Let us suppose that we have tr. invariance and periodic boundaries

$$c_x = \int \frac{dk}{\sqrt{L}} e^{ikx} c_k$$

$\Rightarrow$ BdG description :

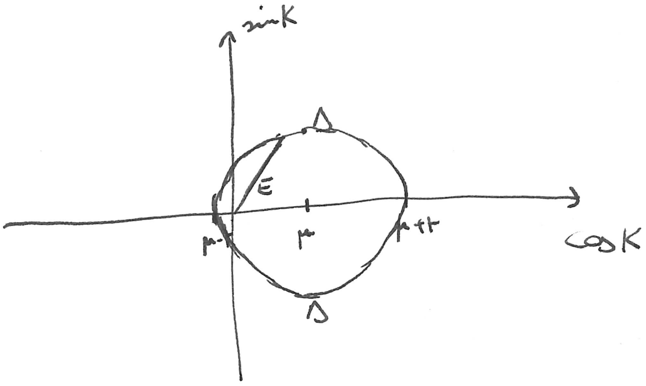
$$H = \sum_k \mu c^\dagger_k c_k - t \cos k\, c^\dagger_k c_k - \frac{\Delta}{2} \sum_{\frac{x k k'}{L}} \left[e^{-ikx} e^{-ik'(x+1)} c^\dagger_k c^\dagger_{k'} + e^{ik'(x+1)+ikx} c_{k'} c_k \right]$$

$$= \sum_k \mu c^\dagger_k c_k - t\cos k\, c^\dagger_k c_k - \frac{\Delta}{2} \left[e^{ik} c^\dagger_k c^\dagger_{-k} + e^{-ik} c_{-k} c_k \right] =$$

$$= \sum_{k \in [-\pi,\pi]} \frac{1}{2} \begin{pmatrix} c^\dagger_k & c_{-k} \end{pmatrix} \begin{pmatrix} \mu - t\cos k & -\frac{\Delta}{2} e^{ik} + \frac{\Delta}{2} e^{-ik} \\ -\frac{\Delta}{2} e^{-ik} + \frac{\Delta}{2} e^{ik} & -\mu + t\cos k \end{pmatrix} \begin{pmatrix} c_k \\ c^\dagger_{-k} \end{pmatrix} =$$

$$= \frac{1}{2} \sum_k \begin{pmatrix} c^\dagger_k & c_{k} \end{pmatrix} \left((\mu - t\cos k) \tau_z + \Delta \sin k\, \tau_y \right) \begin{pmatrix} c_k \\ c^\dagger_{-k} \end{pmatrix}$$

$$E_k = \pm \sqrt{(\mu - t\cos k)^2 + \Delta^2 \sin^2 k}$$



$E = 0$ if $\underbrace{\mu = \pm t}_{\text{critical points}}$ for $\Delta > 0$

Particle-hole sym:

$$H(k) = -\tau_x H^*(-k)\tau_x$$

$\Rightarrow$ if $H(k)|\psi(k)\rangle = E(k)|\psi(k)\rangle$

then

$$H(k)\tau_x|\psi^*(-k)\rangle = -E(k)\tau_x|\psi^*(-k)\rangle$$

If $\tau_x|\psi^*(-k)\rangle = |\psi(k)\rangle \Rightarrow E(k) = 0$

TIME-REVERSAL

$$H(k) = H^*(-k)$$

only if φ is contant

Change of basis: $\mathcal{H}(k) = (\mu - t\cos k)\tau_x + \Delta \sin k\, \tau_y$

$$= \begin{pmatrix} 0 & |E|e^{i\varphi} \\ |E|e^{-i\varphi} & 0 \end{pmatrix}$$

Eigenbasis of τ_x

$\downarrow$

PH

Winding number $= \frac{1}{2\pi}\int_0^{2\pi} \partial_k \varphi \, dk = 0, \pm 1$

$\rightarrow$ 2 phases: $W = 0 \rightarrow$ trivial

$|W| = 1 \rightarrow$ topological $\rightarrow |\mu| < t$

What are these phases?

KITAEV CHAIN

p-wave Hamiltonian :

$$H = +\sum_x \mu\, c_x^\dagger c_x - \frac{t}{2}\sum_x \left(c_x^\dagger c_{x+1} + c_{x+1}^\dagger c_x\right) - \sum_x \left(\frac{\tilde{\Delta}}{2} c_x^\dagger c_{x+1}^\dagger + \frac{\tilde{\Delta}^*}{2} c_{x+1} c_x\right)$$ H

$U(1)$ symmetry is broken $\Rightarrow$ charge is conserved only mod 2
$\Downarrow$
$\mathbb{Z}_2$ symm : $c_x \to -c_x \quad \forall x$

If $\tilde{\Delta} = \Delta e^{i\varphi}$ we apply a gauge tr.: $c_x^\dagger \to e^{i\varphi/2} c_x^\dagger$ Gauge

$$\Rightarrow H = +\sum_x \mu\, c_x^\dagger c_x - \frac{t}{2}\sum_x \left(c_x^\dagger c_{x+1} + c_{x+1}^\dagger c_x\right) - \frac{\Delta}{2}\sum_x \left(c_x^\dagger c_{x+1}^\dagger + c_{x+1} c_x\right)$$

We study the case $\Delta = t$

$$\Rightarrow H = +\sum_x \mu\, c_x^\dagger c_x - i\frac{\Delta}{2}\sum_x \Big[\underbrace{-i(c_x^\dagger - c_x)}_{\gamma_{2x}}\ \underbrace{(c_{x+1} + c_{x+1}^\dagger)}_{\gamma_{2x+1}}\Big]$$

$$\gamma_{2x-1} = c_x + c_x^\dagger \qquad \gamma_{2x} = i\left(c_x - c_x^\dagger\right)$$ Majo

$$\gamma = \gamma^\dagger \quad , \quad \{\gamma_i, \gamma_j\} = 2\delta_{ij}$$

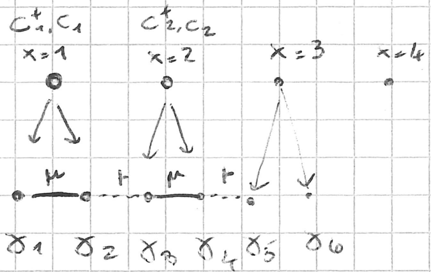
$$c_x = \frac{\gamma_{2x-1} - i\gamma_{2x}}{2} \qquad c_x^\dagger = \frac{\gamma_{2x-1} + i\gamma_{2x}}{2}$$

$$c_x^\dagger c_x = \frac{1 - i\gamma_{2x-1}\gamma_{2x}}{2} \quad \Rightarrow \quad i\gamma_{2x-1}\gamma_{2x} = (-1)^{c_x^\dagger c_x}$$

$\boxed{\text{Ferm. parity}}$

$$H = -\frac{\mu}{2}\sum_x i\gamma_{2x-1}\gamma_{2x} - \frac{t}{2}\sum_x i\gamma_{2x}\gamma_{2x+1}$$

$\longrightarrow$ Staggered configuration



This is the Ising model [via Jordan - Wigner] : $\boxed{JW}$

$$\begin{cases} \gamma_{2r-1} = \sigma_{x,r} \prod_{j<r} \sigma_{z,j} \\ \gamma_{2r} = i\,\underbrace{\sigma_{z,r}\sigma_{x,r}}_{-\sigma_{y,r}} \prod_{j<r} \sigma_{z,j} \end{cases} \rightarrow \quad \gamma = \gamma^{+} \quad \{\gamma_i, \gamma_j\} = 2\delta_{ij}$$

$$i\gamma_{2r-1}\gamma_{2r} = \sigma_{z,r}$$

$$i\gamma_{2r}\gamma_{2r+1} = \sigma_{x,r}\sigma_{x,r+1}$$

$$\Rightarrow \quad H = -i\frac{\mu}{2}\sum_r \gamma_{2r-1}\gamma_{2r} - i\frac{t}{2}\sum_r \gamma_{2r}\gamma_{2r+1} \Rightarrow H = -\frac{\mu}{2}\sum_r \sigma_{z,r} - \frac{t}{2}\sum_r \sigma_{x,r}\sigma_{x,r+1}$$

Majorana $\qquad\qquad\qquad\qquad$ $\boxed{\text{Ising}}$

Fermionic Parity $\prod_r i\gamma_{2r-1}\gamma_{2r} \longleftarrow$ Global symm: $\prod_r \sigma_{z,r}$ $\qquad$ ISING Sym

$$\prod_r i\gamma_{2r-1}\gamma_{2r} = (-1)^F$$

$\langle \sigma_z \rangle = 0$, GS = $|{+}{+}{+}\rangle \pm |{-}{-}{-}\rangle$

Phases

Ferromagnetic phase : Broken Symmetry and degeneracy for $\frac{\mu}{t} \to 0$

Paramagnetic phase : Symmetric phase : No degeneracy $\quad \langle \sigma_z \rangle = 0$

- -

Paramagnetic phase may be understood for $\frac{t}{\mu} \to 0$

$\overset{\mu}{(\gamma_1\ \gamma_2)}\ \overset{\mu}{(\gamma_3\ \gamma_4)}\ \overset{\mu}{(\gamma_5\ \gamma_6)} \Rightarrow \quad i\gamma_1\gamma_2 = 1 \quad \ldots \quad i\gamma_{2r-1}\gamma_{2r} = 1$

$[c^{+}_r c_r = 0]$

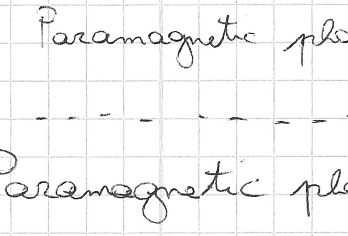
Ferromagnetic phase : $\rightarrow$ TOPOLOGICAL PHASE

Zero Modes

$\gamma_1\ \overset{t}{(\gamma_2\ \gamma_3)}\ \overset{t}{(\gamma_4\ \gamma_5)}\ \overset{t}{(\gamma_6\ \gamma_7)}\ \gamma_8 \quad \Rightarrow$ Uncoupled Majorana modes for $\mu \to 0$

$i\gamma_1\gamma_{2L}$ specify the GS $\qquad$ [γ_1 and γ_8 are sym which do not commute!]

Protection

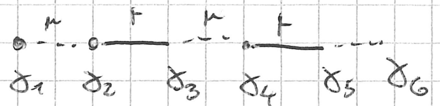


$\rightarrow$ Protection given by fermionic parity

No operator which conserves the F may allow transitions

No quadratic operator in γ's! No local measurement can distinguish F!!

TOPOLOGICAL PHASE $\mu < t$



$$-i\frac{\mu}{2}\gamma_1\gamma_2 \; - i\frac{t}{2}\gamma_2\gamma_3 \; - i\frac{\mu}{2}\gamma_3\gamma_4 \; - i\frac{t}{2}\gamma_4\gamma_5 \; \ldots$$

Zero Mode ?

$[\Gamma, H] = 0$?

$$[\gamma_1, H] = -i\mu\gamma_2$$

$$[\gamma_2, H] = +i\mu\gamma_1 - it\gamma_3$$

$$[\gamma_3, H] = it\gamma_2 - i\mu\gamma_4$$

$\vdots$

$$[\gamma_{2r-1}, H] = it\gamma_{2r-2} - i\mu\gamma_{2r}$$

$$[\gamma_{2r}, H] = i\mu\gamma_{2r-1} - it\gamma_{2r+1}$$

$$\left[\gamma_1 + \frac{\mu}{t}\gamma_3, H\right] = -i\frac{\mu^2}{t}\gamma_4$$

$$+\frac{\mu^2}{t^2}\gamma_5 \qquad = -i\frac{\mu^3}{t^2}\gamma_6$$

$$\Gamma_L = \sum_{m=0}\left(\frac{\mu}{t}\right)^m \gamma_{2m+1} \qquad \Gamma_L^2 = \sum_{m=0}^{N}\left(\frac{\mu}{t}\right)^{2m}$$

$$\Rightarrow [\Gamma_L, H] = \left(\frac{\mu}{t}\right)^{N-1}\mu\,\gamma_{2N}$$ → decays exp with N

Γ_L is localized $= \sum_{m=0} e^{-\frac{m}{\xi}}\gamma_{2m+1}$ $\quad \boxed{\xi^{-1} = \ln\frac{t}{\mu}}$

The same is true on the right side :

$$\Gamma_R = \sum_{m=0}\gamma_{2N-2m}\left(\frac{\mu}{t}\right)^m$$

$$\{\Gamma_L, \Gamma_R\} = 0$$ once normalized ...

→ In the thermodynamic limit they are zero-energy Majorana modes

Effective H $= i e^{-\frac{N}{\xi}}\Gamma_L\Gamma_R$

DISORDER in the TOPOLOGICAL PHASE:

$\langle \mu \rangle < \langle t \rangle$

$-i\frac{\mu_1}{2}\gamma_1\gamma_2 - i\frac{t_1}{2}\gamma_2\gamma_3 - i\mu_2\gamma_3\gamma_4 \dots$

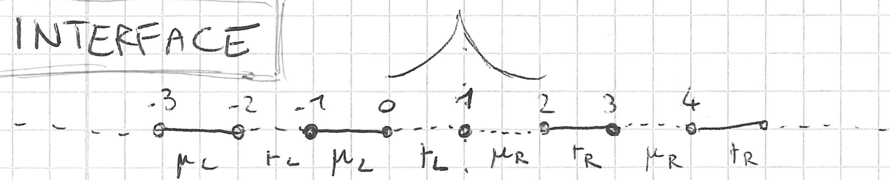
I am adding a disorder that conserves the fermionic parity and it is quadratic in $c, c^\dagger$

$$\Rightarrow \Gamma_L = \sum_{m=0} \left[\prod_{i=1}^{m} \left(\frac{\mu_i}{t_i}\right)\right] \gamma_{2m+1}$$

If $\Delta \neq t$ another two-Majorana term appears: $\propto \boxed{\Delta - t \;\gamma_{2x-1}\gamma_{2x+2}}$

→ still the modes survive

INTERFACE



$t_L < \mu_L$ TRIVIAL

$t_R > \mu_R$ TOPOL.

$$[\gamma_1, H] = i t^L \gamma_0 - i \mu^R \gamma_2$$

$$\Gamma = \sum_{\gamma < 1} \left(\frac{t^L}{\mu^L}\right)^{\gamma} \gamma_{2\gamma-1} + \sum_{\gamma \geq 1} \left(\frac{\mu^R}{t^R}\right)^{\gamma} \gamma_{2\gamma+1} + \gamma_1$$

Two different length-scales: $\left(\ln \frac{\mu^R}{t^R}\right)^{-1}, \left(\ln \frac{t^L}{\mu^L}\right)^{-1}$

TOPOLOGICAL ORDER for MAJORANAS

0. Gap

1. GS degeneracy with open boundary conditions

$$\left[\begin{array}{ll} \rightarrow \ \mathbb{Z}_2 \text{ symmetry} & P = (-1)^{\sum c_r^\dagger c_r} = \prod_r i\gamma_{2r-1}\gamma_{2r} \\ P|\psi_+\rangle = |\psi_+\rangle & P|\psi_-\rangle = -|\psi_-\rangle \end{array}\right]$$

2. $\forall$ local operator V_r

$$\langle \psi_i | V_r | \psi_j \rangle = \bar{V}\delta_{ij} + c(r, i, j)$$

c : decays exponentially in the bulk

Either operator in the bulk or overlap with Majoranas

$\rightarrow$ Robustness against local perturbations in the bulk

3. Local indistinguishability.

$\forall$ observable O_r local:

$$\langle \psi_i | O | \psi_j \rangle = \bar{O}\,\delta_{ij} + o(L, i, j)$$

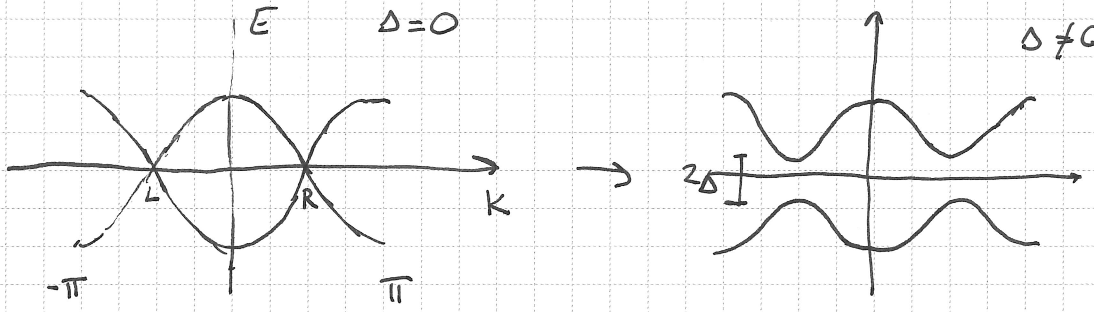
o : decays with system size

$[O, P] = 0 \Rightarrow$ observable.

$$H_{BdG} = (\mu - t\cos k)\,\tau_z + \Delta \sin k\,\tau_y$$

$$E = \pm\sqrt{(\mu - t\cos k)^2 + \Delta^2 \sin^2 k}$$

$\mu = 0 \rightarrow$ center of the topological phase



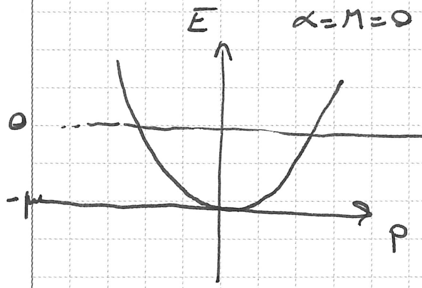
$$H_{eff} = v\psi_L^\dagger i\partial_x \psi_L - v\psi_R^\dagger i\partial_x \psi_R + \Delta \psi_R^\dagger \psi_L^\dagger(-i) + i\Delta \psi_L \psi_R$$

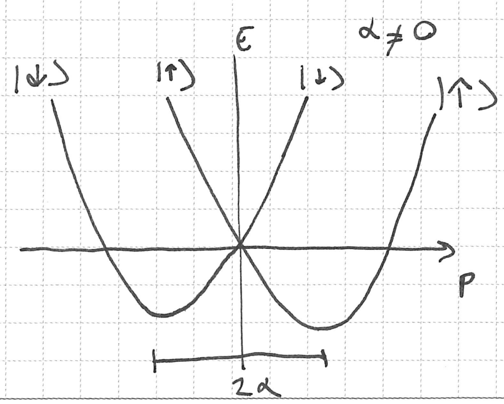
If I do not have p-wave SC, I need

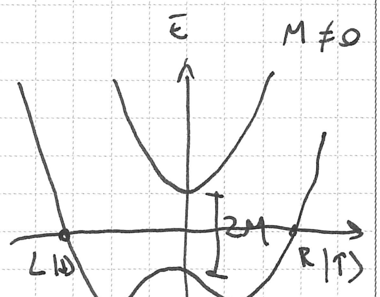
$\psi^\dagger_{R,\downarrow}\ \psi^\dagger_{L,\uparrow} \rightarrow$ opposite spin for left and right components.

RECIPE: Spin-orbit coupling:

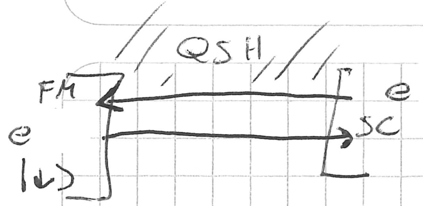
$$H_0 = \frac{p^2}{2m} - \frac{\alpha p \sigma_z}{m} + \underbrace{\frac{gB\sigma_x}{2}}_{M\sigma_x} - \mu$$







REALIZATION IN SC-QSH



$|\uparrow\rangle \quad H_0 = v\left(i\psi_L^\dagger \partial_x \psi_L - i\psi_R^\dagger \partial_x \psi_R\right)$

$T \to$ Backscattering : $+M\psi_R^\dagger\psi_L + h.c.$

U(1) Charge Conserv $\to$ S.C : $-\Delta \psi_R^\dagger \psi_L^\dagger + h.c.$

Basis : $(\psi_\uparrow, \psi_\downarrow, -\psi_\downarrow^\dagger, \psi_\uparrow^\dagger)$ $\qquad [\Gamma, H] = H_{BdG}\Gamma$

$H_{BdG} = iv\partial_x \sigma_z\tau_z + \Delta\tau_x + M\sigma_x \qquad [+\mu\tau_z]$

$$H_{BdG} = \frac{1}{2}\begin{pmatrix} iv\partial_x & M & \Delta & 0 \\ M & -iv\partial_x & 0 & \Delta \\ \Delta & 0 & -iv\partial_x & M \\ 0 & \Delta & M & iv\partial_x \end{pmatrix}$$

- ~~Right~~: $M = 0$, $\Delta(x) = \Delta$ for $x > 0$

Zero modes : $\begin{pmatrix} iv\partial_x & \Delta \\ \Delta & -iv\partial_x \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 0$

$a = -i e^{-\int \frac{\Delta}{v} dx}$

$b = e^{-\int \frac{\Delta}{v} dx} \quad \Rightarrow \quad \Gamma_R = -i\psi_\uparrow e^{-\int} - \psi_\downarrow^\dagger e^{-\int}$

$\Gamma_R^\dagger = i\psi_\uparrow^\dagger e^{-\int} - \psi_\downarrow e^{-\int}$ $\Big\}$ Bound state!

- ~~Right~~ : $M(x) = M \quad \Delta(x) = 0$

$\begin{pmatrix} iv\partial_x & M \\ M & -iv\partial_x \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 0 \qquad a = i e^{\int M/v\, dx}$

$b = e^{\int M/v\, dx}$

$\Gamma_L = i\psi_\uparrow e^{\int} + \psi_\downarrow e^{\int}$

$\Gamma_L^\dagger = -i\psi_\uparrow^\dagger e^{\int} + \psi_\downarrow^\dagger e^{\int}$

- In $x = 0$

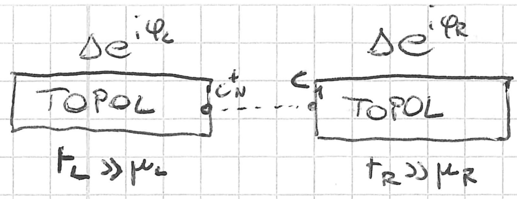
$\alpha\Gamma_R + \beta\Gamma_R^\dagger = \gamma\Gamma_L + \delta\Gamma_L^\dagger$

$\alpha\begin{pmatrix} -i \\ 0 \\ 1 \\ 0 \end{pmatrix} + \beta\begin{pmatrix} 0 \\ -1 \\ 0 \\ i \end{pmatrix} = \gamma\begin{pmatrix} i \\ 1 \\ 0 \\ 0 \end{pmatrix} + \delta\begin{pmatrix} 0 \\ 0 \\ -1 \\ -i \end{pmatrix}$

Only one mode $\quad \Gamma_R + \Gamma_R^\dagger + \Gamma_L + \Gamma_L^\dagger = \Gamma = \Gamma^\dagger$

Localized Majorana zero mode

FRACTIONAL JOSEPHSON



$$H_{int} = \frac{J}{2} C^+_{N,L} C_{1,R} + h.c.$$

$$J \ll t_L, t_R$$

- In the usual case the tunneling of e is suppressed by Δ
- In this case, beside the Cooper Pair tunneling, we have:

$$C_N^+ = \frac{e^{i\varphi_L/2}}{2}\left(\gamma_{2N-1} + i\gamma_{2N}\right) \qquad C_1 = \frac{e^{-i\varphi_R/2}}{2}\left(\gamma_1 - i\gamma_2\right)$$

γ_2 and γ_{2N-1} strongly coupled

<u>Perturbatively</u> $[J \ll t]$

$$\frac{J}{2} C^+_N C_1 + h.c. = \frac{J}{4}\cos\Big(\overbrace{\frac{\varphi_L - \varphi_R}{2}}^{\delta\varphi_{SC} = \Delta B/\phi_0}\Big)\ \underbrace{i\gamma_{2N}\gamma_1}_{\text{Parity of the junction!}} \qquad \phi_0 = \frac{hc}{2e}$$

$\rightarrow$ 4π periodicity

- H is 2π periodic in φ_L and φ_R but the GS is not!

The same can be seen with bosonization

$$\psi_L = :e^{i(\varphi-\theta)}: \qquad \psi_R = :e^{i(\varphi+\theta)}: \qquad \mathcal{H}_0 = \frac{v}{2\pi}\int dx\,(\partial_x\varphi)^2 + (\partial_x\theta)^2$$

$$-\Delta\,\psi_R^+\psi_L^+ + h.c. = -\Delta\cos 2\varphi$$

$$-\Delta e^{i\varphi_L}\psi_R^+\psi_L^+ + h.c. = -\Delta\cos(2\varphi - \varphi_L)$$

$$\Rightarrow \varphi_1 \approx \frac{\varphi_L}{2} \qquad \varphi_2 \approx \frac{\varphi_R}{2} \quad \rightarrow \text{pinned}$$

If I vary $\varphi_R \rightarrow \varphi_R + 2\pi \Rightarrow \varphi_2 \rightarrow \varphi_2 + \pi$ and $\mathcal{H}_0$ changes $\rightarrow$ twist

$\Rightarrow$ 4π periodicity