Introduction to one-dimensional models

M. Burrello
Niels Bohr International Academy and Center for Quantum Devices,
Niels Bohr Institute, University of Copenhagen, 2100 Copenhagen, Denmark.
(Dated: May 9, 2023)

These notes are aimed at providing a useful compendium for an introductory study of Luttinger liquids and bosonization. Here I will collect material from many sources, including the books by Bruus and Flensberg (**BF**, Chap. 19) and by Altland and Simons (**AS**, Chap. 4). For the students interested in diving deeper into the physics of 1D models, there are many books and references about it, for example Giamarchi's book [1], this very good paper by Cazalilla [2], this review [3] and the Bible of bosonization [4].

I. INTRODUCTION

One-dimensional (1D) quantum systems offer an intriguing arena to study many exotic phenomena which often do not have a clear counterpart in higher dimensions. Many systems can be engineered that, at low enough temperature, behave like 1D systems: they include nanowires, chains and ladders of trapped atoms, photonic waveguides, and even the edges of particular 2D materials, such as quantum Hall systems and topological insulators.

The peculiarity of 1D systems, in my opinion, is given by three main aspects:

- **The role of interactions.-** If you consider a classical 1D system of two non-interacting particles on a line, in relative motion with each other, these particles necessarily cross at a certain point in the past or in the future. This is very different from 2D and 3D system and emphasizes that, during their evolution, these particles must at a certain point get close to each other. Therefore, when considering local interactions, their role cannot be neglected. This is, at a very basic level, the reason for which interactions are hardly "weak" in 1D systems and very often the perturbative strategies to deal with interactions are less effective in 1D than in 2D and 3D.

- **Exactly solvable models.-** Despite the problems of perturbative strategies, there is a large class of interacting problems that can be exactly solved in 1D systems. They are called "integrable models" and their study is beyond the scope of these notes. However it is interesting to note that often these exactly solvable models offer a very good starting point to describe more general systems that can be defined, on an effective level, as perturbed versions of known solvable models. In these notes we will deal with Luttinger liquids which define a class of systems that can be described starting from the exactly solvable Tomonaga-Luttinger model, which describes interacting fermions with a linear dispersion.

- **Critical points and conformal invariance.-** The critical points of the renormalization group are gapless models that display scale invariance. In 1D, a system invariant under scaling is also conformally invariant. By exploiting this mathematical property it is possible to give a full classification of all the gapless models in 1+1D (conformal field theory). This goes totally beyond the scope of these notes, but it is at the basis of many strategies to solve interacting 1D problems.

The aim of the techniques we will describe in these notes is the description of **interacting 1D models**. The difficulty in dealing with interacting models can be appreciated by considering the difference between non-interacting and interacting fermions. A non-interacting model of fermions can be expressed in terms of a Hamiltonian which is quadratic in the fermionic operators. In the worst-case scenario, such Hamiltonian will scale as $L \times L$, where L is the number of fermionic modes which is in general proportional with the size of the system. In case of translational invariance things become even easier and the dispersion relations can be easily calculated. These quadratic models include not only free fermions, but also mean field approximations such as the BdG equations. The scenario is dramatically different when the Hamiltonian is not quadratic any longer. This is the situation, for example, when we include quartic density-density interactions as in the following chain of spinless fermions:

$$H = -t \sum_r \left(c^\dagger_{r+1} c_r + H.c. \right) + U \sum_r c^\dagger_r c_r c^\dagger_{r+1} c_{r+1} \, . \tag{1}$$

Here the t term specified the kinetic energy, whereas the U term is a nearest-neighbor density-density interaction. This kind of Hamiltonians may require, in the worst-case scenario, to diagonalize matrices of dimension $2^L \times 2^L$, which is in general a very tough job. Our aim is to find effective tools for the description of this kind of problems.

The family of Hamiltonians similar to (1), furthermore, do not describe only fermionic systems, but also many spin systems through a non-local transformation called Jordan-Wigner. Therefore the techniques we will deal with are very powerful and general and may be applied not only to fermionic but also to bosonic problems.

My aim is to cover the following topics in the next sections:

1. Non-interacting 1D Fermi systems and the 1D Dirac Hamiltonian: it is a necessary preliminary part.
2. The Luttinger liquid problem: we can solve a specific gapless interacting model of fermions in terms of free bosons.
3. Phenomenological bosonization: what can we learn from the Luttinger model? How to rewrite fermionic fields?
4. Field-theoretical bosonization: technical dictionary to map fermionic fields into bosonic (vertex) operators.
5. Spin chains and Jordan-Wigner transformation: how can we apply bosonization to spin chains? The XXZ example.
6. Sine-Gordon model and its RG.

II. PRELIMINARIES: 1D DIRAC HAMILTONIAN

The starting point of our analysis is a 1D non-interacting fermionic system. Therefore, it is important to understand what we mean by 1D. Obviously our world is 3D, thus a description of a system in terms of a single space dimension is an approximation. This approximation relies on the fact that, for low enough temperatures, and low enough energy scales for the motion along the direction $\hat{x}$, we can consider the degrees of freedom along the transverse directions $\hat{y}$ and $\hat{z}$ frozen in their ground state. To specify better this idea, imagine that we have a box-shaped potential along $\hat{y}$ and $\hat{z}$ such that $V(y,z)=0$ in a rectangular region $L_y \times L_z$ and very strong outside. What happens is that we obtain two energy scales $E_{y,z}$ proportional to $1/L_y^2$ and $1/L_z^2$ which separate the ground state of the transverse modes with their first excited states. If L_y and L_z are small enough, we can assume that these gaps $E_{y,z}$ are much larger than the temperature, the kinetic energy of the particles along $\hat{x}$ and the interactions among them. In practice $E_{y,z}$ will be the largest energy scale in our system, so large that, in the study of the low-energy physics, we can forget about it. The same would be true with any confinement potential along the transverse direction which is strong enough: for example, a harmonic confinement would define another energy scale $\hbar\Omega$ and in the limit $\Omega \to \infty$ the system becomes effectively 1D.

This kind of 1D approximation is very useful to describe a huge variety of physical systems, ranging from nanowires, to chains or ladders of ultracold atoms, from optical waveguides to the edge states of 2D topological materials and many many others.

Our prototype for a 1D system of non-interacting spinless fermions is the following:

$$H_0 = -t \sum_r \left[c_{r+a}^\dagger c_r + \text{H.c.} \right] + \mu \sum_r c_r^\dagger c_r \tag{2}$$

This Hamiltonian describes fermions hopping along a chain. The length a is the lattice spacing and it defines a natural ultraviolet cutoff $\Lambda_{max} = \pi/a$ for the momenta, since the Brillouin zone is given by $k \in [-\pi/a, \pi/a)$. In the following, I will use very often units such that $a = 1$. μ is the chemical potential of the system. By doing a Fourier transform we get:

$$H_0 = -t \sum_k 2\cos k \, c_k^\dagger c_k + \mu \sum_k c_k^\dagger c_k \tag{3}$$

Exercise II.1 *Derive Eq. (3) from Eq. (2) by using the following definition for the fermionic operators:*

$$c_r = \frac{1}{\sqrt{L}} \sum_k e^{ikr} c_k \tag{4}$$

For the most mathematically inclined: the exact result is obtained for periodic boundary conditions.

From Eq. (3) we get that the dispersion relation is $E(k) = -2t\cos k$ and the chemical potential fixes the value of the Fermi momentum:

$$-2t\cos(k_F) + \mu = 0\,, \quad \Rightarrow \quad k_F = \arccos\left(\frac{\mu}{2t}\right)\,. \tag{5}$$

In a system of length L with N fermions we have:

$$k_F = \pi\frac{N}{L} \equiv \pi\rho_0 \tag{6}$$

where we defined the average density $\rho_0 = N/L$.

When we consider the physics at low temperature and energies, we can linearize the dispersion relation close to the "Fermi surface": the "Fermi surface" is simply the pair of points $k = \pm k_F$. In a neighborhood of these points we can approximate the dispersion with:

$$H_L = \sum_{k\approx -k_F} -v_F(k+k_F)c_k^\dagger c_k\,, \tag{7}$$

$$H_R = \sum_{k\approx k_F} v_F(k-k_F)c_k^\dagger c_k\,, \tag{8}$$

where we introduced the Fermi velocity:

$$v_F = \left.\frac{\partial E(k)}{\partial k}\right|_{k=k_F} = 2t\sin k_F\,. \tag{9}$$

To define a description in continuum space we introduce two fermionic chiral fields $\tilde\psi_L(x)$ and $\tilde\psi_R(x)$ such that we obtain:

$$H_{L/R} = \mp v_F\int dx\,\tilde\psi^\dagger_{L/R}(x)\left(-i\partial_x \pm k_F\right)\tilde\psi_{L/R}(x) \tag{10}$$

Exercise II.2 *Verify that Eq. (10) is consistent with (7) and (8) through the definition:*

$$\tilde\psi_{L/R}(x) = \int\frac{dk}{\sqrt{2\pi}}e^{ikx}c_{L/R}(k) \tag{11}$$

For the purpose of this exercise, consistently with the linearization of the spectrum, you may consider the integral from $-\infty$ to $+\infty$.

It is useful to redefine the fields and the previous Hamiltonian by the following (gauge) transformation:

$$\tilde\psi_{L/R}(x) = e^{\mp ik_F x}\psi_{L/R}(x) \tag{12}$$

This definition splits the fields into two contributions: the terms $e^{\mp ik_F x}$ define the **fast oscillating behavior** of the fields, namely it is a phase that changes on the length scale of a, whereas $\psi_{L/R}(x)$ is a **slow varying field** which is supposed to vary on length scales much larger than a. This is based on the assumption that the important low energy physics is dictated by what happens at $k \approx \pm k_F$.

Based on these definitions, we are now able to recast the Hamiltonian in the following form:

$$H_0 = H_L + H_R = \int dx\; iv_F\psi_L^\dagger\partial_x\psi_L - iv_F\psi_R^\dagger\partial_x\psi_R \tag{13}$$

Exercise II.3 *Derive Eq. (13) from Eqs. (10) and (12).*

If we introduce a spinor $\psi = (\psi_L, \psi_R)^T$ we finally obtain:

$$H_0 = \int dx\, iv_F\psi^\dagger\sigma_z\partial_x\psi \tag{14}$$

which is the 1D Dirac Hamiltonian. Here σ_z is the Pauli matrix. The corresponding Lagrangian is:

$$\mathcal{L} = \int dx\;\psi^\dagger\left(i\partial_t - iv_F\sigma_z\partial_x\right)\psi \tag{15}$$

Exercise II.4 *Verify that* (14) *derives from* (15) *through the standard relation* $H = \sum_{a=L/R} \Pi_a \partial_t \psi_a - \mathcal{L}$ *and* $\Pi_a = \partial \mathcal{L}/\partial(\partial_t \psi_a)$.

Exercise II.5 *The role of the transformation* (12) *(and the related boundaries for the values of k in the integrals) may result not intuitive when we consider the fact that we are starting from a lattice model like Eq.* (2)*. To understand better why we can model the Fermi surface of Hamiltonian* (2) *with a Dirac cone, consider the following model (Su-Schrieffer-Heeger model):*

$$H_{SSH} = -t_1 \sum_r \left[c^\dagger_{2r+a} c_{2r} + \text{H.c.} \right] - t_2 \sum_r \left[c^\dagger_{2r+2a} c_{2r+a} + \text{H.c.} \right] . \tag{16}$$

Define a suitable unit cell, a suitable Brillouin zone, and calculate its spectrum. What happens for $t_1 = t_2$*?*

The equation of motions for the chiral fields are:

$$(i\partial_t - i v_F \partial_x)\, \psi_L = 0\,, \quad \Rightarrow \quad \psi_L(x + v_F t)\,, \tag{17}$$
$$(i\partial_t + i v_F \partial_x)\, \psi_R = 0\,, \quad \Rightarrow \quad \psi_R(x - v_F t)\,. \tag{18}$$

This demonstrates indeed that the fields ψ_L and ψ_R depend only on $x+v_F t$ and $x-v_F t$ respectively, which justifies their definition as "chiral". It is useful to derive the correlation functions of these fields at zero temperature. Consistently with the linear dispersion relation in Eq. (13), the corresponding Fermi sea is such that the occupied left and right states have momenta $k > 0$ and $k < 0$ respectively. The left correlation function results:

$$\left\langle \psi^\dagger_L(y,t)\psi_L(x,0) \right\rangle_0 = \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, e^{-iky+ikx} \left\langle e^{iH_L t} c^\dagger_L(k) e^{-iH_L t} c_L(k) \right\rangle_0 =$$
$$\int_{-\infty}^{\infty} \frac{dk}{2\pi}\, e^{-iky+ikx} e^{iE_{GS}t} \left\langle c^\dagger_L(k) e^{i\left(\sum_k v_F k c^\dagger_L(k) c_L(k)\right)t} c_L(k) \right\rangle_0 = \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, e^{-iky+ikx} e^{-iv_F kt}\, \Theta(k) \to -\frac{1}{2\pi} \frac{i}{y - x + v_F t}\,. \tag{19}$$

In this calculation we first applied a Fourier transform of the fields similar to Eq. (11) [applied to the fields ψ, which have been translated in momentum space, though]. The time dependence of $\psi^\dagger_L$ has been accounted for by introducing a suitable unitary evolution $\psi^\dagger_L(y,t) = e^{iH_L t}\psi^\dagger_L(y,0)e^{-iH_L t}$, with H_L defined as in Eq. (13). The correlation function is taken over the ground state, in which only states with positive momenta are occupied; its energy is E_{GS}. By applying the operator $c_L(k)$ for positive momenta, a hole excitation with energy $v_F k$ over the ground state is created, such that in the second last step $e^{i\left(\sum_k v_F c^\dagger_L(k) c_L(k)\right)t} \to e^{-i(E_{GS}+v_F k)t}$. For negative k, instead, the ground state is annihilated. Therefore, we need to introduce the Heaviside step function $\Theta(k)$, and the time evolution results in the phase $e^{-iv_F kt}$ acquired by the excitation. The final integration can be performed by introducing a regularization e^{-bk} and sending $b \to 0$.

Analogously we get:

$$\left\langle \psi^\dagger_R(y,t)\psi_R(x,0) \right\rangle_0 = \int_{-\infty}^{0} \frac{dk}{2\pi}\, e^{-iky} e^{+ikv_F t} e^{ikx} \to -\frac{1}{2\pi} \frac{i}{x - y + v_F t}\,. \tag{20}$$

The key aspect of these correlation functions is that they decay algebraically with the space and time distance, and we will obtain again this behavior with the bosonized constructions in section V. These correlation functions have been determined for the linearized field with an idealized Fermi sea. In the original picture, the momenta for left and right movers would be translated by $\mp k_F$ respectively, without causing a qualitative change of the result.

We could refine better this result by considering that, in the original picture, the occupied right and left movers had momenta $0 < k < k_F$ and $0 > k > -k_F$ respectively. By translating this from the $\tilde{\psi}$ to the ψ fields, we would get integration intervals $[-k_F, 0]$ and $[0, k_F]$ for ψ_R and ψ_L respectively, which would provide oscillating numerators in the previous correlation functions, without affecting the algebraic decay. We will adopt a similar choice for the momenta of the left and right modes in the next section.

III. THE LUTTINGER MODEL

The bosonization technique allows us to describe interacting (thus quartic) models of fermions (but also interacting bosonic systems) in terms of a quadratic Hamiltonian of bosonic operators. These bosonic operators are non-local in the original fermionic ones, but provide a simple description of the system and are helpful to evaluate observables, such as the correlation functions, and gain an approximate understanding of the phase diagrams of the original models through renormalization group techniques. At the basis of the bosonized description lies the Luttinger model: a model of interacting and linearly dispersing fermions that can be exactly solved. The solution of this problem was found by Mattis and Lieb in 1965, but its final formalization in terms of bosonization was built in the '70 (Coleman and Mandelstam, Matthis and Luther, Haldane).

Let us start from a Hamiltonian of the form (1) where we consider a density-density interaction:

$$H = H_0 + H_{\text{int}} \equiv H_0 + U \sum_r c^\dagger_{r+1} c_{r+1} c^\dagger_r c_r \tag{21}$$

where H_0 is the free Hamiltonian of Eq. (2) (to be precise the exactly solvable Luttinger liquid is obtained in the case of the linear dispersion as in Eqs. (7,8)). The interaction term reads:

$$H_{\text{int}} = U \sum_r c^\dagger_{r+1} c_{r+1} c^\dagger_r c_r = \sum_{\{k\},r} \frac{U}{L^2} e^{-ik_1(r+1)} e^{ik_2(r+1)} e^{-ik_3 r} e^{ik_4 r} c^\dagger_{k_1} c_{k_2} c^\dagger_{k_3} c_{k_4} =$$

$$= \frac{U}{L} \sum_{\{k\}} \delta\left(k_1 - k_2 + k_3 - k_4\right) e^{-i(k_1-k_2)} c^\dagger_{k_1} c_{k_2} c^\dagger_{k_3} c_{k_4} \,. \tag{22}$$

Here the Dirac δ imposes momentum conservation and it is convenient to define $q \equiv k_2 - k_1 = k_3 - k_4$. In this way the interaction term assumes the general form:

$$H_{\text{int}} = \frac{1}{L} \sum_{k,k',q} V(q) c^\dagger_{k-q} c_k c^\dagger_{k'+q} c_{k'} \,. \tag{23}$$

If we consider our results from the previous section we could redo this calculation by defining:

$$c^\dagger_r = \frac{1}{\sqrt{L}} \left(\sum_{k>-k_F} e^{-i(k+k_F)r} c^\dagger_{R,k} + \sum_{k<k_F} e^{-i(k-k_F)r} c^\dagger_{L,k} \right) . \tag{24}$$

Here, as in Eq. (13), I translated the momenta by $\pm k_F$ for the left and right modes. With these notation, $k = 0$ corresponds to the Fermi surface, and I am considering new extrema in the summation to account for this translation. If you substitute the previous equation into H_{int} you obtain 16 terms. Of these 16 terms, all the terms displaying a fast oscillating behavior $e^{\pm 2ik_F r}$ or $e^{\pm 4ik_F r}$ vanish when summing along r due to momentum conservation. Only 6 terms remain, which display only a "slow" oscillating behavior and do not depend on k_F. This is true for generic k_F, for $k_F = \pi/2$ (half-filling), $4k_F = 2\pi$ and additional "umklapp" terms appear. For $k_F \neq \pi/2$, the explicit calculation shows that the system conserves the chirality, namely $c^\dagger_R c_R$ and $c^\dagger_L c_L$ are conserved quantities.

These 6 terms can be distinguished into two groups: $q \approx 0$ and $q \approx 2k_F$.

Exercise III.1 *Show that the first kind, $|q| \ll k_F$, includes the two terms $c^\dagger_{R,k+q} c_{R,k} c^\dagger_{L,k'-q} c_{L,k'}$, $c^\dagger_{R,k+q} c_{R,k} c^\dagger_{R,k'-q} c_{R,k'}$ and the two terms with $L \leftrightarrow R$; and that for $|q| \approx 2k_F$, instead, one finds the last two terms $c^\dagger_{L,k-q} c_{R,k} c^\dagger_{R,k'+q}, c_{L,k'}$ and the opposite term with $L \leftrightarrow R$ and $q \to -q$. These operators are discussed and represented in sections 19.4.1 - 19.4.3 of BF.*

It is now useful to define, for $|q| \ll k_F$ the following density operators:

$$\rho_R(q) = \sum_{k>-k_F} c^\dagger_{R,k} c_{R,k+q} \,, \tag{25}$$

$$\rho_L(q) = \sum_{k<k_F} c^\dagger_{L,k} c_{L,k+q} \,. \tag{26}$$

Under the assumption that $V(q)$ varies slowly with q (it is not strictly necessary, but it simplifies our discussion), we can simply consider two scattering amplitudes $V(q \approx 0)$ and $V(q \approx 2k_F)$ and we obtain:

$$H_{\rm int} \approx \frac{U}{L} \sum_q \left[V(0) - V(2k_F)\right] \left[\rho_R(q)\rho_L(-q) + \rho_L(q)\rho_R(-q)\right] + V(0) \left[\rho_R(q)\rho_R(-q) + \rho_L(q)\rho_L(-q)\right] \tag{27}$$

This is a general form meant as an example: for each specific problem (for example on the lattice) one can derive the proper interaction coefficient. The key aspect is that the interaction is always quadratic in the density operators ρ. More detail can be found in BF, section 19.4. Starting from a generic Hamiltonian, quadratic in the ρ operators, we can solve it. The first step to do so is to calculate all the commutators $\left[\rho_{R/L}(q), \rho_{R/L}(q')\right]$. Let us check, as an example:

$$\begin{aligned}
[\rho_R(q), \rho_R(-q)] &= \sum_{k,k'>-k_F} \left[c_k^\dagger c_{k+q}, c_{k'}^\dagger c_{k'-q}\right] = \sum_{k,k'>-k_F} c_k^\dagger c_{k+q} c_{k'}^\dagger c_{k'-q} - c_{k'}^\dagger c_{k'-q} c_k^\dagger c_{k+q} = \\
&= \sum_{k,k'>-k_F} c_{k'}^\dagger c_k^\dagger c_{k+q} c_{k'-q} + \delta(k'-k-q) c_k^\dagger c_{k'-q} - c_{k'}^\dagger c_k^\dagger c_{k+q} c_{k'-q} - \delta(k'-q-k) c_{k'}^\dagger c_{k+q} = \\
&= \sum_{k>\max(-k_F,-k_F-q)} c_k^\dagger c_k - \sum_{k>\max(-k_F,-k_F-q)} c_{k+q}^\dagger c_{k+q} = \begin{cases} \texttt{If } q>0 & \sum_{-k_F<k<-k_F+q} c_k^\dagger c_k = q\frac{L}{2\pi} \\ \texttt{If } q<0 & -\sum_{-k_F<k<-k_F-q} c_k^\dagger c_k = -|q|\frac{L}{2\pi} \end{cases} = \frac{qL}{2\pi}
\end{aligned} \tag{28}$$

The final result is justified by the fact that we are considering a state in which the single-particle right states are occupied for $k < 0$ and we assume $|q| \ll k_F$. Therefore, $c_k^\dagger c_k |0\rangle = |0\rangle$ for the interval we are considering.

Exercise III.2 *By following the previous example show that:*

$$[\rho_R(q), \rho_R(-q')] = \delta_{qq'} \frac{L}{2\pi} q\,, \qquad [\rho_R(q), \rho_L(-q')] = 0\,, \qquad [\rho_L(q), \rho_L(-q')] = -\delta_{qq'} \frac{L}{2\pi} q \tag{29}$$

The definition of the density operators ρ and their commutation relations allow us to introduce the following bosonic operators:

$$b_p = \sqrt{\frac{2\pi}{L|p|}} \left(\Theta(p)\rho_R(p) + \Theta(-p)\rho_L(p)\right)\,, \tag{30}$$

$$b_p^\dagger = \sqrt{\frac{2\pi}{L|p|}} \left(\Theta(p)\rho_R(-p) + \Theta(-p)\rho_L(-p)\right)\,; \tag{31}$$

where I introduced the Heaviside step function $\Theta(p)$.

Exercise III.3 *Show that the b operators are honest bosonic operators that fulfill:*

$$\left[b_p, b_{p'}^\dagger\right] = \delta_{pp'} \tag{32}$$

It is important to emphasize that the $b, b^\dagger$ operators are quadratic in the original fermionic operators and, in turn, any interaction of the form (27) is quadratic in the operators b and $b^\dagger$. To summarize we see that:

$$\text{For } p>0: \qquad \rho_R(p) = \sqrt{\frac{Lp}{2\pi}} b_p\,, \qquad \rho_R(-p) = \sqrt{\frac{Lp}{2\pi}} b_p^\dagger\,, \tag{33}$$

$$\text{For } p<0: \qquad \rho_L(p) = \sqrt{\frac{L|p|}{2\pi}} b_p\,, \qquad \rho_L(-p) = \sqrt{\frac{L|p|}{2\pi}} b_p^\dagger\,. \tag{34}$$

This implies:

$$\frac{1}{L}V(p)\rho_R(-p)\rho_R(p) \to \frac{|p|}{2\pi}V(p)b_p^\dagger b_p \quad \text{for } p>0\,, \tag{35}$$

$$\frac{1}{L}V(p)\rho_L(-p)\rho_L(p) \to \frac{|p|}{2\pi}V(p)b_p^\dagger b_p \quad \text{for } p<0\,, \tag{36}$$

$$\frac{1}{L}V(p)\rho_R(-p)\rho_L(p) \to \frac{|p|}{2\pi}V(p)b_p^\dagger b_{-p}^\dagger \quad \text{for } p>0\,. \tag{37}$$

With these rules one can diagonalize any interaction Hamiltonian of the previous forms in terms of bosons with a linear dispersion. The major point now is to understand what happens to the kinetic part of the Hamiltonian H_0. H_0 is only quadratic in the fermionic operators $c, c^\dagger$, therefore it is a priori not clear how we can express it as a function of the bosonic operators $b, b^\dagger$. To understand it, we must calculate the following commutator (we assume for simplicity $p_0>0$):

$$\begin{aligned}
[b_{p_0}, H_0] &= \left[\sqrt{\frac{2\pi}{Lp_0}}\rho_R(p_0), \sum_{k>-k_F} v_F k c_{R,k}^\dagger c_{R,k}\right] = \sqrt{\frac{2\pi}{Lp_0}}\sum_{k,k'>-k_F}\left[c_{R,k'}^\dagger c_{R,k'+p_0}, v_F k c_{R,k}^\dagger c_{R,k}\right] = \\
&= \sqrt{\frac{2\pi}{Lp_0}}\sum_{k,k'>0} v_F(k-k_F)\left(c_{R,k'}^\dagger c_{R,k}\delta(k'+p_0-k) - \delta(k-k')c_{R,k}^\dagger c_{R,k'+p_0}\right) = \\
&= \sqrt{\frac{2\pi}{Lp_0}}\left[\sum_{k>-k_F+p_0} v_F k c_{R,k-p_0}^\dagger c_{R,k} - \sum_{k>-k_F} v_F k c_{R,k}^\dagger c_{R,k+p_0}\right] = \\
&= \sqrt{\frac{2\pi}{Lp_0}}\sum_{k>-k_F} v_F p_0 c_{R,k}^\dagger c_{R,k+p_0} = v_F p_0 b_{p_0}\,.
\end{aligned} \tag{38}$$

Therefore we find that the commutator $[b_q, H_0]$ is proportional to b_q. This is the algebraic behavior of a simple harmonic oscillator! Therefore we can redefine:

$$H_0 = \sum_{p\neq 0} v_F|p| b_p^\dagger b_p\,. \tag{39}$$

This relation tells us, essentially, that both the fermionic and this bosonic H_0 generate the same time evolution. This conclusion is surprising: we are stating that the quadratic Dirac Hamiltonian of the fermions is equivalent to a quadratic Hamiltonian of the bosonic operators b and $b^\dagger$. Also this piece of the Hamiltonian defines bosons with a linear dispersion. The final striking result, thus, is that the fermionic Hamiltonian $H_0 + H_{\text{int}}$ becomes a quadratic and easily solvable Hamiltonian in terms of $b, b^\dagger$ operators. This relies on the linear dispersion of the massless Dirac Hamiltonian H_0, but, in general, we know that this is the behavior expected in 1D fermionic systems close to their Fermi surface. This is the solution of the Luttinger problem and it is the pillar of the bosonization techniques we will discuss in the next sections. In particular, putting together Eqs. (35,36,37,39), we obtain an Hamiltonian of the following form:

$$H(p) = \sum_{p>0}|p|\,(b_p^\dagger, b_{-p})\begin{pmatrix} v_F + \frac{V(p)}{2\pi} & \frac{V(p)}{2\pi} \\ \frac{V(p)}{2\pi} & v_F + \frac{V(p)}{2\pi}\end{pmatrix}\begin{pmatrix} b_p \\ b_{-p}^\dagger\end{pmatrix}; \tag{40}$$

its spectrum can be derived through a Bogoliubov transformation of the b operators and it reads:

$$E(p) = |p|v_F\sqrt{1+\frac{V(p)}{\pi v_F}}\,, \tag{41}$$

such that the effective velocity of the system depends on the interaction $V(p)$.

Exercise III.4 *The Hamiltonian (27) gives rise to a slightly more general quadratic Hamiltonian than Eq. (40), since the off-diagonal interaction terms $\rho_R\rho_L$ have different coefficients than the diagonal. Find the energy spectrum for Eq. (27) as a function of $V(0)$ and $V(2k_F)$ by using Eqs. (33) and (34).*

IV. PHENOMENOLOGICAL BOSONIZATION

In the previous section we saw that an interacting model of spinless 1D fermions can be recast into a non-interacting model of bosons and we derived this result in momentum space. In this section, instead, we will begin from a real space description and we will focus on the role of the density operator, which constitutes, essentially, the main observable of the system for a spinless model. This section is mostly inspired by Chap. 3 in [1].

Let us consider a 1D model of N spinless particles, either bosons or fermions, sitting in the positions $\{x_j\}$. The density operator can be defined as:

$$\rho(x) = \sum_j^N \delta(x - x_j)\,. \tag{42}$$

We will label with L the system size such that, the average density is $\rho_0 = N/L$. In the case of fermions we also know $\rho_0 = k_F/\pi$. We define with $\bar{d} = L/N = \rho_0^{-1}$ the average interparticle distance and, essentially, we may think that, if the system is uniform enough, the positions x_j will be only slightly displaced from their average position by a fluctuation u_j:

$$x_j = \bar{d}j + u_j\,. \tag{43}$$

Our purpose is to obtain a low-energy field theoretical description of this model and, to indirectly describe the density, we define a so-called labeling field θ_l, which fulfills the condition:

$$\theta_l(x_j) = 2\pi j\,. \tag{44}$$

The field $\theta_l(x)$, thus, is essentially counting the number of particle before the point x, but it is a continuous field. We also impose that θ_l is monotonically increasing. We introduce the following function f:

$$f(x) = -i\left(e^{i\theta_l(x)} - 1\right)\,.$$

f vanishes for each $x = x_j$. Therefore, we obtain:

$$\delta\left(f(x)\right) = \sum_{j\in\text{zeros of } f} \frac{1}{|\partial_x f(x_j)|}\delta(x - x_j) = \sum_{j\in\text{zeros of } f} \frac{1}{|\partial_x \theta(x_j)|}\delta(x - x_j)\,.$$

Therefore we can rewrite the density as:

$$\rho(x) = \sum_j \delta(x - x_j) = \sum_j |\partial_x\theta_l(x)|\,\delta(\theta_l(x) - 2\pi j) \tag{45}$$

Since we are considering θ_l monotonically increasing, we can avoid the modulo in this definition. Now we apply Poisson's summation formula and we obtain:

$$\rho(x) = \frac{\partial_x\theta_l(x)}{2\pi}\sum_{p=-\infty}^{+\infty} e^{ip\theta_l(x)}\,. \tag{46}$$

I remind that Poisson summation formula states:

$$\sum_n s(t + 2\pi n) = \frac{1}{2\pi}\sum_k \hat{s}\left(\frac{k}{2\pi}\right)e^{ikt}\,;$$

where s and $\hat{s}$ are a sufficiently regular function and its Fourier transform. At this point we redefine the labeling field to emphasize the role of fluctuations and we set:

$$\theta_l(x) = 2\pi\rho_0 x - 2\theta(x) = 2k_F x - 2\theta(x)\,, \tag{47}$$

where the last equality holds for fermions. This equation define the field:

$$\theta(x) = -\frac{\theta_l(x)}{2} + \pi\rho_0 x\,. \tag{48}$$

The density operator now reads:

$$\rho(x) = \left(\rho_0 - \frac{\partial_x\theta(x)}{\pi}\right)\sum_p e^{i2p(\pi\rho_0 x - \theta(x))} \tag{49}$$

We observe that:

- Eq. (49) suggests $[\theta(x), \theta(x')] = 0$, since we want the densities in different positions to commute.
- The slow oscillating part of the density, which is the dominating part if we do some local averaging, is:

$$\rho(x) = \rho_0 - \frac{\partial_x\theta(x)}{\pi}\,. \tag{50}$$

So far we manipulated the definition of the density operator. In the next we will introduce the creation and annihilation operators of the particles in the system. We will start from the bosonic case, which is less common but a bit simpler, and then we will modify our result to describe the fermionic case.

A. Bosonization of bosons

We want to define a bosonic field ψ_B such that $\rho(x) = \psi_B^\dagger(x)\psi_B(x)$, which is the standard definition of the density. In particular, up to an irrelevant phase, we define:

$$\psi_B^\dagger(x) = \sqrt{\rho(x)}e^{-i\varphi(x)}\,. \tag{51}$$

This is a standard ansatz, based on the introduction of a phase field φ. You may have encountered a similar situation to describe Cooper pairs in a superconductor, or a Bose-Einstein condensate. The equation (51) defines the bosonization of a bosonic field, namely we are translating the bosonic field $\psi_B^\dagger$ into the exponential of another bosonic field φ. This exponential is technically called "vertex operator" in the conformal field theory literature. Since we are dealing with bosons, we want to fulfill the following commutation relations:

$$\left[\psi_B(x), \psi_B^\dagger(x')\right] = \delta(x - x')\,. \tag{52}$$

Let us understand what this equation implies. From eqs. (51) and (52) we derive:

$$e^{i\varphi(x)}\sqrt{\rho(x)}\sqrt{\rho(x')}e^{-i\varphi(x')} - \sqrt{\rho(x')}e^{-i\varphi(x')}e^{i\varphi(x)}\sqrt{\rho(x)} = \delta(x - x')\,; \tag{53}$$

in particular, for $x = x'$ we get:

$$e^{i\varphi(x)}\rho(x)e^{-i\varphi(x)} - \rho(x) = \delta(x - x')\,, \tag{54}$$

thus a good solution for our problem is given by imposing:

$$\left[\rho(x'), e^{-i\varphi(x)}\right] = \delta(x' - x)e^{-i\varphi(x)}\,. \tag{55}$$

This indeed reveals that the operator $e^{-i\varphi(x)}$ creates a boson. In particular we can consider the slow varying modes and we obtain:

$$\left[\rho_0 - \frac{\partial_x\theta(x')}{\pi}, e^{-i\varphi(x)}\right] = \delta(x' - x)e^{-i\varphi(x)}\,; \tag{56}$$

where ρ_0 is just a constant. This shows that we can consider $\partial_x\theta/\pi$ as the conjugate field of φ and, brutally, we can think about $\partial_x\theta/\pi = \Pi_\varphi = -i\partial_\varphi$. More properly, we can recast this relation into **one of the fundamental equations of bosonization**:

$$\left[\frac{\partial_x\theta(x')}{\pi}, \varphi(x)\right] = -i\delta(x - x')\,. \tag{57}$$

This is the fundamental commutation relation relating the fields θ and φ. It is an equal-time commutation relation: both the fields are here taken at the same time. It states indeed that $\partial_x\theta/\pi$ is the canonically conjugate operator of φ, thus $\partial_x\theta(x) \propto \partial_t\varphi(x)$. In the following (also for the fermionic fields) we will always consider the relation (57) true.

To the purpose of doing calculations, it may be helpful to recast the fundamental equation (57) in a different form. In this case there are several possibility (with pros and cons) and I choose the following convention:

$$[\theta(x'), \varphi(x)] = -i\pi\Theta(x' - x)\,. \tag{58}$$

Here I am using the Heaviside step function and I define it in the following way:

$$\Theta(r) = \begin{cases} \Theta(r > 0) = 1 \\ \Theta(r = 0) = 0 \text{ for bosons}\,; \quad \Theta(r = 0) = 1/2 \text{ for fermions} \\ \Theta(r < 0) = 0 \end{cases} \tag{59}$$

So far we considered the slow-oscillating modes only, but the fast-oscillating term do not change the picture. We have indeed:

$$e^{i2p\theta(x')}e^{i\varphi(x)} = e^{i\varphi(x)}e^{i2p\theta(x')}e^{i2\pi p\Theta(x'-x)}\,; \tag{60}$$

this equation states that the fast-oscillating operators $e^{i2p\theta(x')}$ commute with $e^{i\varphi(x)}$ when $x \neq x'$. This is because, for $x \neq x'$, $\Theta = 0, 1$. For $x = x'$, to be rigorous, further regularization would be needed. The previous prescription for bosons, $\Theta(0) = 0$, provides the correct result. Hereafter we assume that the equations (55,57) hold.

Exercise IV.1 *Recall that the Campbell-Baker-Haussdorf formula reads:*

$$e^A e^B = e^{A+B+\frac{1}{2}[A,B]+\ldots} \tag{61}$$

where, in our case, the ... are never relevant because $[A, B]$ will always be a number. Using the CBH formula derive Eq. (60).

We are now in the position to rewrite our definition of $\psi_B^\dagger$ as a function of φ and θ. From the previous equations we get:

$$\psi_B^\dagger(x) = \sqrt{\rho_0 - \frac{\partial_x\theta(x')}{\pi}}\sqrt{\sum_p e^{i2p(\pi\rho_0 x - \theta(x))}}e^{-i\varphi(x)}\,. \tag{62}$$

To simplify we brutally exploit the fact that $\sqrt{\delta} = \delta$ and we remember that the $\sum_p$ was coming from the Poisson summation. Therefore it follows:

$$\psi_B^\dagger(x) = \sqrt{\rho_0 - \frac{\partial_x\theta(x')}{\pi}}\sum_p e^{i2p(\pi\rho_0 x - \theta(x))}e^{-i\varphi(x)}\,; \tag{63}$$

this is a sufficiently rigorous definition, based on all the commutation relations we derived so far, and it finally defines the bosonic field $\psi_B^\dagger$ in a fully bosonized form as a function of exponentials of φ and θ.

Very often one assumes that the fast oscillating modes with $p \neq 0$ are totally negligible with respect to the $p = 0$ contribution. This is an assumption dictated by renormalization group considerations and scaling dimensions, and it is usually very well justified. On a practical level, keeping into account the higher harmonics $p \neq 0$ allows for a better description of systems in which the free-particle dispersion is not linear and may account for interesting multi-particle processes. However these elements are usually negligible because, by averaging over space, the fast oscillating terms $e^{\pm 2ip\pi\rho_0 x}$ go to zero very fast. Therefore one usually considers only the $p = 0$ term.

Exercise IV.2 *There is a second brutal approximation, which is less rigorous (but it may still be justified by RG arguments): sometimes one can neglect the fluctuations of the density with respect to ρ_0: $\partial_x\theta \ll \rho_0$ in the prefactor of ψ_B. In this case $\psi_B^\dagger \approx \sqrt{\rho_0}e^{-i\varphi}$. Consider the Hamiltonian:*

$$H = \frac{\hbar^2}{2m}\int dx \left(\partial_x\psi_B^\dagger\right)\left(\partial_x\psi_B\right) + U\int dx\,(\rho(x))^2 \tag{64}$$

1. *By using this brutal approximation for ψ_B and Eq. (50) for ρ, express H as a function of φ and θ.*

2. *By using that $\partial_x\theta/\pi$ and φ are canonically conjugate fields, find the Lagrangian for φ and its equations of motion.*

Hint: the canonical conjugation is the relation Eq. (57). In this way you can obtain the canonical "momentum" operator Π such that $[\varphi(x),\Pi(x')] = i\hbar\delta(x-x')$. This allows you in turn to get the relation between Π and $\partial_t\varphi$ needed in the Legendre transformation to obtain the Lagrangian.

B. Bosonization of fermions

We want now to define fermionic operators ψ and $\psi^\dagger$ as a function of the fields φ and θ. To this purpose a useful choice is to maintain valid the equation (57), and modify the bosonic operators ψ_B and $\psi_B^\dagger$ to get fermionic rather than bosonic equal-time commutation relations:

$$\left\{\psi(x),\psi^\dagger(x')\right\} = \delta(x-x')\,. \tag{65}$$

The way we proceed is based on the Campbell-Baker-Hausdorff formula: in the bosonic case, in Eq. (60) we saw that the exponentials $e^{i2p\theta}$ entering the definition (63) of the bosonic field ψ_B commute with the operator $e^{i\varphi}$; this is because $2p$ is even. If we modify $2p \to 2p+1$, instead, we obtain, for $x' > x$ an anticommutation relation. We exploit this observation and we define the fermionic field in the following way:

$$\psi^\dagger(x) = \sqrt{\rho_0 - \frac{\partial_x\theta(x')}{\pi}}\sum_p e^{i(2p+1)(\pi\rho_0 x-\theta(x))}e^{-i\varphi(x)}\,. \tag{66}$$

For $x \neq x'$ we can evaluate:

$$\begin{aligned}\left\{\psi(x),\psi^\dagger(x')\right\} = {}& \sqrt{\rho_0 - \frac{\partial_x\theta(x')}{\pi}}\sum_p e^{i(2p+1)\left(\pi\rho_0 x'-\theta(x')\right)}e^{-i\varphi(x')}e^{i\varphi(x)}\sum_{p'} e^{i(2p'+1)(\pi\rho_0 x-\theta(x))}\sqrt{\rho_0 - \frac{\partial_x\theta(x)}{\pi}} + \\ & + e^{i\varphi(x)}\sum_{p'} e^{i(2p'+1)(\pi\rho_0 x-\theta(x))}\sqrt{\rho_0 - \frac{\partial_x\theta(x)}{\pi}}\sqrt{\rho_0 - \frac{\partial_x\theta(x')}{\pi}}\sum_p e^{i(2p+1)\left(\pi\rho_0 x'-\theta(x')\right)}e^{-i\varphi(x')}\,.\end{aligned} \tag{67}$$

To calculate this we consider (57) and the relation $[\varphi(x),\varphi(x')] = 0$; therefore, we can modify the first line in Eq. (67) by moving the operator $e^{i\varphi(x)}$ on the extreme left and the operator $e^{-i\varphi(x')}$ on the extreme right. These operators commute with each other and, furthermore $[e^{i\varphi(x)},\partial_x\theta(x')] = 0$ for $x \neq x'$ (from eq. (57)). Therefore, when we try to recast the first line of (67) in a form ordered as its second line, we acquire the following phases from the CBH formula:

$$e^{(2p+1)[\theta(x'),\varphi(x)]}e^{(2p'+1)[\varphi(x'),\theta(x)]} = e^{-i(2p+1)\pi\Theta(x'-x)+i(2p'+1)\Theta(x-x')} = -1\,, \quad \text{for } x \neq x'\,. \tag{68}$$

Here I applied equation (58), and the result is given for any p, p' integer. This demonstrates that indeed Eq. (65) holds for $x \neq x'$. Also in this case, the situation for $x = x'$ would require additional regularization, and, for the moment, we neglect this technical aspect. It is easy to see, though, that there is some contribution proportional to the $\delta(x-x')$ appearing due to $[\partial_x\theta,\varphi]$. From this, we conclude that the field (66) fulfills the commutation relation (65), and we rewrite it in the following form:

$$\psi^\dagger(x) = \sum_{p\geq 0}\sqrt{\rho_0 - \frac{\partial_x\theta(x)}{\pi}}e^{i(2p+1)(k_F x-\theta(x))}e^{-i\varphi(x)} + \sqrt{\rho_0 - \frac{\partial_x\theta(x)}{\pi}}e^{-i(2p+1)(k_F x-\theta(x))}e^{-i\varphi(x)} \tag{69}$$

where I am separating modes with a positive and negative momentum and I used $k_F = \pi\rho_0$.

At this point it is useful to introduce two approximations, similarly to what we did for the bosonic case:

1. We assume that the $p = 0$ terms in Eq. (69) dominate over all the others. This is justified based on renormalization group results we will study later. The main idea behind it is that the terms with $p \neq 0$ provide only minor (negligible) corrections to the correlation functions of the fields, thus, in the end, to the expectation value of any observable. To consider only the terms in $e^{\pm ik_F x}$ is essentially related to consider physical systems, as a function of the fields $\psi,\psi^\dagger$, in which the dispersion relation of the free fermions is well approximated by a linear behavior. In some special and not very common cases, one uses higher harmonics to include non-linear effects in the dispersion, or to model multi-particle scattering processes. From now on, however, we will always deal with the "standard" prescription that includes only the $e^{\pm ik_F x}$ contributions.

2. In the definition of these fields we consider the density fluctuation $\partial_x\theta/\pi$ negligible with respect to ρ_0.

Under these two approximations, I can slightly modify my former definition of the fermionic fields (I change some constant phases and the normalization) and I redefine:

$$\psi^\dagger(x) = N\left[e^{ik_F x}e^{-i(\varphi(x)+\theta(x))} + e^{-ik_F x}e^{-i(\varphi(x)-\theta(x))}\right]. \tag{70}$$

By comparing this definition with our results for the Dirac field, you realize that the first term in (70) is the creation operator for the left movers, whereas the second term refers to the right modes. In particular I define the so-called "vertex operators":

$$\psi_L^\dagger = Ne^{-i(\varphi(x)+\theta(x))}, \qquad \psi_R^\dagger = Ne^{-i(\varphi(x)-\theta(x))}. \tag{71}$$

About the normalization N, from the previous equation we expect $N \propto \sqrt{\rho_0} \propto \bar{d}^{-1/2}$ where $\bar{d}$ is the average particle separation. I observe that a better way of normalizing $\psi^\dagger$ is obtained by substituting $\bar{d}$ with the lattice spacing a, meant as the shortest length scale (UV cutoff). In any case, for reasonable systems $\bar{d}$ and a are of the same order of magnitude.

It is useful to observe that we defined $\psi^\dagger$ based on the commutation relations (65) and (57). Following our redefinition of the fields in Eq. (70), it is useful to address the form of the density operator, which allows us to fix the value of the normalization N. Let us consider $x > x'$ and define:

$$\rho(x) = \lim_{x\to x'} \psi^\dagger(x)\psi(x'). \tag{72}$$

This is a sort of "point-splitting" procedure which helps in regularizing the definition of the operator. We get:

$$\begin{aligned}\psi^\dagger(x)\psi(x') = N^2\Big[&e^{ik_F(x-x')}e^{-i(\varphi+\theta)(x)}e^{i(\varphi+\theta)(x')} + e^{-ik_F(x-x')}e^{-i(\varphi-\theta)(x)}e^{i(\varphi-\theta)(x')} + \\ &+ e^{ik_F(x+x')}e^{-i(\varphi+\theta)(x)}e^{i(\varphi-\theta)(x')} + e^{-ik_F(x+x')}e^{-i(\varphi-\theta)(x)}e^{i(\varphi+\theta)(x')}\Big],\end{aligned} \tag{73}$$

where the first line includes the slow-varying modes and the second the fast-oscillating terms (FO). Instead of taking the limit $x \to x'$, we consider a situation in which $x - x' = \bar{d}$. Our assumption is that the slow-varying fields do not change much on a length scale of $\bar{d}$ (or equivalently of a), therefore we can apply a first-order Taylor expansion of the fields:

$$\theta(x) \approx \theta(x') + \bar{d}\partial_x\theta(x'), \qquad \varphi(x) \approx \varphi(x') + \bar{d}\partial_x\varphi(x'). \tag{74}$$

From the previous equations we get:

$$\begin{aligned}\psi^\dagger(x)\psi(x') &\approx N^2\left[e^{ik_F\bar{d}-i(\partial_x\varphi+\partial_x\theta)\bar{d}+\frac{1}{2}[\theta(x),\varphi(x')]} + e^{-ik_F\bar{d}-i(\partial_x\varphi-\partial_x\theta)\bar{d}-\frac{1}{2}[\theta(x),\varphi(x')]} + \mathrm{FO}\right] \approx \\ &\approx N^2\left[-i\left(1+ik_F\bar{d}-i(\partial_x\varphi+\partial_x\theta)\bar{d}\right) + i\left(1-ik_F\bar{d}-i(\partial_x\varphi-\partial_x\theta)\bar{d}\right) + \mathrm{FO}\right] = 2N^2k_F\bar{d} - 2N^2\partial_x\theta\bar{d} + \mathrm{FO}.\end{aligned} \tag{75}$$

In conclusion we recover that, for $\bar{d}$ (or a) small enough, the fluctuations of the density operator are still proportional to $\partial_x\theta$. We fix $N^2 = 1/2\pi\bar{d}$ in such a way that we obtain:

$$\rho(x) = \frac{k_F}{\pi} - \frac{\partial_x\theta(x)}{\pi} + \mathrm{FO}; \tag{76}$$

this is consistent with Eqs. (49) and (50). An important thing to observe is that $N = \sqrt{1/2\pi\bar{d}}$, which means that $\psi^\dagger(x)$ scales with $\bar{d}^{-1/2}$ which is consistent with the correlation functions of the Dirac theory calculated at the end of Sec. II. This will be discussed better in the next section, after a proper analysis of the bosonic fields φ and θ.

Exercise IV.3 *Consider the definition of the density operator (76). Apply the CBH formula and verify that the leading contribution of the fast oscillating terms results:*

$$\mathrm{FO} = \frac{i}{2\pi\bar{d}}\left[e^{-2ik_Fx}e^{2i\theta(x)} - e^{2ik_Fx}e^{-2i\theta(x)}\right]. \tag{77}$$

For practical purposes, when dealing with lattice systems, we will typically substitute $\bar{d} \to a$ in the normalization of these fermionic fast-oscillating terms.

Exercise IV.4 *Consider the definition of the density operator (76) and its fast oscillating terms in Eq. (77).*

1. *Calculate the operator $\rho(x)\rho(x')$ with a suitable Taylor expansion to order $(\partial_x\theta)^2$. Consider small distances $x-x'\approx a$, such that you can Taylor expand the fields φ and θ at first order, and the exponential at second order to get the required terms in $(\partial_x\theta)^2$. In doing so, separate fast and slow oscillating parts.*
2. *What happens for $k_F=\pi/2a$?*

As a check: consider the density density interaction $H_{\rm int}=\tilde{U}\int dx\,\rho(x)\rho(x-a)$; for $k_F\neq\pi/2a$, the slow oscillating terms must yield:

$$H_{\rm int,slow}=\int dx\,\frac{\tilde{U}}{\pi^2}\left(1-\cos(2k_Fa)\right)\left(\partial_x\theta\right)^2+\frac{\tilde{U}}{a\pi^2}\sin(2k_Fa)\partial_x\theta-\frac{2\tilde{U}k_F}{\pi^2}\partial_x\theta \tag{78}$$

The second and third terms can be integrated out and give only a constant contribution (which is zero for systems with a conserved number of particles!). $\tilde{U}$ has units of energy times distance (differently from U in Eq. (21)).

V. FIELD-THEORETICAL (AXIOMATIC) BOSONIZATION

In this section I finally give you the main rules to translate fermionic into bosonic fields. These are justified by our phenomenological analysis of the previous section and here I will try to analyze on a more mathematical basis these rules and their implications. The main prescription to bosonize the (spinless) fermionic field is the one we derived in the previous section:

$$\psi^\dagger(x)=\frac{1}{\sqrt{2\pi a}}\left[e^{ik_Fx}e^{-i(\varphi(x)+\theta(x))}+e^{-ik_Fx}e^{-i(\varphi(x)-\theta(x))}\right], \tag{79}$$

$$\psi(x)=\frac{1}{\sqrt{2\pi a}}\left[e^{-ik_Fx}e^{i(\varphi(x)+\theta(x))}+e^{ik_Fx}e^{i(\varphi(x)-\theta(x))}\right], \tag{80}$$

where a labels the lattice spacing of the system (the inverse of the UV cutoff in the momenta). To be able to use this prescription, we need four fundamental rules (we already saw two of them, the other two will be derived in this section):

1. **CBH formula**:

$$e^Ae^B=e^{A+B+\frac{1}{2}[A,B]} \tag{81}$$

   In our case $[A,B]$ is always a number, therefore this is enough.

2. **Commutation relations** of the bosonic fields:

$$\left[\frac{\partial_x\theta(x')}{\pi},\varphi(x)\right]=-i\delta(x-x')\quad\Rightarrow\quad[\theta(x'),\varphi(x)]=-i\pi\Theta(x'-x) \tag{82}$$

   Keep in mind that (57) is more fundamental than (58); the latter is my arbitrary but useful choice.

3. **Correlation function of vertex operators**:

$$\begin{aligned}\left\langle e^{i\sum_j a_j\varphi(x_j)+i\sum_k b_k\theta(x_k)}\right\rangle&=\exp\left[-\frac{\sum_j a_j^2}{2}\left\langle\varphi(x_j)^2\right\rangle-\frac{\sum_k b_k^2}{2}\left\langle\theta(x_k)^2\right\rangle\right.\\&\left.-\sum_{j<j'}a_ja_{j'}\left\langle\varphi(x_j)\varphi(x_{j'})\right\rangle-\sum_{k<k'}b_kb_{k'}\left\langle\theta(x_k)\theta(x_{k'})\right\rangle\right]=\\&=\exp\left[-\frac{1}{2}\left\langle\left(\sum_j a_j\varphi(x_j)\right)^2\right\rangle-\frac{1}{2}\left\langle\left(\sum_k b_k\theta(x_k)\right)^2\right\rangle\right]\end{aligned} \tag{83}$$

4. **Correlation functions of the bosonic fields**:

$$\langle\varphi(x)\varphi(0)\rangle \approx -\frac{1}{2K}\ln x\,, \qquad \langle\theta(x)\theta(0)\rangle \approx -\frac{K}{2}\ln x\,. \tag{84}$$

These four equations are what you need to operate with bosonization on a practical level. In the next section we will extend them to the spinful case to really gain a complete set of tools. By looking at the third and fourth of these equations, you see that they are meant to calculate correlation functions. Thus they will allow you to predict the behavior of most of the observables. Here I introduced the so-called Luttinger parameter K (sometimes it is also called g) which, as we will see at the end of the section, is meant to codify the interactions amplitude.

A. From the lattice to bosonization

A standard situation which is often encountered is the bosonization of an interacting lattice model, as in the case of Eq. (21). In the lattice formulations, the hopping amplitudes J and U are energy scales, and the bosonization procedure must be consistent with their dimensionality. In particular, in going from the lattice to the continuum we may approximate:

$$-t\sum_r \left[c_r^\dagger c_{r+a} + \text{H.c.}\right] \to -t\int \frac{dx}{a}\left[c_x^\dagger c_{x+a} + \text{H.c.}\right] \to -t\int dx\left[\psi^\dagger(x)\psi(x+a) + \text{H.c.}\right]\,. \tag{85}$$

In the notation of Eqs. (79,80) the factor $1/a$ has already been included in the field definition. This results in $\psi^\dagger\psi$ being a density operator (not a number operator). Concerning the interactions, some additional care is required. Let us consider H_{int} in Eq. (21):

$$U\sum_r c_{r+a}^\dagger c_{r+a} c_r^\dagger c_r \to U\int \frac{dx}{a} c_{x+a}^\dagger c_{x+a} c_x^\dagger c_x \to \int dx\, aU\rho(x+a)\rho(x)\,; \tag{86}$$

where in the last step, we applied $c_x^\dagger c_x \to a\rho(x)$. The operator ρ, then, can be expressed based on Eq. (76).

In the next, we will discuss more in detail the properties of the bosonic fields θ and φ, and their correlations.

B. Massless Klein-Gordon fields

Our starting point is inspired by the results of the previous sections: in Sec. III we saw that the exact solution of the Luttinger model is given by a free bosonic problem with linear dispersion and from exercises IV.2 and IV.4 you should already be convinced that Hamiltonians including the terms $(\partial_x\varphi)^2$ and $(\partial_x\theta)^2$ are particularly important. Therefore we consider the following Lagrangian in Minkowski space:

$$\mathcal{L} = \frac{\hbar}{2\pi}\int dx\,\frac{(\partial_t\theta)^2}{u} - u(\partial_x\theta)^2\,. \tag{87}$$

From now on I set $\hbar = 1$. u is a velocity and you can consider it as the Fermi velocity of the Fermionic model you want to describe. From this Lagrangian we derive:

$$\partial_t\frac{\partial\mathcal{L}}{\partial(\partial_t\theta)} + \partial_x\frac{\partial\mathcal{L}}{\partial(\partial_x\theta)} = 0 \tag{88}$$

$$\Rightarrow \partial_t^2\theta - u^2\partial_x^2\theta = 0 \tag{89}$$

$$\Rightarrow (\partial_t - u\partial_x)(\partial_t + u\partial_x)\theta = 0 \tag{90}$$

Equation (89) is the massless Klein-Gordon equation and it obviously implies $E^2 = u^2p^2$ (as in the massless Dirac equation). It can be rewritten in the form (90) to emphasize that there will be a right and a left solution (as in the massless Dirac equation). From the previous Lagrangian we define the conjugate field:

$$\Pi = \frac{\partial\mathcal{L}}{\partial(\partial_t\theta)} = \frac{\hbar}{\pi u}\partial_t\theta. \tag{91}$$

The canonical quantization procedure implies:

$$[\Pi(x), \theta(x')] = \left[\frac{\hbar}{\pi u}\partial_t\theta(x), \theta(x')\right] = -i\hbar\delta(x-x')\,. \tag{92}$$

Now we define the dual field of θ, which not surprisingly is φ. In order to be the dual field of θ, φ must fulfill the following relations [they are valid for $K=1$ only! See Eq. (120) for the general form]:

$$\partial_t\theta = u\partial_x\varphi\,, \qquad \partial_x\theta = \frac{\partial_t\varphi}{u}\,. \tag{93}$$

Exercise V.1 *Verify that the requirements for the dual field are trivially compatible with the Klein-Gordon equation (89).*

Based on the equations (93), let us verify the chiral behavior of the vertex operators. Let us define right and left bosonic fields as appering in equation (79,80):

$$\varphi_R = \varphi - \theta\,, \qquad \varphi_L = \varphi + \theta\,. \tag{94}$$

From all the previous equation we derive:

$$(\partial_t + u\partial_x)\,\varphi_R = \partial_t\varphi - u\partial_x\theta - \partial_t\theta + u\partial_x\varphi = 0\,; \tag{95}$$

therefore we recover the result that φ_R depends on $ut-x$ only. In the same way one verifies that φ_L is a function of $ut+x$ only. Consistently with the Dirac field we conclude that right and left operators give rise to:

$$\psi^\dagger(x) = \frac{1}{\sqrt{2\pi a}}\left[e^{-ik_F x}e^{-i\varphi_R} + e^{ik_F x}e^{-i\varphi_L}\right]\,. \tag{96}$$

Let us proceed by defining the Hamiltonian of the system:

$$H = \int dx\Pi(x)(\partial_t\theta)(x) - \mathcal{L} = \frac{\hbar}{2\pi}\int dx\,\frac{(\partial_t\theta)^2}{u} + u(\partial_x\theta)^2 = \frac{\hbar u}{2\pi}\int dx\,(\partial_x\varphi)^2 + (\partial_x\theta)^2\,. \tag{97}$$

For $u = v_F$, this corresponds to the Dirac Hamiltonian. The duality relations between θ and φ allow to express this Hamiltonian as a function of φ only and also φ fulfills the Klein-Gordon equation. Therefore we obtain:

$$\partial_t^2\varphi - u^2\partial_x^2\varphi = 0 \quad \Rightarrow \quad \partial_t\partial_x\theta - \partial_x\partial_t\theta = 0 \quad \Rightarrow \quad \partial_t\rho(x) + \partial_x\frac{\partial_t\theta}{\pi} = 0\,. \tag{98}$$

The last equation becomes the continuity equation for the system if we impose the following definition of the current:

$$j = -\frac{\partial_t\theta}{\pi} = -u\frac{\partial_x\varphi}{\pi}\,. \tag{99}$$

C. Correlation functions

In order to estimate the expectation value of any observable, we need to calculate the correlation functions of the physical fermionic fields which, in turn, are determined by the correlation functions of the bosonic fields φ and θ. In the following we are going to derive the equations (83) and (84) which are necessary for this purpose.

For the calculation of the correlation functions we heavily rely on the fact that the free Lagrangian corresponds to a Gaussian partition function. Let us rewrite the Lagrangian in Euclidean time $\tau = it$, we get:

$$\mathcal{L} = \frac{1}{2\pi}\int dx\,\frac{(\partial_t\theta)^2}{u} - u(\partial_x\theta)^2 \quad \rightarrow \quad -\frac{1}{2\pi}\int dx\,\frac{(\partial_\tau\theta)^2}{u} + u(\partial_x\theta)^2\,. \tag{100}$$

We define the partition function as:

$$Z(0) = \int D_\theta \exp\left[-\frac{1}{2\pi}\int dk\,d\omega\;\theta(-k)\left(\frac{\omega^2}{u} + uk^2\right)\theta(k)\right]\,, \tag{101}$$

where we are using the short-hand notation $\theta(-k) = \theta(-\omega, -k)$, given by a Fourier transform on both time and space. It is convenient to define a generating function (see Sec. 3.2 of Flensberg's notes):

$$Z(\eta) = \int D_\theta \exp\left[- \int dk\, d\omega\, \frac{1}{2\pi} \theta(-k) \left(\frac{\omega^2}{u} + u k^2 \right) \theta(k) + \eta(-k)\theta(k) \right] , \tag{102}$$

again with the convention $\eta(-k) = \eta(-\omega, -k)$. The correlation function of the θ field in momentum space is given by:

$$\langle \theta(k)\theta(-k) \rangle = \lim_{\eta \to 0} \partial_{\eta(k)} \partial_{\eta(-k)} \frac{Z(\eta)}{Z(0)} . \tag{103}$$

Now let us exploit that Z is Gaussian and define the momentum space Green's function:

$$G(k) = \frac{\pi u}{\omega^2 + u^2 k^2} , \tag{104}$$

with $G(k) = G(-k)$. We get:

$$Z(\eta) = \int D_\theta \exp\left[-\frac{1}{2} \int dk\, d\omega \, \left(\frac{\theta(k)}{G^{1/2}(k)} + G^{1/2}(k)\eta(k) \right) \left(\frac{\theta(-k)}{G^{1/2}(k)} + G^{1/2}(k)\eta(-k) \right) + \frac{1}{2} \int dk\, d\omega\, \eta(k)\eta(-k)G(k) \right] . \tag{105}$$

Therefore, we get, as expected:

$$\langle \theta(k)\theta(-k) \rangle = G(k) . \tag{106}$$

Furthermore:

$$\frac{Z(\eta)}{Z(0)} = \exp\left[\frac{1}{2} \int dk\, d\omega\, \eta(k)\eta(-k)G(k) \right] . \tag{107}$$

With a Fourier transform we obtain:

$$\frac{Z\left(\eta\right)}{Z(0)} = \exp\left[\frac{1}{2} \int d^2x\, d^2x'\, \eta(x) G(x, x') \eta(x') \right] . \tag{108}$$

We calculate now the correlation function:

$$G(x, \tau, x', \tau') = \langle \theta(x, \tau)\theta(x', \tau') \rangle = \int \frac{dk\, d\omega}{4\pi^2} G(k) e^{i\omega(\tau - \tau') - ik(x - x')} = \int \frac{dk\, d\omega}{4\pi^2} \frac{\pi u}{\omega^2 + u^2 k^2} e^{i\omega(\tau - \tau') - ik(x - x')} =$$
$$= \int_0^\infty p dp \int_0^{2\pi} \frac{d\alpha}{4\pi} \frac{1}{p^2} e^{ipr\cos\alpha} = \int_0^\infty dp \frac{J_0(pr)}{2p} \tag{109}$$

where we defined $\vec{p} = (\omega/u, -k)$ and $\vec{r} = (u\left[\tau - \tau'\right], x - x')$ such that $r = \sqrt{(x - x')^2 + u^2(\tau - \tau')^2}$. One of the problems of massless Klein-Gordon fields is that this integral diverges for both $p \to 0$ and $p \to \infty$. Luckily we are dealing with the effective description of a model on a lattice, which provides us with reasonable and physical cut-offs $\Lambda_{\min} = 2\pi/L$ and $\Lambda_{\max} = 2\pi/a$. Therefore the previous expression becomes (where I take this approximations from Eq. 3.9 in [4]):

$$\int_{\Lambda_{\min}}^{\Lambda_{\max}} dp \frac{J_0(pr)}{2p} \approx -\frac{1}{4} \ln\left(\frac{r^2 + \Lambda_{\max}^{-2}}{\Lambda_{\min}^{-2}} \right) \approx -\frac{1}{2} \ln r + \frac{1}{2} \ln \Lambda_{\min}^{-1} . \tag{110}$$

All the approximations we are exploiting here are valid only for $a \ll r \ll L$, which is the regime of validity of our correlation functions. Totally different results would be obtained, for example, for small intervals $\Lambda_{\min} \lesssim \Lambda_{\max}$ (which is not an interesting regime now, but will be useful in renormalization group calculations). Eq. (110) becomes Eq. (84) at equal times; we can indeed conclude:

$$\langle \theta(x)\theta(x') \rangle \approx -\frac{1}{2} \ln|x - x'| + \text{const.} , \tag{111}$$

where the constant depends on the cutoffs, in particular on L. From the previous equations we also derive:

$$\left\langle \theta^2(x) \right\rangle = \frac{1}{2} \ln\left(\frac{\Lambda_{\max}}{\Lambda_{\min}} \right) . \tag{112}$$

Finally we note that the same relations apply to the field φ, since the Lagrangian is the same. In principle, there is another correlation one should check and results:

$$\langle \theta(x)\varphi(x')\rangle \approx -i\frac{\pi}{2}\Theta(x-x')\,; \tag{113}$$

this correlation does not carry much information but the usual commutation relation between θ and φ. For practical purposes one can ignore it after considering the right phases given by the CBH formula.

Let us proceed now to calculate Eq. (83), which can be easily derived from the equations we already wrote. In particular let us consider the correlations (108). In this equation we substitute $\eta(x) = i\sum_{k=1}^{M} b_k \delta(x-x_k)$ and we look at the definition of $Z(\eta)$; we get:

$$\left\langle e^{i\sum_k b_k\theta(x_k)}\right\rangle = e^{-\frac{1}{2}\left\langle\left(\sum_k b_k\theta(x_k)\right)^2\right\rangle} = e^{-\frac{1}{2}\sum_k b_k^2\langle\theta^2(x_k)\rangle - \sum_{k<k'} b_k b_{k'}\langle\theta(x_k)\theta(x'_k)\rangle}\,. \tag{114}$$

This demonstrates (83) for the part concerning θ. The same result is obtained for φ, thus leading to (83).

Let us consider in particular what happens with two vertex operators:

$$\left\langle e^{i\alpha\varphi(x_1)}e^{-i\beta\varphi(x_2)}\right\rangle = e^{\alpha\beta\langle\varphi(x_1)\varphi(x_2)\rangle - \frac{\alpha^2+\beta^2}{2}\langle\varphi^2\rangle} = e^{\alpha\beta\left(-\frac{1}{2}\ln|x_1-x_2|\right)+\text{const.}} = \frac{A}{|x_2-x_1|^{\alpha\beta/2}}\,; \tag{115}$$

this shows how the bosonization techniques allows me to obtain the space dependent behavior of the correlation functions, up to a non-universal constant (A in this case) which does not depend on the positions.

Now we can conclude by calculating the correlation functions of the fermionic fields; let us consider the following example:

$$\begin{aligned}\left\langle \psi_L^\dagger(x')\psi_L(x)\right\rangle = \frac{1}{2\pi a}\left\langle e^{-i(\varphi+\theta)(x')}e^{i(\varphi+\theta)(x)}\right\rangle = \frac{-i}{2\pi a}\left\langle e^{-i\left(\varphi(x')-\varphi(x)+\theta(x')-\theta(x)\right)}\right\rangle = \\ = \frac{-iA}{2\pi a}e^{\langle\varphi(x')\varphi(x)\rangle+\langle\theta(x')\theta(x)\rangle} = \frac{-iA}{2\pi a}\frac{1}{|x'-x|^{\frac{1}{2}+\frac{1}{2}}}\,.\end{aligned} \tag{116}$$

Here A is a non-universal constant, and we recovered the result (20), which is a good sanity check. The predictions of the correlation functions obtained by the bosonized fields match the Dirac theory in the free case.

Exercise V.2 *Based on the calculation of correlation functions (consider simply left-left and right-right correlations), verify that the approximation of neglecting $p \neq 0$ in Eq. (69) is reasonable when considering large enough $|x-x'|$. Observe also that the "average" correlation function $G(r) = \sum_x G(x, x+r)$ of fermionic observables in a translationally invariant system helps to erase the fast-oscillating terms.*

D. The Luttinger parameter

So far we considered the definition of the Hamiltonian of the system based on a non-interacting picture. The role of interactions, however, is easily encoded in a parameter K, called the Luttinger parameter. Consider the modified Hamiltonian:

$$H = \frac{\hbar v_F}{2\pi}\int dx\, K(\partial_x\varphi)^2 + \frac{(\partial_x\theta)^2}{K} \tag{117}$$

where v_F is the Fermi velocity of the interacting system. We can redefine:

$$\varphi' = \sqrt{K}\varphi\,, \qquad \theta' = \frac{\theta}{\sqrt{K}}\,. \tag{118}$$

In terms of these new fields, the previous Hamiltonian is mapped into a new non-interacting Hamiltonian:

$$H = \frac{\hbar v_F}{2\pi}\int dx\,(\partial_x\varphi')^2 + (\partial_x\theta')^2\,. \tag{119}$$

The transformation from φ, θ to φ', θ' implies that the commutation relations for these fields are not canonical any longer, and, for example, it modifies the duality relations:

$$\partial_t \theta = K v_F \partial_x \varphi \,, \qquad \partial_x \theta = K \frac{\partial_t \varphi}{v_F} \,. \tag{120}$$

Concerning density and current operators, consistently with the field construction (79,80), we have:

$$\rho = \rho_0 - \frac{\partial_x \theta}{\pi} = \rho_0 - \frac{K \partial_t \varphi}{\pi v_F} \,, \tag{121}$$

$$j = -\frac{\partial_t \theta}{\pi} = -K v_F \partial_x \varphi \,. \tag{122}$$

It is important to stress, however, that the commutation equation (57) remains invariant, and, based on it, one can derive the Lagrangian for φ or θ which is a non-interacting Lagrangian. Therefore we get:

$$\langle \varphi'(x)\varphi'(0)\rangle = -\frac{1}{2}\ln|x| = K \langle \varphi(x)\varphi(0)\rangle \,, \tag{123}$$

$$\langle \theta'(x)\theta'(0)\rangle = -\frac{1}{2}\ln|x| = \frac{1}{K} \langle \theta(x)\theta(0)\rangle \,. \tag{124}$$

From this equations we conclude:

$$\langle \varphi(x)\varphi(0)\rangle = -\frac{1}{2K}\ln|x| \,, \tag{125}$$

$$\langle \theta(x)\theta(0)\rangle = -\frac{K}{2}\ln|x| \,. \tag{126}$$

These trivial equations allows for the solution of the interacting problems: a system with $K \neq 1$ is intrinsically non-interacting and, despite that, we can represent it with quadratic Hamiltonians of the bosonic fields. The effects of interactions are mostly encoded in the K parameter. For instance, by using Eqs. (125,126) it is easy to find the dependence on K of the two-point correlation function (116). To understand better how the K parameters enters the description of the interacting systems, consider the following key exercise.

Exercise V.3 *Consider a one dimensional chain of fermions with generic $k_F \neq \pi/2a$ and a nearest-neighbor interaction as in Hamiltonian (21). Based on the standard bosonization prescription (79,80) and on the result (78) (thus only the slow oscillating part of the interaction), derive the slow-oscillating part of the Hamiltonian as a function of θ and φ, starting from the free Hamiltonian H_0 and adding the interaction. Use a second order Taylor expansion considering the lattice spacing a as a small parameter. Verify that you get:*

$$H = \int dx \, \frac{2ta \sin k_F a}{2\pi} \left[(\partial_x \varphi)^2 + (\partial_x \theta)^2\right] + \int dx \, \frac{\tilde{U}}{\pi^2} \left(1 - \cos(2k_F a)\right) (\partial_x \theta)^2 \,. \tag{127}$$

From this equation, calculate the value of K and the velocity u as a function of $\tilde{U} = Ua$ and v_F, where $v_F = (2ta/\hbar)\sin k_F a$ is the Fermi velocity of the non-interacting model H_0.

What is the additional (slow-oscillating) interaction term that appears for $k_F = \pi/2a$?

Let us now consider a superconducting system, to get a feeling about how to use all this machinary.

Exercise V.4 *Take the previous chain for $k_F \neq \pi/2a$. Let us suppose that our chain is put in proximity with a p-wave superconductor. Therefore we include an additional term in the tight-binding model:*

$$H_{\rm sc} = -\Delta \sum_r \left(c_r^\dagger c_{r+a}^\dagger + c_{r+a} c_r \right) \,, \tag{128}$$

with real $\Delta > 0$.

1. *Find the bosonized description of the previous term (as usual, consider the lattice spacing a to be a small parameter).*
2. *Based on Eq. (125), what is the scaling dimension of the operator you found in $H_{\rm sc}$ as a function of K? When is it relevant?*

3. *Consider a situation in which $H_{\rm sc}$ is relevant. Based on the bosonized description, what does it happen when Δ becomes large compared to the kinetic energy and interaction term (on a "semiclassical" level)? In particular, what happens to the field φ? How many semiclassical minima of $H_{\rm sc}$ as a function of φ are there?*

4. *What do you expect from correlations of the kind $\left\langle \psi_R^\dagger(x)\psi_L^\dagger(x+a)\psi_R(y)\psi_L(y+a)\right\rangle$ (consider only the dominant term in the limit $|x-y|\gg a$)?*

5. *Suppose now that the SC order parameter acquires a position dependent phase:*

$$H_{\rm sc} = -\sum_r \left(\Delta e^{i\phi r} c_r^\dagger c_{r+a}^\dagger + \Delta e^{-i\phi r} c_{r+a} c_r\right), \tag{129}$$

with $\Delta > 0$ and $\phi \ll 1/a$. What is the bosonized description of this interaction?

6. *If Δ is strong, what happens semiclassically to φ? Remember that the current is proportional to $j \propto \partial_x\varphi$. What do you conclude?*

7. *Imagine to progressively increase ϕ: what happens to the kinetic energy? What happens to the current?*

8. *Imagine that ϕ increases a lot, such that $v_F\phi^2 \gtrsim \Delta/a$. What happens to the system? What do you expect to see in the current?*

9. *If you consider that ϕ is proportional to a magnetic field $\phi \propto B$, can you relate the previous observations with a known SC effect (even though we are only in 1D, and we do not truly have long-range superconducting order)?*

Exercise V.5 *Let us consider the Hamiltonian:*

$$H = \frac{\hbar v}{2\pi}\int dx\, K(\partial_x\varphi)^2 + \frac{(\partial_x\theta)^2}{K} + A\cos(\alpha\theta) + B\cos(\beta\varphi) \tag{130}$$

1. *Determine for which values of the Luttinger parameter K the A and B terms are relevant/irrelevant in the RG sense. Hint: split the cosines into exponentials to evaluate their scaling dimension through their correlation functions.*

2. *This part is inspired by models with $\mathbb{Z}_p$ symmetry [5]. Consider $\alpha = \beta = \sqrt{2p}$ with $p \in \mathbb{N}$. For which values of p is it possible that there exist a K such that both the terms are irrelevant?*

3. *What are, in your opinion, the implications on the phase diagram of such a system as a function of p and K? Which gapped and gapless phases could you expect?*

Exercise V.6 *Calculate the Luttinger parameter K for the bosonic system in Ex. IV.2. For spinless bosonic systems in the non-interacting limit, how does K behave?*

Exercise V.7 *Several one-dimensional systems can be modelled as locally interacting bosons. These systems include ultracold bosonic atoms in 1D optical lattices or 1D arrays of superconducting islands connected by Josephson junctions, in which the Cooper pairs can be thought as bosons hopping on a discrete chain. To describe these systems, we may consider the following model for bosons on a chain [Bose-Hubbard model]:*

$$H = -t\sum_r \left[b_{r+a}^\dagger b_r + H.c.\right] + U\sum_r n_r^2\,. \tag{131}$$

Here the operators b and $b^\dagger$ are standard bosonic operators and $n_r = b_r^\dagger b_r$ measures the number of particles in

the site r. $U > 0$ represents a local repulsive interaction. In the following, let us assume that we can vary the density of the system $\rho_0 = N/L = \sum_r n_r/L$ as we wish.

1. *Concerning the kinetic term, bosonize it through the basic approximation $b_r \to \psi = \sqrt{\rho_0} e^{i\varphi}$ as in Ex. IV.2. Concerning the interaction, consider $n_r = a\rho$ and express the operator ρ via Eq. (49) by taking into account the harmonics $p = -1, 0, 1$ only. Write the bosonized Hamiltonian, including the fast-oscillating terms that you obtain in this way. You are supposed to find additional terms with respect to what you found in Ex. IV.2.*

2. *Analyze the behavior of these additional interactions as a function of ρ_0 and U. Focus on the ones potentially more relevant in the RG sense. Based on the result of Ex. V.6, do you think there are regimes in which these additional operators become relevant? For which values of ρ_0 and U do you expect they may give rise to phase transitions?*

If you arrived here, the description you found is a quite elegant way of studying the Mott - superfluid phase transitions in these systems.

VI. WHAT ABOUT THE SPIN?

So far we analyzed in detail the construction of fermionic operators for spinless fermions. Now we want to double our description to account for this degree of freedom and, as we will see, this requires the introduction of a final element, the set of so-called *Klein factors* κ_s.

Let us start by labelling with $s = \uparrow, \downarrow$ the spin degree of freedom. The natural way of extending our previous construction is to double the number of bosonic fields: therefore we consider now two pairs of dual fields φ_s, θ_s. We assume that fields with different spin commute with each other: essentially, what we are doing is to double the previous construction to two species that, at first, are not interacting:

$$\psi_s^\dagger(x) = \frac{1}{\sqrt{2\pi a}} \left[e^{ik_F x} e^{-i(\varphi_s(x)+\theta_s(x))} + e^{-ik_F x} e^{-i(\varphi_s(x)-\theta_s(x))} \right] , \tag{132}$$

$$\psi_s(x) = \frac{1}{\sqrt{2\pi a}} \left[e^{-ik_F x} e^{i(\varphi_s(x)+\theta_s(x))} + e^{ik_F x} e^{i(\varphi_s(x)-\theta_s(x))} \right] . \tag{133}$$

Here a particular care must be devoted to the definition of the Fermi momentum: if the two spin species have the same dispersion and are subject to the same chemical potential, the Fermi momentum is the same for both species and it amounts to $k_F = \pi\rho_0/2$ where, now, $\rho_0 = \rho_{0,\uparrow} + \rho_{0,\downarrow} = N/L$ where N is the *total* number of fermions and a factor $1/2$ has been introduced in k_F to account for spin degeneracy. In other situations (for example in the presence of spin-orbit coupling) the Fermi momenta may be different for the two species, and left and right movers may be translated to different positions of the Brillouin zone. Therefore, before starting a bosonization procedure with spinful fermions, it is always useful to check what are the values of the momenta which characterize the points in the Fermi surface for left and right movers of the two spin species. In full generality, therefore, one should consider a set of four Fermi momenta $k_{F,L/R,s}$ that must enter the previous definition. The commutation relations of the bosonic fields (82) are now modified in the following way:

$$\left[\frac{\partial_x \theta_{s'}(x')}{\pi}, \varphi_s(x) \right] = -i\delta(x-x')\delta_{ss'} \quad \Rightarrow \quad [\theta_{s'}(x'), \varphi_s(x)] = -i\delta_{ss'}\pi\Theta(x'-x) . \tag{134}$$

This poses a problem: when we consider fermionic fields belonging to different species, they commute instead of anticommuting. For example, $[\psi_\uparrow, \psi_\downarrow] = 0$, which is clearly wrong. We must correct this commutation relation. There are several techniques to do so: the easiest way to solve this problem is the introduction of a set of operators κ_s which are essentially "artificial" (meaning that they are mostly meant to keep track of the correct signs, but their physical counterpart is not very clear) and obey the rules of the Clifford algebra:

$$\{\kappa_s, \kappa_{s'}\} = 2\delta_{ss'} , \quad \kappa_s^2 = 1 , \qquad \kappa_s = \kappa_s^\dagger = \kappa_s^{-1} . \tag{135}$$

You may recognize that these operators behave like virtual Majorana modes. If you need to account for larger spins or many species, it is enough to extend the set of possible values of s by maintaining the previous rules. The correct

definition of the fermionic operators now reads:

$$\psi_s^\dagger(x) = \frac{\kappa_s}{\sqrt{2\pi a}}\left[e^{ik_F x}e^{-i(\varphi_s(x)+\theta_s(x))} + e^{-ik_F x}e^{-i(\varphi_s(x)-\theta_s(x))}\right], \tag{136}$$

$$\psi_s(x) = \frac{\kappa_s}{\sqrt{2\pi a}}\left[e^{-ik_F x}e^{i(\varphi_s(x)+\theta_s(x))} + e^{ik_F x}e^{i(\varphi_s(x)-\theta_s(x))}\right]. \tag{137}$$

Let me mention that other possibilities are known and a bit more physical to solve the commutation problem. The main one exploits the introduction of a non-local fermionic parity operator by substituting the Klein factors in the following way:

$$\kappa_\uparrow \to e^{i\pi N_\downarrow}, \qquad \kappa_\downarrow \to 1. \tag{138}$$

Here I am introducing the total number operator of down fermions $\pi N_\downarrow = \theta_\downarrow(-\infty) - \theta_\downarrow(+\infty) + \text{const.}$ which is consistent with the rule we adopted $\rho_\downarrow = \rho_0 - \partial_x\theta_\downarrow/\pi$. For finite size systems $N_\downarrow$ is indeed related to the difference of the boundary conditions of the $\theta_\downarrow$ field. The operator $e^{i\pi N_\downarrow}$ is nothing else that the fermionic parity of the down fermions, therefore it anticommutes with both $\psi_\downarrow$ and $\psi_\downarrow^\dagger$, thus solving the problem of the commutation relations. The price we pay is that $e^{i\pi N_\downarrow}$ is a non-local operator, and it may be inconvenient to use it for several applications. Depending on the system you may choose to adopt the more standard, but less intuitive Klein factors, or the introduction of this more physical fermionic parity.

The physics of spinful fermions is obviously very rich. Here we focus on the specific case in which there is a substantial symmetry between $\uparrow$ and $\downarrow$ particles, such that the system is invariant under this exchange. Therefore, in this case, the spin species share the same parameters K and v_F when we describe the problem without interactions between up and down states. When we include the interaction between up and down states, the Hamiltonian of the system usually assumes the general form:

$$H = \sum_s \frac{v_F}{2\pi}\int dx\,\left[K(\partial_x\varphi_s)^2 + \frac{(\partial_x\theta_s)^2}{K}\right] + \int dx\,\left[g_\varphi(\partial_x\varphi_\uparrow)(\partial_x\varphi_\downarrow) + g_\theta(\partial_x\theta_\uparrow)(\partial_x\theta_\downarrow)\right]. \tag{139}$$

Other interaction terms depending on the fields (rather than their derivatives) may appear. We focus, however, on this quadratic part of the Hamiltonian. To diagonalize this Hamiltonian we introduce the so-called charge (ρ) and spin (σ) fields:

$$\theta_\rho = \frac{\theta_\uparrow + \theta_\downarrow}{\sqrt{2}}, \quad \theta_\sigma = \frac{\theta_\uparrow - \theta_\downarrow}{\sqrt{2}}, \tag{140}$$

$$\varphi_\rho = \frac{\varphi_\uparrow + \varphi_\downarrow}{\sqrt{2}}, \quad \varphi_\sigma = \frac{\varphi_\uparrow - \varphi_\downarrow}{\sqrt{2}}. \tag{141}$$

This is essentially a canonical and unitary change of basis, which preserves the commutation relations.

Exercise VI.1 *Verify all the commutation relations for the spin and charge fields based on the previous equations.*

The meaning of these fields is to separate spin and charge degrees of freedom in such a way that: $\rho_\rho = -\partial_x\theta_\rho/\pi$ is the total density of the system, $j_\rho \propto -\partial_x\varphi_\rho/\pi$ is the total current; whereas $\partial_x\theta_\sigma$ and $\partial_x\varphi_\sigma$ are related to the spin density and current. In terms of the spin and charge degrees of freedom the Hamiltonian (139) assumes a diagonal form:

$$H = \sum_{a=\rho,\sigma}\frac{1}{2\pi}\int dx\,\left[u_a K_a(\partial_x\varphi_a)^2 + \frac{u_a}{K_a}(\partial_x\theta_a)^2\right], \tag{142}$$

with:

$$K_{\rho/\sigma} = \sqrt{\frac{v_F K^2 \pm \pi K g_\varphi}{v_F \pm \pi K g_\theta}}, \tag{143}$$

$$\frac{u_{\rho/\sigma}}{2\pi} = \sqrt{\left(\frac{v_F K}{2\pi} \pm \frac{g_\varphi}{2}\right)\left(\frac{v_F}{2\pi K} \pm \frac{g_\theta}{2}\right)}. \tag{144}$$

The second equation has a very important physical consequence: charge and spin degrees of freedom travel with different velocities! More in general there is a perfect *separation* between charge and spin degrees of freedom: the two

sectors are dictated by completely separated Hamiltonians. This demonstrates that, for interacting 1D system, the quasiparticle excitations do *not* carry both a charge and a spin. Rather, there are two different kinds of excitations for charge and spin. This is reflected also from the correlation functions of spin and density operators which decay in general in different ways.

Exercise VI.2 *Derive* (142),(143) *and* (144) *from* (139).

VII. XXZ MODEL AND JORDAN-WIGNER TRANSFORMATION

When we began definining the bosonization procedure, we started from 1D fermionic models, which are typically defined on a chain, and we set up their continuum limit through the Dirac model. In this section, we start again from a chain model, but, this time, we consider a spin-1/2 quantum chain. In particular we analyze the so called XXZ model, defined by the Hamiltonian:

$$H_{\mathrm{XXZ}} = -J_{xy} \sum_r \left(\sigma_{x,r}\sigma_{x,r+1} + \sigma_{y,r}\sigma_{y,r+1}\right) + J_z \sum_r \sigma_{z,r}\sigma_{z,r+1}\,, \tag{145}$$

where we consider the system in the thermodynamic limit. Here the spin operators $\sigma_{i,r}$ are Pauli operators acting on the site r. Differently from fermionic chains, all the operators in different sites always commute. The XXZ model describes several materials which behave, effectively, as 1D quantum antiferromagnets; for instance, neutron scattering in $SrCo_2V_2O_8$ displays the dispersion of many of the massive excitations in its antiferromagnetic phase [6].

Here we are interested in the theoretical basis of the study of this model. The objectives of the following analysis are the following:

- Learn that spin chains (with suitable conservation rules) are equivalent to fermionic 1D systems (Jordan-Wigner transformation). This will be very useful in the study of 1D fermionic systems with topological order.
- Understand that the sine-Gordon model is very general.
- Having a further example of the usefulness of renormalization group and scaling.

Most of the material in this section can be found in more detail in Chapter 6 of [1], and several other places.

A. How can we turn a spin chain into a fermionic model?

To begin our analysis of the Hamiltonian (145), let us first observe its symmetries. It is easy to see that the Hamiltonian commutes with the following string symmetry:

$$P_z = \prod_r \sigma_{z,r} = e^{i\pi \sum_r (\sigma_{z,r}-1)/2}\,; \tag{146}$$

additionally, H_{XXZ} also commutes with the analogous P_x and P_y operators. We observe that $P_i^2 = \mathbb{1}$, such that each of the P_i symmetries is a $\mathbb{Z}_2$ symmetry of the system.

It is a bit harder to show that the Hamiltonian also commutes with the operator:

$$Q_z = \sum_r \sigma_{z,r}\,, \qquad [H_{\mathrm{XXZ}}, Q_z] = 0\,. \tag{147}$$

To show the previous commutation relation let us observe that Q_z trivially commutes with the J_z term, and let us focus on the J_{xy} term. It is convenient to introduce the single spin operators:

$$\sigma^+ = \frac{1}{2}\left(\sigma_x + i\sigma_y\right) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\,, \tag{148}$$

$$\sigma^- = \frac{1}{2}\left(\sigma_x - i\sigma_y\right) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\,. \tag{149}$$

These operators are such that:

$$\left[\sigma_z, \sigma^{\pm}\right] = \pm 2\sigma^{\pm} \,. \tag{150}$$

Given these operators, we can rewrite:

$$\sigma_{x,r}\sigma_{x,r+1} + \sigma_{y,r}\sigma_{y,r+1} = 2\left(\sigma_r^+\sigma_{r+1}^- + \sigma_r^-\sigma_{r+1}^+\right) \,. \tag{151}$$

From the previous equations we derive:

$$\begin{aligned}\left[\sigma_{z,r} + \sigma_{z,r+1}, \sigma_{x,r}\sigma_{x,r+1} + \sigma_{y,r}\sigma_{y,r+1}\right] = 2\left[\sigma_{z,r} + \sigma_{z,r+1}, \sigma_r^+\sigma_{r+1}^- + \sigma_r^-\sigma_{r+1}^+\right] = \\ = 4\left(\sigma_r^+\sigma_{r+1}^- - \sigma_r^-\sigma_{r+1}^+ - \sigma_r^+\sigma_{r+1}^- + \sigma_r^-\sigma_{r+1}^+\right) = 0 \,,\end{aligned} \tag{152}$$

and this proves that $[Q_z, H_{\mathrm{XXZ}}] = 0$. Q_z can be considered the generator of the rotations of all the spins around the $\hat{z}$ direction, therefore we get that, in general, all the operators $e^{i\alpha Q_z}$ are symmetries of the system. This constitutes a U(1) family of global symmetries. In the limit $J_z = -J_{xy}$ we recover the Heisenberg model, which, instead, has a full SU(2) rotation symmetry.

It is interesting to observe that, for $|J_z| \gg |J_{xy}|$ we expect the system to behave as a (gapped) antiferromagnet (for $J_z > 0$) or ferromagnet (for $J_z < 0$), whereas the limit $|J_{xy}| \gg |J_z|$ is more complicated to deal with, and we will focus on this regime in the following. For simplicity we will also restrict to $J_z > 0$ and $J_{xy} > 0$; the sign of J_{xy}, however, is not influential because it can be flipped by a proper redefinition of the local basis of the even spins.

The Hamiltonian in terms of $\sigma^{\pm}$ and σ_z can be interpreted as a Hamiltonian of hard-core bosons, namely a chain of (interacting) bosons in 1D, with an infinite onsite repulsion, such that, in each site, there can be either 0 or 1 bosons. This mapping can simply be obtained from:

$$b_r = \sigma_r^+ \,, \quad b_r^\dagger = \sigma_r^- \,, \quad n_r \equiv b_r^\dagger b_r = \frac{1 - \sigma_{z,r}}{2} \,. \tag{153}$$

The hard-core boson constraint is implied by the relations $b^2 = b^{\dagger 2} = 0$. Hard-core boson operators commute if taken on different sites, but the study of the bosonic Hamiltonian may be difficult due to the fact that we have an infinite onsite repulsion. To overcome this problem the strategy is to map the XXZ model into a model of fermions instead.

The mapping is done by introducing fermionic spinless operators $c_r^\dagger$ and c_r such that:

$$\sigma_{z,r} = 1 - 2n_r = 1 - 2c_r^\dagger c_r = (-1)^{c_r^\dagger c_r}. \tag{154}$$

This means that $|\uparrow\rangle_r \to |0\rangle_r$ and $|\downarrow\rangle_r \to |1\rangle_r$ (this is the convention we choose here, the opposite convention can be used as well with a bit of care). In particular this also implies:

$$P_z = (-1)^{\sum_r c_r^\dagger c_r} \,, \tag{155}$$

thus P_z is the fermionic parity (which must be conserved); and:

$$Q_z = -2\sum_r c_r^\dagger c_r + L \tag{156}$$

which implies the conservation of the number of fermions. The main problem to be solved is the fact that $\{c_r, c_r'\} = 0$ whereas $[b_r, b_r'] = 0$. To solve this problem we introduce the so-called Jordan-Wigner transformation:

$$c_r^\dagger = \sigma_r^- \prod_{j<r} \sigma_{z,j} \,, \tag{157}$$

$$c_r = \sigma_r^+ \prod_{j<r} \sigma_{z,j} \,. \tag{158}$$

The string $\mathcal{L}_r = \prod_{j<r} \sigma_{z,j}$ is called Jordan-Wigner string and it enforces the correct anticommutation rules. Observe that $\mathcal{L}_r^2 = \mathbb{1}$. Crucially, the c and $c^\dagger$ operators are non-local in terms of the spin operators: this is a key ingredient to change their statistics from bosonic to fermionic. The Jordan-Wigner mapping, however, is a unitary mapping of the Hilbert space from spins to fermions, and thus it preserves the spectrum of the system.

Exercise VII.1 *We derive the properties of the Jordan-Wigner transformation.*

1. *Calculate* $\{\sigma^{\pm}, \sigma_z\}$.
2. *Using the previous relations, show that:*

$$\{c_{r_1}, c_{r_2}\} = \{c^\dagger_{r_1}, c^\dagger_{r_2}\} = \{c^\dagger_{r_1}, c_{r_2}\} = 0\,, \quad \textit{for} \quad r_1 > r_2\,. \tag{159}$$

3. *Using the previous relations, show that:*

$$\{c^\dagger_r, c_r\} = 1\,. \tag{160}$$

The previous commutation rules show that c_r and $c^\dagger_r$ are properly defined fermionic operators. It is also possible to derive the inverse transformation: from Eqs. (157) and (158) is easy to see that:

$$c^\dagger_r c_r = \sigma^- \sigma^+ = \frac{1 - \sigma_{z,r}}{2}\,, \tag{161}$$

consistently with Eq. (154). In particular we have:

$$\mathcal{L}_r = (-1)^{\sum_{j=1}^{r-1} c^\dagger_j c_j} \tag{162}$$

such that the Jordan-Wigner string simply accounts for the fermionic parity of the system from the first site to the site $r-1$ (here I relaxed the thermodynamic limit just to be more precise; the Jordan-Wigner transformation works straightforwardly with finite systems with open boundary conditions). From the previous relations we easily derive:

$$\sigma^-_r = (-1)^{\sum_{j=1}^{r-1} c^\dagger_j c_j} c^\dagger_r\,, \tag{163}$$

$$\sigma^+_r = (-1)^{\sum_{j=1}^{r-1} c^\dagger_j c_j} c_r\,. \tag{164}$$

Therefore we get:

$$\sigma^-_{r+1}\sigma^+_r = c^\dagger_{r+1} e^{i\pi c^\dagger_r c_r} c_r = c^\dagger_{r+1} c_r\,. \tag{165}$$

From Eqs. (154) and (165) we obtain:

$$H_{\mathrm{XXZ}} = -2J_{xy} \sum_r \left(c^\dagger_{r+1} c_r + c^\dagger_r c_{r+1} \right) + 4J_z \sum_r \left(c^\dagger_{r+1} c_{r+1} - 1 \right) \left(c^\dagger_r c_r - 1 \right)\,. \tag{166}$$

This Hamiltonian corresponds to the Hubbard model (1) with $t = 2J_{xy}$ and $U = 4J_z$ up to a shift of the chemical potential $\mu \to \mu - 8J_z$. We demonstrated that the XXZ model is thus equivalent to the Hubbard model through the non-local Jordan-Wigner mapping.

We observe that the Jordan-Wigner mapping is, in general, non local: all the spin operators which do not commute with Q_z become non-local fermionic operators (due to the presence of a Jordan-Wigner string). However, local operators which preserve Q_z are mapped into local operators that preserve the fermionic number (like in the case of Eq. (165)).

Once we understood that the XXZ model is nothing else than the Hubbard model, we are ready to bosonize it following our strategy of the previous sections, and, in particular, you should have already calculated everything in the Exercises IV.4 and V.3.

For the spin models that preserve Q_z and do not have an explicit magnetic field term coupled to σ_z, the magnetization of the ground state vanishes, such that $Q_z|\psi_{\mathrm{GS}}\rangle = 0$. This implies, from Eq. (156) that the system is at half-filling:

$$Q_z|\psi_{\mathrm{GS}}\rangle = 0 \implies \left(L - 2\sum_{r=1}^{L} c^\dagger_r c_r \right) |\psi_{\mathrm{GS}}\rangle = 0 \implies \frac{N}{L}|\psi_{\mathrm{GS}}\rangle = \frac{1}{2}|\psi_{\mathrm{GS}}\rangle \implies k_F = \frac{\pi}{2a}\,, \tag{167}$$

where $N = \sum_r c^\dagger_r c_r$ is the number of fermions in the system. Therefore, when analyzing this system it is crucial to consider the umklapp term of the fermionic interaction, that arises from the fast oscillating terms in the density operators with $k_F = \pi/2a$.

From the results of Ex. IV.3 and Ex. V.3, we get that the bosonized Hamiltonian for the XXZ model assumes the form:

$$H_{\mathrm{XXZ}} = \frac{1}{2\pi}\int dx\, \left[v_F K(\partial_x\varphi)^2 + \frac{v_F}{K}(\partial_x\theta)^2\right] + \frac{2J_z}{\pi^2 a^2}\int dx\, \cos(4\theta) \tag{168}$$

where the Luttinger parameter and the Fermi velocity can be approximated with the expression found in Ex. V.3. In particular:

$$K \approx \sqrt{\frac{\pi J_{xy}}{\pi J_{xy} + 4J_z}}\,. \tag{169}$$

Otherwise consider that the XXZ model is integrable and the exact values for K and v_F are known (see, for example, Chap. 6 of [1]):

$$\frac{J_z}{J_{xy}} = -\cos\frac{\pi}{2K}\,. \tag{170}$$

From the previous Hamiltonian, we understand that the sine-Gordon term is relevant when $4K < 2$. Thus we expect that:

- for $K > 1/2$, thus for small J_z, the interaction term is irrelevant and the system is gapless: we are in the Luttinger Liquid phase.
- for $K < 1/2$, thus for large enough J_z, the interaction term opens a gap: the system is in the antiferromagnetic phase. The field θ can be considered semiclassically pinned to one of the minima of $\cos(4\theta)$, such that $\theta(x)$ can be regarded as a constant, and the charge fluctuations $\partial_x\theta \sim 0$ are strongly suppressed.
- The phase transition at $K = 1/2$ is called Berezinskii-Kosterlitz-Thouless (BKT) phase transition. It corresponds to the Heisenberg symmetric point based on the exact solution (170). This is a very special critical point in which the divergence of the correlation length is not algebraic but exponential and it corresponds to an "infinite-order" phase transition. Due to the fact that the correlation length grows very fast when approaching the critical point, the gapped phase is a bit peculiar: all the correlation functions should decay exponentially in this phase but, sometimes, the correlation length is so large that could be measured only for irrealistically large systems. A similar issue may characterize the gapless phase, in which all the correlations decay algebraically. Mermin-Wagner theorem prevents this phase to be truly considered "ordered", however, for certain 2D classical ferromagnets, a finite magnetization is consistently measured and to verify the algebraic decay predicted by the Mermin-Wagner theorem one should consider a sample "bigger than the state of Texas" (Bramwell and Holdsworth, PRB 1994). The Mermin-Wagner theorem is definitely true, but, in many cases related to the universality class of the sine-Gordon model, it is simply not relevant. More information can be found in AS pages 471-474.

VIII. THE RENORMALIZATION GROUP ANALYSIS OF THE SINE-GORDON MODEL

A. General considerations about BKT transition

In the previous sections we obtained that both the fermionic spinless chain at half filling and the XXZ model fall in the universality class of the sine-Gordon model and, therefore, display a particular phase transition between a gapless and a gapped phase called Berezinskii-Kosterlitz-Thouless, which, as we will show in this section, can be considered an infinite-order phase transition.

The peculiarity of this phase transition is even more remarkable in 2D classical systems (remember that a 1+1 D quantum system can be mapped into a 2D classical system by considering their partition functions). When approaching these problems, we need to be careful: roughly speaking, there are two broad classes of problems that may manifest this transition, and, deep down, they correspond to the following:

1. Sine-Gordon term $\cos\beta\theta$ (scaling dimension $\beta^2 K/4$): in this case the ordered/gapped phase is the phase at low K and, if you consider the quadratic Hamiltonian written in terms of θ only, you may convince yourself that the classical 2D analog of such a situation has a temperature such that $T \propto K$. In these cases the low-K ordered and gapped regime corresponds to low temperatures. θ here is considered as a phase and the ordered phase is the one without vortices.

2. Sine-Gordon term $\cos\beta\varphi$ (scaling dimension $\beta^2/4K$): this case is the dual of the previous. Here the phase at large K is the gapped phase. Very often this "duality" means that the operator $e^{-i2\varphi}$ is an operator that creates vortices [the commutations are such that $e^{-2i\varphi}$ causes a jump by 2π to the field θ; in 2 classical dimensions this 2π-jump is a vortex]. This is a common situation in which θ is the superfluid phase or the direction of the magnetization for the XY model, and φ is a "disorder" operator: when $\cos\beta\varphi$ dominates, this means that the system has deconfined vortices that can freely nucleate and move. In this case the gapped phase is thus the disordered phase at high temperature (and it is still true that $K \propto T$). This second scenario is indeed the one typical of the main physical examples of the BKT transition, namely the 2D ^{4}He superfluid and the classical 2D XY model, which were the models studied by Berezinskii, Kosterlitz and Thouless.

Therefore, for every system, you must understand whether you are in the first or second scenario to define what is "ordered", "disordered", gapped or gapless (which may be a bit confusing). The 2D classical XY and superfluid models (and, in general, the second kind of sine-Gordon models) are the ones in which the peculiarity of the BKT phase transitions are more evident: at low temperature the system is gapless but quasi-ordered, namely the magnetization of the XY dipoles is locally oriented in the same direction, with the possibility, however, of having gapless excitations (similar to spin-waves) which destroy (as a power law) the long-range order, consistently with Mermin-Wagner theorem. The high temperature phase, instead, is totally disordered, despite being gapped. In this case, it is evident that no spontaneous symmetry breaking is involved in this phase transition (neither phase is ordered) and, strictly speaking, there is no local order parameter (although, for any finite system size, typically the magnetization on the plane distinguishes the low-temperature quasi-ordered phase). The lack of spontaneous symmetry breaking and local order parameter makes the BKT phase transition in the classical XY model the first example of a topological phase transition, in which the order parameter is associated with non-local objects, which, in this case, correspond to the expectation value of the winding number of the magnetization in closed loops, which reveals the behavior of the vortices in the system.

Historically, this kind of universality class has indeed been introduced to study the behavior of vortices in two-dimensional ^{4}He superfluids. Then it has been extended to study particular thin 2D superconductors (especially granular superconductors or Josephson junction arrays). It is important to notice that the standard type 2 superconductors do not belong to this universality class due to the role of Coulomb interaction: the Anderson mechanism gaps the Goldstone mode due to the Coulomb interaction, thus driving the system away from the massless Gaussian model, independently on the sine-Gordon term.

These models can be approximated with two-dimensional classical models. At low temperature these systems are described by assuming a substantial order of the superfluid/superconducting phase, which means that the creation of vortices is suppressed, and when vortices nucleate as thermal excitations, they are bound in a vortex-antivortex pair. At high temperatures, instead, when the vortices are nucleated they can almost freely propagate (we could say that the vortices "condense" since $\langle e^{-i2\varphi}\rangle \neq 0$), thus the phase becomes disordered, and its correlation function decay exponentially. This roughly describes superfluids, or superconducting systems where the Coulomb interaction can be neglected, thus BKT appears as a transition between a disordered and gapped phase at high T and a (quasi-)ordered low T phase.

To roughly sketch the physics of the 2D classical models displaying this kind of transition, let us consider the classical XY model. The XY model is a classical planar Heisenberg model that describes ferromagnets in 2D. We define it on a square lattice and, on each site, we consider a continuous degree of freedom corresponding to a unitary vector in 2D: $\vec{S}_{\vec{r}} = (\cos\theta_{\vec{r}}, \sin\theta_{\vec{r}})$. This vector may represent a spin polarization or a superfluid phase.

The corresponding Hamiltonian, in the presence of a magnetic field, reads:

$$H = -J_{xy} \sum_{\langle \vec{r},\vec{r}' \rangle} \vec{S}_{\vec{r}} \cdot \vec{S}'_{\vec{r}'} \tag{171}$$

With the introduction of the variables θ we get:

$$H = -J_{xy} \sum_{\langle \vec{r},\vec{r}' \rangle} \cos\left(\theta_{\vec{r}} - \theta_{\vec{r}'}\right) . \tag{172}$$

Going in the continuum, and assuming that the field θ varies slowly with respect to the lattice spacing, we can approximate the Hamiltonian as:

$$H = \int d^2r \, J_{xy} \left[(\partial_x\theta)^2 + (\partial_y\theta)^2 \right] . \tag{173}$$

Roughly speaking, the gradient along $\hat{y}$ gives rise to the $(\partial_t\theta)^2$ going to the quantum 1+1D description with $J_{xy}/T \propto 1/K$. To obtain the mapping into the sine-Gordon model, however, it is necessary to consider that θ is defined only

modulo 2π and, in 2D, vortex configurations may appear. To account for this, the operators $e^{i\varphi}$ and $e^{-i\varphi}$ must be introduced and, the correct way of doing so, is based on considering first an operator that defines the density of vortices, then introducing the logarithmic interaction between vortices, and finally getting read of this interaction with a kind of Hubbard-Stratonovich description based on the field φ. The whole process is a bit complicated and it requires some clever manipulation of the partition function of the model. You may gain some idea about it by looking at Chapter 8.6 of AS.

Another realization of the sine-Gordon model is related to the one dimensional arrays of superconducting islands with Josephson junctions connecting them. Such models are typically represented by Hamiltonians of the kind:

$$H_{\rm JJ} = \sum_{i\in \text{ SC islands}} E_c \left(Q_i + q_{\rm ind}\right)^2 + \sum_{\langle i,j\rangle} \frac{1}{C}\left(Q_i + q_{\rm ind}\right)\left(Q_j + q_{\rm ind}\right) - E_J \sum_{\langle i,j\rangle} \cos\left(\varphi_i - \varphi_j\right) , \tag{174}$$

where φ_i is the superconducting phase of the SC island i, Q_i is the charge operator associated to the island i. The first term in the Hamiltonian is the charging energy of a single island, the second is the cross-capacitance term between neighboring islands, and the last is the Josephson coupling between neighboring island. Since $[Q_j, \varphi_j] = i$, from a comparison with Eq. (55), Q can be considered as proportional to the density ρ and also this system can be mapped into a sine-Gordon model (with $\cos(2p\theta)$ interactions) when modelling Q through Eq. (49) and considering an integer average density (see also Ex. V.7). In this case, the low K gapped regime corresponds to Mott insulators ($E_J \ll E_C$) and the high K regime is a Luttinger liquid modeling the superconducting phase.

B. RG analysis of the sine-Gordon model

In the following, we want to study more in detail the phase diagram of the quantum sine-Gordon model. The starting point is the action in Euclidean time:

$$S = \frac{1}{2\pi}\int d^2r \left[\frac{K}{u}\left(\partial_\tau\varphi\right)^2 + Ku\left(\partial_x\varphi\right)^2\right] + \int d^2r\, g\cos\left(\beta\varphi\right) . \tag{175}$$

In this case I have chosen the sine-Gordon term associated to the field φ. Similar results would be obtained by considering a $\cos\alpha\theta$ term, with the only difference given by $K \to K^{-1}$. In full generality you will have both terms and, to start your RG analysis, you should consider the most relevant.

Considering the action (175) we observe that the scaling dimension of $\cos\beta\varphi$ is:

$$D_g = \frac{\beta^2}{4K} . \tag{176}$$

Therefore the term is relevant when $\frac{\beta^2}{4K} < 2$, and we expect to enter in a gapped phase for large values of K. This scaling prediction will be made more rigorous by analyzing the RG flow. The objective of the RG analysis is to determine the flow of the K, v and g parameters, and we will do it by considering the quadratic part of the Hamiltonian as the unperturbed Hamiltonian, and the cosine potential as a "small" perturbation we will treat at **second order**. Ideally this implies that our analysis is valid until $|g| < \Omega$ where Ω is a threshold beyond which our model is no longer valid. Typically $\Omega \approx 2t$ is set by the bandwidth of a system or another energy scale associated to the kinetic energy in the case of the bosonization of fermionic systems. In the following I will denote by S_0 the first term in (175) and by S_I the cosine interaction term.

The idea behind the Wilsonian renormalization group is to consider a cutoff in momentum Λ and a small scale parameter l such that we can rescale the cutoff to a smaller value $\tilde{\Lambda} = \Lambda e^{-l}$ corresponding to $\tilde{\Lambda}/\Lambda \approx 1 - dl$. Following the standard approach we can split the field φ into a "slow" and a "fast" component. The first includes all the momenta smaller than $\tilde{\Lambda}$, the second the momenta included in the shell $\tilde{\Lambda} < k < \Lambda$:

$$\varphi(x,t) = \varphi_{\mathsf{s}}(x,t) + \varphi_{\mathsf{f}}(x,t) . \tag{177}$$

To understand the renormalization flow, we must derive an effective action for the slow modes only, by averaging over the fast modes. One obtains:

$$S_{\rm eff}(\tilde{\Lambda}) = S_0(\varphi_{\mathsf{s}}) - \ln\left\langle e^{-S_I(\varphi_{\mathsf{s}}+\varphi_{\mathsf{f}})}\right\rangle_{\mathsf{f}} \approx S_0(\varphi_{\mathsf{s}}) + \underbrace{\langle S_I(\varphi_{\mathsf{s}}+\varphi_{\mathsf{f}})\rangle_{\mathsf{f}}}_{\mathcal{A}} - \frac{1}{2}\left(\underbrace{\left\langle S_I^2(\varphi_{\mathsf{s}}+\varphi_{\mathsf{f}})\right\rangle_{\mathsf{f}}}_{\mathcal{B}} - \underbrace{\langle S_I(\varphi_{\mathsf{s}}+\varphi_{\mathsf{f}})\rangle_{\mathsf{f}}^2}_{\mathcal{A}^2}\right) + \dots \tag{178}$$

In this expression, the average values are taken integrating over the fast modes. In the following we will evaluate the values of $\mathcal{A}$ and $\mathcal{B}$ to obtain S_{eff}.

It is useful to consider the following relations:

$$\left\langle \varphi_{\mathsf{f}}^2(x)\right\rangle_{\mathsf{f}} = \int_{\tilde{\Lambda}<k<\Lambda} \frac{d^2k}{4\pi}\frac{1}{Kk^2} = \int_{\tilde{\Lambda}}^{\Lambda}\frac{dk}{2}\frac{1}{Kk} = \frac{1}{2K}\ln\frac{\Lambda}{\tilde{\Lambda}}\,, \tag{179}$$

$$\left\langle \varphi_{\mathsf{f}}(x,t)\varphi_{\mathsf{f}}(x',t')\right\rangle_{\mathsf{f}} = \int_{\tilde{\Lambda}}^{\Lambda} dk \frac{J_0(kr)}{2Kk} \approx \frac{C(r)}{2K}\ln\frac{\Lambda}{\tilde{\Lambda}} \tag{180}$$

where the logarithm captures the scaling behavior, and $C(r)$ is a short-range function of $r=\sqrt{v^2(t-t')^2+(x-x')^2}$ (to be more precise, in this case with a sharp cutoff, $C\approx J_0(\Lambda r)$ and it is not really short-ranged, but it can be made short-ranged with better cutoffs, as shown in Appendix A). Relation (180) is analogous to (109) but in the limit $\tilde{\Lambda}\approx\Lambda$. Here and in the following t and t' are Euclidean times. In particular you can imagine C as a peaked function (see Fig. 2) defining a typical small length scale that I will call α in the following. For practical purposes we can set $\alpha = a$ in the following.

1. *First-order terms*

Let us now calculate the first-order term $\mathcal{A}$:

$$\mathcal{A} = \frac{g}{2}\left\langle e^{i\beta(\varphi_{\mathsf{s}}+\varphi_{\mathsf{f}})}+e^{-i\beta(\varphi_{\mathsf{s}}+\varphi_{\mathsf{f}})}\right\rangle_{\mathsf{f}} = \frac{g}{2}\left[e^{i\beta\varphi_{\mathsf{s}}}\left\langle e^{i\beta\varphi_{\mathsf{f}}}\right\rangle_{\mathsf{f}} + e^{-i\beta\varphi_{\mathsf{s}}}\left\langle e^{-i\beta\varphi_{\mathsf{f}}}\right\rangle_{\mathsf{f}}\right] = \frac{g}{2}\left[e^{i\beta\varphi_{\mathsf{s}}}e^{-\frac{\beta^2}{2}\left\langle\varphi_{\mathsf{f}}^2\right\rangle_{\mathsf{f}}} + e^{-i\beta\varphi_{\mathsf{s}}}e^{-\frac{\beta^2}{2}\left\langle\varphi_{\mathsf{f}}^2\right\rangle_{\mathsf{f}}}\right] =$$

$$= g\cos\left(\beta\varphi_{\mathsf{s}}\right)e^{-\frac{\beta^2}{2}\frac{1}{2K}\ln\frac{\Lambda}{\tilde{\Lambda}}} = g\cos\left(\beta\varphi_{\mathsf{s}}\right)\left(\frac{\tilde{\Lambda}}{\Lambda}\right)^{\frac{\beta^2}{4K}} = g\cos\left(\beta\varphi_{\mathsf{s}}\right)\left(1-\frac{\beta^2}{4K}dl\right)\,. \tag{181}$$

We thus obtain the first order contribution:

$$\mathcal{A} = \int d^2x\, g\cos\left(\beta\varphi_{\mathsf{s}}(x)\right)\left(1-\frac{\beta^2}{4K}dl\right) = \int d^2x'\left[1+\left(2-\frac{\beta^2}{4K}\right)dl\right] g\cos\left(\beta\varphi'_{\mathsf{s}}(x')\right)\,. \tag{182}$$

Here I am using that, in 1+1D, $\varphi(x)=\varphi'(x')$ and $d^2x = d^2x'(1+2dl)$. The scaling dimension $D_g=\frac{\beta^2}{4K}$ appears naturally in the first order term. The first order RG equation for g indeed reads:

$$\frac{dg}{dl} = \left(2-\frac{\beta^2}{4K}\right)g\,. \tag{183}$$

This confirms that the cosine interaction is relevant for $D_g<2$ and, indeed, at first order we find:

$$g(l) = g_0 e^{(2-D_g)l}\,; \tag{184}$$

here g_0 is the bare value of g in the initial Hamiltonian.

2. *About the gap*

The bosonization procedure works only for energy scales below a certain threshold, which we may consider to be the bandwidth of the system $\Delta^* = 4t$ (or some fraction of it). As a consequence, we must consider our RG flow only for values of the coupling constants below this threshold, and we may define a maximum l^* for the flow parameter such that:

$$g(l^*) = g_0 e^{(2-D_g)l*} = \Delta^*/a\,. \tag{185}$$

When g reaches the threshold Δ^*, we may safely assume that it drove the system away from the Luttinger liquid critical point (the length scale a here appears because we defined a as an energy density). l^* plays the role of the end of our RG flow.

If we want to give a brutal estimate of the gap of the system, we must consider that an energy gap always evolve like the inverse of a length. Thus we have $\Delta(l) = e^l \Delta_0$, where Δ_0 can be considered as the actual gap of the system in the original length scale. From this we derive:

$$\Delta_0 = e^{-l^*}\Delta^* = \left(\frac{a g_0}{\Delta^*}\right)^{\frac{1}{2-\frac{\beta^2}{4K}}} \Delta^* . \tag{186}$$

In particular, for the free Dirac Hamiltonian with a mass $m = g_0$, we have $\beta = 2$ and $K = 1$ which gives $\Delta_0 = g_0 = m$, consistently with the gap in the free model.

The equation of the gap is very interesting because it displays one of the peculiarities of the BKT transitions. Differently from "standard" phase transition the behavior of the gap is continuous and all its derivatives are continuous as well and approach zero when the gap closes. To see this clearly let us consider the parameter $x = 2 - \frac{\beta^2}{4K}$, such that $x > 0$ in the gapped phase and $x = 0$ at the transition. The gap is proportional to $e^{-\lambda/x}$ for $x > 0$ and vanishes for $x < 0$ ($\lambda = -\ln(g_0/\Delta^*) > 0$). In the limit $x \to 0^+$ all the derivatives of the gap tend to zero and the gap is not an analytic function. This hints to the fact that the ground state energy of the system is continuous across this phase transition and all its derivatives are as well. One could therefore consider the BKT phase transition as an "infinite-order" phase transition, which is usually hard to detect because of this continuity. The behavior of the correlation length ξ on the gapped phase is exactly the opposite of the gap: ξ diverges at the critical point for $x \to 0^+$ as $e^{\lambda/x}$, thus faster than any power.

3. *Second order*

By taking the square of the first-order result we get:

$$\mathcal{A}^2 = \int d^2x_1' d^2x_2' \left[1 + \left(4 - 2\frac{\beta^2}{4K}\right) dl\right] g^2 \cos\left(\beta\varphi_{\mathsf{s}}'(x_1')\right) \cos\left(\beta\varphi_{\mathsf{s}}'(x_2')\right) . \tag{187}$$

$\mathcal{A}^2$ will cancel several terms in $\mathcal{B}$. We must calculate:

$$\begin{aligned}\mathcal{B} = \left\langle S_I^2 \right\rangle_{\mathsf{f}} = \frac{g^2}{4} \left\langle \int d^2x_1 d^2x_2 \sum_{\mu,\nu=\pm1} \left[e^{i\mu\beta(\varphi_{\mathsf{s}}(x_1)+\varphi_{\mathsf{f}}(x_1))} e^{i\nu\beta(\varphi_{\mathsf{s}}(x_2)+\varphi_{\mathsf{f}}(x_2))} \right] \right\rangle_{\mathsf{f}} = \\ = \frac{g^2}{4} \int d^2x_1 d^2x_2 \sum_{\mu,\nu=\pm1} e^{i\beta(\mu\varphi_{\mathsf{s}}(x_1)+\nu\varphi_{\mathsf{s}}(x_2))} \left\langle e^{i\beta(\mu\varphi_{\mathsf{f}}(x_1)+\nu\varphi_{\mathsf{f}}(x_2))} \right\rangle_{\mathsf{f}} ;\end{aligned} \tag{188}$$

we have:

$$\left\langle e^{i\beta(\mu\varphi_{\mathsf{f}}(x_1)+\nu\varphi_{\mathsf{f}}(x_2))} \right\rangle_{\mathsf{f}} = e^{-2\frac{\beta^2}{2}\left\langle\varphi_{\mathsf{f}}^2\right\rangle_{\mathsf{f}}} e^{-\beta^2\mu\nu\langle\varphi_{\mathsf{f}}(x_1)\varphi_{\mathsf{f}}(x_2)\rangle_{\mathsf{f}}} = e^{-\beta^2\frac{1}{2K}\ln\frac{\Lambda}{\tilde\Lambda}} e^{-\beta^2\mu\nu\frac{1}{2K}\ln\frac{\Lambda}{\tilde\Lambda}C(x_1-x_2)} , \tag{189}$$

where we used Eqs. (179,180). We get:

$$\mathcal{B} = \frac{g^2}{4}\left(\frac{\tilde\Lambda}{\Lambda}\right)^{\frac{\beta^2}{2K}} \int d^2x_1 d^2x_2 \sum_{\mu,\nu=\pm1} \left(\frac{\tilde\Lambda}{\Lambda}\right)^{\frac{\beta^2\mu\nu}{2K}C(x_1-x_2)} e^{i\beta(\mu\varphi_{\mathsf{s}}(x_1)+\nu\varphi_{\mathsf{s}}(x_2))} . \tag{190}$$

By substituting $\tilde\Lambda/\Lambda = 1 - dl$ and $d^2x = (1+2dl)d^2x'$ we get:

$$\mathcal{B} = \frac{g^2}{4}\left[1 + \left(4 - \frac{\beta^2}{2K}\right) dl\right] \int d^2x_1' d^2x_2' \sum_{\mu,\nu=\pm1} \left(1 - \frac{\beta^2\mu\nu}{2K} C(x_1'-x_2')dl\right) e^{i\beta\left(\mu\varphi_{\mathsf{s}}'(x_1')+\nu\varphi_{\mathsf{s}}'(x_2')\right)} , \tag{191}$$

where, as usual, we disregard terms of order dl^2. We observe that the terms independent on C match the terms in $\mathcal{A}^2$. We get:

$$\frac{\mathcal{A}^2 - \mathcal{B}}{2} = \frac{g^2 dl}{8} \int d^2x_1' d^2x_2' \sum_{\mu,\nu=\pm1} \frac{\beta^2\mu\nu}{2K} C(x_1'-x_2') e^{i\beta\left(\mu\varphi_{\mathsf{s}}'(x_1')+\nu\varphi_{\mathsf{s}}'(x_2')\right)} , \tag{192}$$

Here we proceed by splitting the two terms with $\mu = \nu$ from the two terms with $\mu = -\nu$.

4. The term $\mu = \nu$

Let us analyze first the terms with $\mu = \nu$. In this case we get the following integral :

$$\frac{\beta^2 g^2 dl}{8K} \int d^2x_1' d^2x_2' C(x_1' - x_2') \cos\left(\beta\varphi_s'(x_1') + \beta\varphi_s'(x_2')\right) \approx \frac{\gamma\beta^2 g^2 dl}{8K} \int d^2x' \cos 2\beta\varphi_s'(x') \,. \tag{193}$$

The approximation we did is justified by the fact that C can be made very localized (with suitable cutoffs better than the sharp one we are using, see App. A) and, in this situation, the dominant and most relevant part of this term is given by the substitution $C(x_1' - x_2') \approx \gamma\delta(x_1' - x_2')$, with $\gamma \approx \alpha^2/u$. The previous term tells us that, at second order, the operator $\cos 2\beta\varphi$ emerges. This operator, however, commutes with S_I and it is less relevant than S_I, therefore we can neglect it in the RG flow. For completeness, however, we should include a new term in S_I of the form:

$$S_I' = S_I + \int d^2x h \cos 2\beta\varphi \,, \tag{194}$$

with a second order RG equation dictated by:

$$\frac{dh}{dl} = (2 - D_h)\, h + \frac{\beta^2 g^2 \gamma}{8K} \tag{195}$$

where $D_h = \beta^2/K$ is the scaling dimension of the new (less relevant) operator and the second term is obtained from (193). The initial condition of the flow must be $h(l = 0) = 0$ since h does not appear in the original action.

5. The term $\mu = -\nu$

Let us continue our analysis with the last term $\mu = -\nu$ in Eq. (192). In this case, if we simply take $C(x_1' - x_2') \approx \gamma\delta(x_1' - x_2')$ we do not get anything. The most relevant term must therefore be obtained by considering a better expansion for the localized function C. The way of dealing with this is to change variables from x_1', x_2' to center of mass and relative coordinate:

$$x_R' = \frac{x_1' + x_2'}{2} \,, \quad x_r' = x_1 - x_2 \,. \tag{196}$$

The correlation function C depends only on $|x_r|$ and we may assume that it is non-negligible only in a neighborhood of size α around $|x_r| = 0$ (see App. A). We observe that α is in general a non-universal quantity of the system and depends on the microscopic behavior of the model, although we can set $\alpha = a$ for practical purposes. Based on this observation, and assuming that α is small, we can approximate:

$$\begin{aligned} \int d^2x_R' d^2x_r' C(|x_r|) \cos\left(\beta\varphi_s'(x_1') - \beta\varphi_s'(x_2')\right) &\approx \frac{\alpha^2}{u} \int d^2x_R' \cos\left(\beta\alpha\partial_{|x_R|}\varphi_s'(x_R')\right) \approx \\ &\approx -\frac{1}{2} \int d^2x_R' \beta^2 \frac{\alpha^4}{u} \left(\partial_{|x_R|}\varphi_s'(x_R')\right)^2 = -\frac{\beta^2\alpha^4}{2u} \int d^2x' \left[\left(\partial_{x'}\varphi_s'(x')\right)^2 + \frac{1}{u^2}\left(\partial_{\tau'}\varphi_s'(x')\right)^2\right] \,, \end{aligned} \tag{197}$$

where we Taylor - expanded the cosine assuming α small and we neglected the constant term. The additional constant α^2/u is given by the integration of C in $dx_r'\, d\tau_r'$. We conclude that the term (197) modifies the quadratic part of the Hamiltonian. In particular, neglecting the subleading operator in Eq. (193) we obtain:

$$\frac{\mathcal{A}^2 - \mathcal{B}}{2} \approx \frac{\beta^4 g^2 \alpha^4 dl}{16Ku} \int d^2x' \left[\left(\partial_{x'}\varphi_s'(x')\right)^2 + \frac{1}{u^2}\left(\partial_{\tau'}\varphi_s'(x')\right)^2\right] \,, \tag{198}$$

such that:

$$S_{\text{eff}} = \int d^2x' \left(\frac{Ku}{2\pi} + \frac{\beta^4 g^2 \alpha^4 dl}{16Ku}\right)\left(\partial_{x'}\varphi_s'(x')\right)^2 + \left(\frac{K}{2\pi u} + \frac{\beta^4 g^2 \alpha^4 dl}{16Ku^3}\right)\left(\partial_{\tau'}\varphi_s'(x')\right)^2 + \mathcal{A} \,. \tag{199}$$

Differentiating S_{eff} we can derive the RG equations for the Luttinger parameter. In particular we obtain:

$$\frac{K'}{2\pi} = \sqrt{\left(\frac{Ku}{2\pi} + \frac{\beta^4 g^2 \alpha^4 dl}{16Ku}\right)\left(\frac{K}{2\pi u} + \frac{\beta^4 g^2 \alpha^4 dl}{16Ku^3}\right)} \approx \frac{K}{2\pi} + \frac{1}{2\pi} dl \frac{\pi g^2 \beta^4 \alpha^4}{8u^2 K} \,, \tag{200}$$

thus:

$$\frac{dK}{dl} = \frac{\beta^4 g^2 \alpha^4 \pi}{8u^2 K}\,, \quad \text{with } \alpha \approx a\,. \tag{201}$$

Analogously one gets:

$$\frac{du}{dl} = 0\,, \tag{202}$$

the velocity does not change under the RG flow, which is consistent with the Lorentz invariance of the system. To summarize the RG equations of the system at second order are:

$$\frac{dK_\varphi}{dl} = \frac{\pi\beta^4 g^2 \alpha^4}{8u^2 K_\varphi}\,, \qquad \frac{dg}{dl} = \left(2 - \frac{\beta^2}{4K_\varphi}\right) g\,; \tag{203}$$

here the subscript φ reminds us that the original cosine term is of the kind $\cos\beta\varphi$. The same calculation starting with the dual term $\cos\beta\theta$ implies instead:

$$\frac{dK_\theta}{dl} = -\frac{\pi\beta^4 g^2 \alpha^4 K_\theta^3}{8u^2}\,, \qquad \frac{dg}{dl} = \left(2 - \frac{\beta^2 K_\theta}{4}\right) g\,, \tag{204}$$

as it can be derived by mapping $K_\varphi \to K_\theta^{-1}$.

C. Kosterlitz and Thouless equations

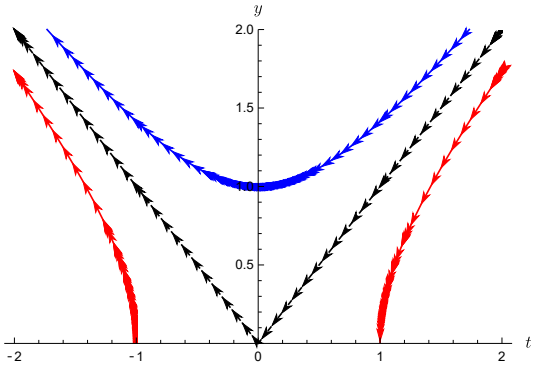

FIG. 1: RG flow of the KT equations for small t referred to a sine-Gordon interaction $\cos\beta\theta$. See the text.

In order to study the flow of the RG group at second order obtained by the previous equations, I analyze in more detail Eq. (204) (a similar study is done in AS page 471-474, but with different notations). We are interested in the behavior close to the phase transition that occurs at $D_g = \beta^2 K_\theta/4 = 2$, thus $K_\theta = 8/\beta^2$, therefore we define the small parameter t and the effective coupling constant y:

$$t \equiv \frac{\beta^2 K_\theta}{4} - 2\,, \qquad y \equiv \frac{4\sqrt{\pi}\alpha^2 g}{u}\,. \tag{205}$$

Now, we use the fact that we consider only the behavior very close to the phase transition, where $t \ll 1$. We substitute the previous equations in (204) and we keep only the dominant term in each of the equations. We get:

$$\frac{dt}{dl} = -y^2\,, \qquad \frac{dy}{dl} = -ty\,. \tag{206}$$

These are the Kosterlitz and Thouless equations. The first step to solve them is to realize that $\mu = t^2 - y^2$ is invariant under the RG flow defined by these equations. Therefore the flow trajectories in the plane (t, y) are always hyperboles and $y = \pm t$ are the separatrices defininig the critical lines. Here we describe the possible behaviors for $y > 0$ (thus $g > 0$) (see Fig. 1).

- If $t > 0$ and $y < t$, then the system flows to $y = 0$. This corresponds trivially to the regime where g is irrelevant and small, that we could explore also at first order. It is the Luttinger liquid phase.
- If $t > 0$ and $y = t$, the system flows to the critical fixed point $t = y = 0$. This is the critical BKT point, corresponding to the Heisenberg model for the XXZ chain.
- If $y > t$, the system flows to $y \to \infty$, independently on t: this is the massive phase.
- If $t < 0$ the system always flows to $y \to \infty$: this is the situation in which g is relevant and the system is driven into the massive phase.

Based on the RG equations and on the conservation of the parameter μ under the RG flow, it is also possible to refine the estimate of the gap we gave at first order in Eq. (186) and of the correlation length, which, in general, grows as $e^{\lambda/\sqrt{x}}$. The general trend, however, remains the same, confirming the essential singularity of the correlation length close to the transition.

Appendix A: The two-point correlation function with smooth cutoffs (courtesy of A. Haller)

In this appendix we consider in more detail the locality of the two-point correlation function defined in Eq. (180) and of the function $C(r)$. For a sharp momentum cutoff, the first integral in (110) returns the Bessel function of the first kind $C(r) = J_0(\Lambda r)$ with asymptotic expression $J_0(\Lambda r) \approx \sqrt{2/(\pi\Lambda r)}\cos(\Lambda r - \pi/4)$. This function has a long algebraic tail and is not a sharp function in r (see also Fig. 2). The origin of this long tail resides in the choice of the momentum cutoff in the integration scheme. Recall that we aim at integrating an infinitesimal shell of large momenta, which allows for a certain degree of freedom in the form of the shell itself. In particular, instead of taking a step function at the cutoffs, we can impose a smoother cutoff at Λ by rewriting the momentum integration as:

$$\int_0^\Lambda dp \to \int_0^\infty dp f_n(p,\Lambda), \quad f_n(p,\Lambda) = \frac{\Lambda^n}{p^n + \Lambda^n}, \quad n \in \mathbb{N}. \tag{A1}$$

The sharp situation is recovered for $n \to \infty$ (see Fig. 2).

In practice, the integration of the fast modes in (180) evaluates to

$$\langle \varphi_{\mathsf{f}}(x,t)\varphi_{\mathsf{f}}(x',t')\rangle_{\mathsf{f}} = \frac{1}{2K}\int_{\tilde\Lambda}^{\Lambda} dp \frac{J_0(pr)}{p} = \frac{1}{2K}\int_0^{\Lambda} dp \frac{J_0(pr)}{p} - \frac{1}{2K}\int_0^{\tilde\Lambda} dp \frac{J_0(pr)}{p} \tag{A2}$$

$$\longrightarrow \frac{1}{2K}\int_0^\infty dp J_0(pr) p^{n-1}\left(\frac{1}{p^n + \tilde\Lambda^n} - \frac{1}{p^n + \Lambda^n}\right) \tag{A3}$$

$$= \frac{1}{2K}\int_0^\infty dp J_0(pr) p^{n-1} \frac{n\Lambda^n}{(\Lambda^n + p^n)^2} dl + O(dl^2), \tag{A4}$$

leading to the modified function C_n which depends on the smoothness n, i.e.

$$C_n(r) = \int_0^\infty dp J_0(pr) p^{n-1} \frac{n\Lambda^n}{(\Lambda^n + p^n)^2}. \tag{A5}$$

For the special case $n = 2$, the integral of the φ_{f}-fields evaluates to $C_2(r) = \Lambda r K_1(\Lambda r)$ that decays exponentially fast (see Fig. 2(c) in red). In particular, the function follows the asymptotic decay $zK_1(z) \sim \sqrt{\pi z/2}\exp(-z)$ and is already negligible for $z = 1$, i.e. $K_1(1) \approx 0.0062$. Therefore, the integration of $C_2(r)$ can be confined to a small interval $r < \alpha$ where $\alpha \sim 2\pi/\Lambda = a$ is a small length scale comparable with the lattice spacing a.

[1] T. Giamarchi, *Quantum physics in one dimension*, Clarendon Press, Oxford 2003.
[2] M. A. Cazalilla, *Bosonizing one-dimensional cold atomic gases*, Journal of Physics B: AMOP 37, S1-S47 (2004), arXiv:cond-mat/0307033
[3] M. A. Cazalilla, R. Citro, T. Giamarchi, E. Orignac and M. Rigol, *One dimensional bosons: From condensed matter systems to ultracold gases*, Rev. Mod. Phys. **83**, 1405 (2011).
[4] A. O. Gogolin, A. A. Nersesyan and A. M. Tsvelik, *Bosonization and Strongly Correlated Systems*, Cambridge University Press, 1998.
[5] P. Lecheminant, A. O. Gogolin and A. A. Nersesyan, Nucl. Phys. B **639**, 502 (2002).
[6] A. K. Bera *et al.*, Nature Physics **16**, 625 (2020).

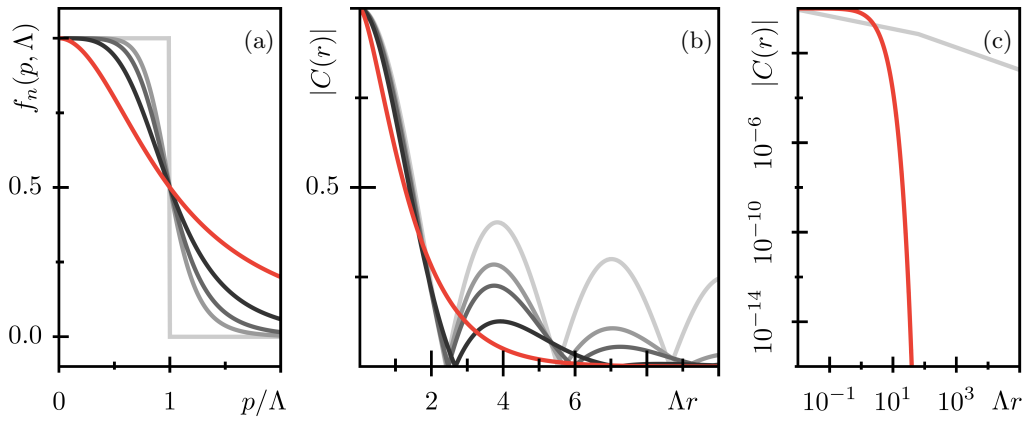

FIG. 2: Panel (a) shows the chosen cutoff $f_n(p,\Lambda)$ with $n \in \{2,4,6,8,\infty\}$ ($n=2$ in red, $n=\infty$ in light gray) for the integral expression in (A1), which results in various approximations of $C(r)$ for several n plotted in (b). Panel (c) highlights the exponentially sharp function $C_2(r) = \Lambda r K_1(r\Lambda)$ compared to the sharp cutoff result $C_\infty(r) = J_0(\Lambda r)$.