Notes on the Renormalization Group

M. Burrello
Niels Bohr International Academy and Center for Quantum Devices,
Niels Bohr Institute, University of Copenhagen, 2100 Copenhagen, Denmark.
(Dated: April 27, 2023)

The study of renormalization group (RG) is a huge subject which encompasses all branches of many-body physics and it is so intricate that mastering all the techniques related to RG is a kind of art. RG is so fundamental that one could present a course only on that. Obviously, in our very limited time, we will consider only a very limited and general introduction to the subject based on the condensed–matter point of view. The basic idea is that the ones of you who will study the thermodynamic behavior of many-body systems, will have a basis to develop a better technical knowledge of the tools they will need in their specific research.

I. LIST OF TOPICS AND REFERENCES

I list here the main points I want to touch in our lectures and I want you to know. For each point, I indicate some of the possible sources you can use to study it (AS: Altland and Simons; C: Cardy; K: Kardar). The follwoing notes are inspired by these sources.

- Introduction to the idea of RG; distinction between microscopic (fast - oscillating) and macroscopic (slow - oscillating) degrees of freedom; definition of coarse graining; definition of rescaling; universality; RG flow and fixed points. AS: 409-412 + Sec. 8.3 ; C: 28-34; K Sec. 4.4
- General theory of the RG and RG flow. AS: Sec. 8.3; C: Sec. 3.3; K: 4.5
- Kadanoff blocking (basic idea of RG in real space); example of the classical 1D Ising chain through transfer matrix (see also the problems in the first section of the first exercise set). AS: Sec. 8.1.3; C: Pages 34-37 + Ex. 3.1; K: Secs. 6.1,6.2,6.3 (this is more detailed).
- General Wilson's RG (momentum space); the Gaussian case; scaling dimensions of the terms in a Lagrangian; scaling of correlation functions; definition of relevant and irrelevant terms. AS: Secs. 8.1 - 8.4; C: scattered everywhere... Cardy does things a bit differently; K: Secs. 4.5, 4.6, 4.7 (see also 5.1, 5.2, 5.5 if you want to see it in full detail)
- Example of second order RG with the sine-Gordon model and the BKT transition: we will do it after bosonization and you find it in Sec. VIII B of my 1D models notes.

II. BASIC CONCEPTS OF THE RG

- We consider the generic Hamiltonian $H(\{g\})$ of a many-body system, typically in the thermodynamic limit (or sufficiently large).

- The renormalization group provides techniques to identify the paradigmatic statistical behavior of the system as a function of the parameters $\{g\}$.

- To determine the statistical/universal behavior of the system we must focus on its macroscopic / long-range features, whereas what happens on the microscopic short-range scale does not directly influence the statistical properties.

- Therefore we distinguish long-range (slowly varying) degrees of freedom and long-range (fast oscillating) degrees of freedom:

$$Z = \sum_{\text{slow}} \left(\sum_{\text{fast}} e^{-\beta H(\text{slow},\, \text{fast})} \right) = \sum_{\text{slow}} e^{-\beta H'(\text{slow})} \tag{1}$$

- In the second equality of the previous equation, we traced out the fast degrees of freedom and this operation corresponds to the so-called *coarse graining*.

- In a coarse graining, I lose information and I reduce the degrees of freedom.

- An RG transformation of the Hamiltonian corresponds to two steps:

  1. a coarse graining;
  2. a rescaling.

- The rescaling is needed to compare the original and final Hamiltonians and corresponds to:

$$x \to x' = x/b\,, \quad k \to k' = kb \tag{2}$$

  where $b > 1$ is a rescaling parameter. We will often write $b = e^l$, and, typically, $l \ll 1$ and I may write dl instead.

- An iteration of the RG procedure is schematically represented as:

$$H\,(\text{fast, slow}) \xrightarrow{\text{CG}} H'\,(\text{slow}) \xrightarrow{\text{Resc.}} H'\,(\text{rescaled slow})\ , \tag{3}$$

$$H\,(\{g\}) \qquad \xrightarrow{\text{RG step}} \qquad H'\,(\{g'\}) \tag{4}$$

- During the RG procedure, different microscopic models may converge to the same renormalized Hamiltonian. This happens when they share the same behavior of the slow degrees of freedom, but different fast/microscopic features.

- This convergence of different microscopic models defines universality classes and thermodynamic phases. Two Hamiltonians are equivalent under RG if they converge to the same **fixed point**:

$$H^{FP}\,(\{g^*\}) \qquad \xrightarrow{\text{RG step}} \qquad H^{FP}\,(\{g^*\}) \tag{5}$$

  A fixed point Hamiltonian is left invariant by the RG steps. A fixed point typically characterizes either a thermodynamic phase of matter or a phase transition.

- Being invariant under RG steps, a fixed point does not display any scale dependence. This typically correspond to either having no energy gaps in the system (usually a critical point / phase transition between different gapped phases), or having an infinite energy gap (usually a trivial thermodynamic phase).

- Fixed points can be stable or unstable under given perturbations.

III. GENERAL SCHEME TO DESCRIBE THE RG FLOW, RELEVANT AND IRRELEVANT OPERATORS

Let us formalize the previous intuitions. We consider a Hamiltonian of the form:

$$H = \sum_i g_i \phi_i \,. \tag{6}$$

In this notation, the g_i parameters constitute the initial (also called *bare*) coupling constants (they correspond to the initial conditions at $l = 0$ of the RG flow). You should instead think about the ϕ_i as being the operators that determine the Hamiltonian. In order to fix the ideas, I am indeed thinking about a quantum field theory Hamiltonian. Typically you derive the coupling constants $\{g\}$ from a microscopic model and you want to know what the thermodynamic phases which may appear in the system are as a function of these parameters. To have a complete description, you should include in the $\{\phi\}$ set also operators which are not present in the bare Hamiltonian, but may appear during the renormalization. Essentially, you typically consider a large set of operators $\{\phi\}$, although several of the bare coupling constants g_i associated with these operators may be zero at the bare level.

During a given RG step from $l = 0$ to $l = \ell = \ln b$, the coupling constants will flow from $\{g\}$ to $\{g'\}$. For simplicity, let us consider a model with a single coupling constant g (everything will be extended to a vector of coupling constants g_i). The renormalized value of the coupling constant is represented as

$$g' = \tilde{R}_{l=\ell}(g) \;\Rightarrow\; g' \approx g + \ell \frac{d}{dl}\, \tilde{R}_l(g)\Big|_{l=0} \,. \tag{7}$$

Essentially, here we are assuming that, in general, the renormalized g', namely $\tilde{R}_\ell(g)$, is a complicated function of g and ℓ. However, it is smooth in ℓ; therefore we Taylor expand $\tilde{R}_\ell(g)$ for $\ell \to 0$ (corresponding to $b \to 1$, with $b = e^\ell$), and we consider only its first order term in ℓ. This is the first expansion we typically apply to determine the RG flow of the coupling constants (at least, in the usual scenario in which $\tilde{R}_\ell(g)$ cannot be exactly calculated).

More in general, we can introduce a vector of coupling constants $\vec{g}$, and we introduce the following vector of functions:

$$\vec{R}(\vec{g}) = \frac{d\vec{g}}{dl} \,. \tag{8}$$

A **fixed point** $\vec{g}^*$ is defined by:

$$\vec{R}(\vec{g}^*) = 0 \,, \tag{9}$$

under the assumptions that all g_i are finite. If one of the coupling constants is infinite, than $\vec{g}^*$ also represents a fixed point.

For finite g_i, the previous equation implies that there are no length scales left in the system. The physical system described by such a fixed point is self similar (roughly speaking, looking at it from different distances it looks always the same). This also means that all length scales diverged, and, the most common scenario at which this happens, is a second-order phase transition. Here there are many details I am hiding. For example, it is also possible that the Hamiltonian decouples in different sectors affecting different degrees of freedom, and only one of these sectors displays a phase transition (while the others remain gapped).

Let us consider now, how the RG flow looks like **close to a fixed point**, such that $\vec{g} - \vec{g}^*$ can be considered "small" and justify a further Taylor expansion in the g parameters. We have:

$$g'_a - g^*_a \approx \ell \frac{d}{dl}\, \tilde{R}_{l,a}(g)\Big|_{l=0} = \ell R_a \left((\vec{g} - \vec{g}^*) + \vec{g}^*\right) \approx \ell \sum_b W_{ab} \left(g_b - g^*_b\right) \,. \tag{10}$$

The approximation linearizes the behavior of the RG flow not only in the scaling parameter l (first Taylor expansion in the previous equation), but also in the distance from the fixed point (second Taylor expansion). The previous approximation is valid close to a fixed point such that Eq. (9) holds; the matrix W characterizes the first (linear) terms in the expansion and it corresponds indeed to:

$$W_{ab} = \frac{\partial}{\partial g_b} \left[\frac{dg'_a}{dl}\right]_{\vec{g}^*} \,. \tag{11}$$

The linearization over the coupling constants g is a first order approximation, which provides very important information about the RG flow. Clearly, in several cases, this approximation is not sufficient, and one needs to consider also higher order terms in the perturbative expansion.

W, however, encodes enough information to obtain a preliminary understanding of the vicinity of a fixed point. In particular, we can define left eigenvectors of W. They correspond to suitable linear superposition of the operators ϕ_i (in the jargoon of conformal field theory, they are also called primary fields). I introduce the left eigenvectors $\vec{\varphi}_\alpha$ of W such that:

$$\sum_a \varphi_{\alpha,a} W_{a,b} = \varphi_{\alpha,b} \lambda_\alpha \,, \tag{12}$$

where λ_α is the eigenvalue associated to the eigenvector $\vec{\varphi}_\alpha$. In particular, the primary operator associated to the eigenvector $\vec{\varphi}_\alpha$ is associated to the point in the g_i plane:

$$v_\alpha = \sum_a \varphi_{\alpha,a} \left(g_a - g_a^*\right) \,. \tag{13}$$

Cardy calls these vectors *scaling variables*, and they are linear combinations of the deviations $(g_a - g_a^*)$ from an RG fixed point. These scaling variables (thus the corresponding primary operators) display a particular behavior under the (first-order approximation of) the RG flow:

$$\frac{dv_\alpha}{dl} = \sum_a \varphi_{\alpha,a} \partial_l \left(g_a - g_a^*\right) = \sum_{a,b} \varphi_{\alpha,a} W_{a,b} \left(g_b - g_b^*\right) = \sum_b \varphi_{\alpha,b} \lambda_\alpha \left(g_b - g_b^*\right) = \lambda_\alpha v_\alpha \,. \tag{14}$$

Here the second equality corresponds to the definition (11) of W, and the third equality to the eigenvalue relation (12). The last step is dictated by the definition (13). The previous relation is fundamental because it shows that the primary operators evolve under the RG flow as:

$$v_\alpha(l) = e^{\lambda_\alpha l} v_\alpha(0) \,. \tag{15}$$

Here l is the renormalization group parameter, and one can see that:

- For $\lambda_\alpha > 0$, the operator v_α grows under the RG flow. Primary operators with $\lambda_\alpha > 0$ are called **relevant**.
- For $\lambda_\alpha < 0$, the operator v_α becomes smaller and smaller under the RG flow. Primary operators with $\lambda_\alpha < 0$ are called **irrelevant**.
- For $\lambda_\alpha = 0$, our first order perturbation theory in $\{g\}$ is not sufficient to establish the behavior of v_α under the RG flow. Second order is necessary. Primary operators with $\lambda_\alpha = 0$ are called **marginal**.

IV. RENORMALIZATION GROUP IN REAL SPACE: KADANOFF BLOCKING

I sketch here the basic concept of the renormalization group seen from the real space perspective. In particular, we focus on a lattice system. To fix the ideas, let us consider as an example the classical Ising model in a 2D square lattice:

$$H = -J \sum_{\langle i,j \rangle} s_i s_j \,, \tag{16}$$

where the sum is taken over nearest neighbors, i and j are vectors representing the position of the spins in the lattice, and $s_i = \pm 1$ is a classical spin 1/2 degree of freedom.

The RG construction by Kadanoff consists in rewriting the previous Hamiltonian based on block of spins of size $b \times b$ (more in general b^D for classical systems in D space dimensions). For each block we may define an average spin:

$$s_{r'}^{(1)} = b^{-D} \sum_{r \in r'} s_r \,; \tag{17}$$

here r' labels the position of the block. The blocks do not overlap and cover the whole original system, such that each spin s_r belongs to a single block characterized by $s_{r'}^{(1)}$. The goal is to find a suitable Hamiltonian $H^{(1)}$ for the $s^{(1)}$ degrees of freedom such that, concerning the average long-distance (small momentum) observable, both H and $H^{(1)}$ provide the same results. In particular I search for $H^{(1)}$ such that:

$$H\left[s\right] \overset{\text{RG step}}{\longrightarrow} H^{(1)}\left[s^{(1)}\right] \tag{18}$$

and the corresponding partition functions are the same:

$$Z(s) = Z\left(s^{(1)}\right) . \tag{19}$$

We have:

$$Z(s) = \sum_s e^{-\beta H[s]} = \int \prod_r Ds_r e^{-\beta H[s]} = \int \prod_r Ds_r \prod_{r'} Ds^{(1)}_{r'} e^{-\beta H[s]} \prod \delta\left(s^{(1)}_{r'} - b^{-D} \sum_{r\in r'} s_r\right) =$$
$$\int \prod_{r'} Ds^{(1)}_{r'} \left[\prod_r Ds_r e^{-\beta H[s]} \prod \delta\left(s^{(1)}_{r'} - b^{-D} \sum_{r\in r'} s_r\right)\right] \equiv \int \prod_{r'} Ds^{(1)}_{r'} e^{-\beta H^{(1)}\left[s^{(1)}\right]} . \tag{20}$$

In the last equality, the term in square brackets is formally $e^{-\beta H^{(1)}\left[s^{(1)}\right]}$, which, in principle, gives us a definition of $H^{(1)}$. The previous relation describes essentially the coarse graining in which we block b^D spins together and we lose information about the microscopic degrees of freedom s, by keeping the information related to the average spin in each block. Concerning the rescaling, we must ensure that the average magnetization remains the same in both description. We have:

$$\left\langle s^{(1)} \right\rangle = Z^{-1} \prod_r Ds_r \prod_{r'} Ds^{(1)}_{r'} e^{-\beta H[s]} \prod \delta\left(s^{(1)}_{r'} - b^{-D} \sum_{r\in r'} s_r\right) \frac{b^D}{L^D} \sum_{r'} s^{(1)}_{r'}$$
$$= Z^{-1} \prod_r Ds_r e^{-\beta H[s]} \frac{b^D}{L^D} b^{-D} \sum_{r'} \left(\sum_{r\in r'} s_r\right) = \langle s\rangle . \tag{21}$$

Here L is the linear system size and it is important to stress that the number of blocks in a finite size system would be L^D/b^D, such that the red factor must be included in the definition of the average $\left\langle s^{(1)}\right\rangle$. That factor, roughly speaking, represents the rescaling of the spin operator. We will discuss this more formally when considering the Wilson RG construction.

V. RENORMALIZATION GROUP IN REAL SPACE: TRANSFER MATRIX

The Kadanoff blocking we saw in the previous section gives us an idealized way of understanding how RG works in real space for lattice systems. Differently from what we considered in the beginning, in the Kadanoff blocking, $b = 2$ or $b = 3$, and it is not really in the form $b = 1 + dl$. Furthermore, the previous construction does not give us a systematic idea about how to derive the Hamiltonian $H^{(1)}$ and, therefore, it is not very useful in practice.

There are, however, several statistical physics problems in which the Hamiltonian $H^{(1)}$ (or, better, the weight $e^{-\beta H^{(1)}}$) can be exactly determined. One of the major examples is provided by systems with short-range interactions that can be described by a so-called **transfer matrix**. In this section we consider the construction of the transfer matrix for the simplest possible case, namely the 1D classical Ising model:

$$H = -J \sum_r s_r s_{r+1} - h \sum_r s_r . \tag{22}$$

Also in this case, we have a ferromagnetic interaction between nearest neighbor classical spin 1/2 ($s_r = \pm 1$). h is a (longitudinal) magnetic field. The partition function of the system reads:

$$Z = \sum_{\{s_r\}} e^{-\beta H} = \sum_{\{s_r\}} e^{\beta J \sum_r s_r s_{r+1} + \beta h \sum_r s_r} . \tag{23}$$

We introduce the transfer matrix to rewrite the partition function Z and be able of suitably applying a coarse graining transformation. First of all we define two vectors:

$$|s_r = \pm 1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} . \tag{24}$$

For a finite size system with N sites, we can rewrite the partition function in the form:

$$\sum_{s_1, s_N} \langle s_N | O(s_N) \prod_r T_{s_{r+1}, s_r} O(s_1) | s_1 \rangle . \tag{25}$$

Here I introduced the transfer matrix T which is defined as:

$$T_{s_{r+1},s_r} = \langle s_{r+1}|e^{-\beta H_r(s_{r+1},s_r)}|s_r\rangle\,, \tag{26}$$

where, in particular:

$$H_r\left(s_{r+1},s_r\right) = -Js_rs_{r+1} - \frac{h}{2}\left(s_r+s_{r+1}\right)\,. \tag{27}$$

The previous description of the partition function captures exactly what happens in the bulk of the system, because $\sum_r H_r = H +$ boundary corrections. The boundary corrections are accounted for by the suitable operators $O(s_1)$ and $O(s_N)$ appearing in Eq. (25).

A. Transfer matrix of the 1D classical Ising model

Eqs. (26) and (25) give the general expression of the partition function in terms of the transfer matrix for open boundary systems. In the specific case of the 1D Ising model we get:

$$T = \begin{pmatrix} e^{\beta J+\beta h} & e^{-\beta J} \\ e^{-\beta J} & e^{\beta J-\beta h}\end{pmatrix} \tag{28}$$

and

$$Z = \sum_{s_1,s_N}\langle s_N|e^{\beta h s_N/2}T^{N-1}e^{\beta h s_1/2}|s_1\rangle\,. \tag{29}$$

The transfer matrix construction becomes even easier in the case of periodic boundary conditions (ring). If $s_1 = s_{N+1}$ we get:

$$Z = \mathrm{Tr}\,\mathrm{T^N} = \lambda_+^{\mathrm{N}} + \lambda_-^{\mathrm{N}}\,, \tag{30}$$

where $\lambda_\pm$ are the eigenvalues of T:

$$\lambda_\pm = e^{\beta J}\cosh\left(\beta h\right) \pm \sqrt{e^{2\beta J}\sinh^2\left(\beta h\right) + e^{-2\beta J}}\,. \tag{31}$$

From the previous equation we exactly know the partition function of the system, which means that we know everything. In particular we get:

$$F = -\beta^{-1}\ln Z \overset{N\to\infty}{\to} -\beta^{-1}N\ln\lambda_+ = -N\left[J+\beta^{-1}\ln\left(\cosh\left(\beta h\right)+\sqrt{\sinh^2\left(\beta h\right)+e^{-4\beta J}}\right)\right]\,. \tag{32}$$

This approximation of F holds because, in the thermodynamic limit $\left(\lambda_-/\lambda_+\right)^N$ becomes negligible. From F one can easily derive the magnetization and any other important observable.

B. Transfer matrix, path integral, and the mapping from 1+1D classical systems to 0+1D quantum systems

The construction of the transfer matrix for classical systems shares many similarities with the construction of an effective action for a quantum system. Essentially, the transfer matrix can be seen as the opposite mapping, with respect to path integral, from a classical model in $1+1$D to a quantum model in $0+1$D. In general this construction justifies a common belief that $d+1$ D classical statistical physics model can be mapped into $(d-1)+1$ D quantum models and vice versa. The trick is given by interpreting:

$$T = e^{-\delta t H_q} \tag{33}$$

where δt in an interval of time in Euclidean time and H_q is a suitable quantum Hamiltonian (a 2×2 Hamiltonian in the case of the Ising model). In this respect Eq. (25) can be interpreted as a path integral construction:

$$Z = \int D_{s_i}\langle s_N|T|s_{N-1}\rangle\langle s_{N-1}|\ldots T|s_1\rangle\,. \tag{34}$$

For periodic boundary conditions this matches the construction of path integrals with Matsubara frequencies.

C. RG with the transfer matrix

Let us go back to the RG construction. It is easy to see that:

$$Z = \mathrm{Tr}\, T^N = \mathrm{Tr}\, (T^b)^{N/b} \,. \tag{35}$$

If you compare this with Eq. (20), you see that Z can be considered both the partition function of the Ising model with N sites and transfer matrix T, and the partition function of the rescaled Ising model with N/b Kadanoff's blocks and transfer matrix T^b. Therefore a comparison between T and T^b allows us to understand how the Hamiltonian, and its coupling constants, must be renormalized in an RG step with parameter b. The full calculation with generic J and h is a bit boring, and you find it, for instance, in the AS book (sec 8.1.3). We restrict here to the case $h = 0$:

$$T(h=0)_{s_1,s_2} = e^{\beta J s_1 s_2} = \cosh \beta J \left[1 + s_1 s_2 \tanh \beta J\right] \,. \tag{36}$$

I label $c = \cosh \beta J$ and $v = \tanh \beta J$. We get:

$$T = c \begin{pmatrix} 1+v & 1-v \\ 1-v & 1+v \end{pmatrix} \,, \tag{37}$$

$$T^2 = 2c^2 \begin{pmatrix} 1+v^2 & 1-v^2 \\ 1-v^2 & 1+v^2 \end{pmatrix} \,, \tag{38}$$

$$T^3 = 2^2 c^3 \begin{pmatrix} 1+v^3 & 1-v^3 \\ 1-v^3 & 1+v^3 \end{pmatrix} \,. \tag{39}$$

You can show by induction that:

$$c \to 2^{n-1} c^n \,, \quad v \to v^n \,. \tag{40}$$

These are our (discrete) flow equations! The general structure is of the kind:

$$e^{\beta J s_1 s_2} \to e^{\beta' A' + \beta' J' s'_1 s'_2} \,. \tag{41}$$

Here A' is an uninteresting energy shift. The important aspect is to determine $\beta' J'$, which can be done by considering:

$$v^n = \tanh \beta' J' = \tanh^n \beta J \,, \tag{42}$$

in particular, we set $n = b$ and we can take $b = 2$:

$$v' = v^2 \,. \tag{43}$$

Where $v, v' \in [0, 1]$. A simple analysis of the parabola (Newton's method for recursion) easily shows (see Fig. 1) that there are only 2 fixed points:

- $v^* = 1$: this is an unstable fixed point since $v = 1 - \epsilon$ would flow away from it iteration after iteration. This fixed point corresponds to $\beta J \to \infty$, which is the zero-temperature limit (or the case with infinite interactions). It represents a ferromagnetic/ordered fixed point which, in 1D, is not stable.

- $v^* = 0$: this is a stable fixed point, since $v = \epsilon$ would flow towards $v = 0$. This fixed point corresponds to $\beta J \to 0$, which is the infinite-temperature limit (or the case with no interactions). This is a paramagnetic/disordered fixed point.

The fact that in $1D$ the system flows to $v^* = 0$ is telling us that, essentially, the J interaction is irrelevant around this fixed point. In the end, we applied this complex machinery to find that the 1D classical Ising model has a trivial behavior: it always flows to the infinite temperature limit unless the temperature vanishes. In more complex situations (or in 2D classical Ising ferromagnets) the ferromagnetic fixed point would be stable as well and a third unstable fixed point separateing the ordered and disordered phase would appear.

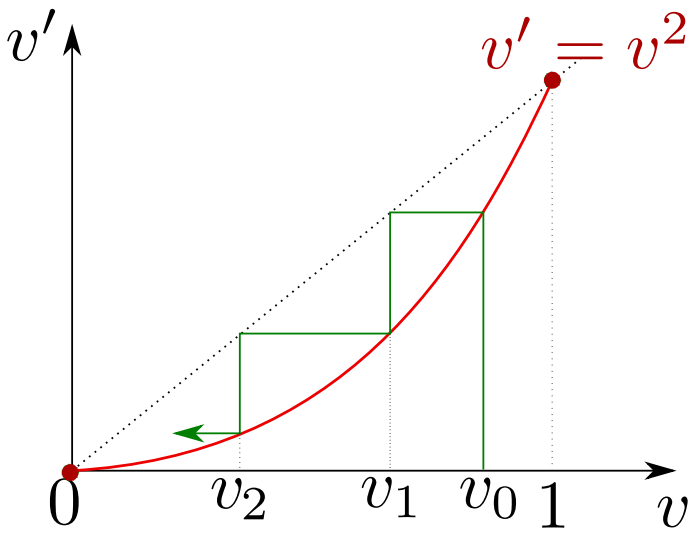

FIG. 1: Schematic plot of the iteration of a few RG steps for the classical 1D Ising model. The red curve is the result in Eq. (43) of the Kadanoff blocking on the v parameter of the transfer matrix. When we begin from a bare value v_0, the first iteration results in $v_1 = v_0^2$, the second in $v_2 = v_0^4$ and so on. Newton's iteration method shows that there is only a single attractive fixed point at $v = 0$. $v = 1$ constitutes instead an unstable fixed point.

D. Remark on transfer matrix and correlation length

We saw that the transfer matrix is equivalent to the partition function, in the sense that it carries all the information of the system. A very important example is given by the correlations and the correlation length. The two point correlation function of the system is indeed given by:

$$\langle W_i W_{i+r} \rangle = Z^{-1} \sum_{\{s\}} W_i(s) W_{i+r}(s) e^{-\beta H(s)} \, . \tag{44}$$

where W are arbitrary local operators, for instance $W_r = s_r$. We can rewrite the connected correlation function in a form of the kind:

$$\langle W_i W_{i+r} \rangle_c = \frac{\sum_{s_1, s_N} \langle s_N | O(s_N) T^{N-r-i-1} \langle s_{i+r} | WT | s_{i+r-1} \rangle T^r \langle s_i | WT | s_{i-1} \rangle T^{i-1} | s_1 \rangle}{\sum_{s_1, s_N} \langle s_N | T^{N-1} | s_1 \rangle} - \langle W_i \rangle \langle W_{r+i} \rangle \, . \tag{45}$$

It is possible to show that the leading term in the previous expression scales as $(\lambda_-/\lambda_+)^r = e^{-r/\xi}$. In general, here, λ_+ is the largest eigenvalue of T, and λ_- the second largest, and we are considering r large. This shows that the correlation length scales as:

$$\xi \propto 1/\ln(\lambda_+/\lambda_-) \, . \tag{46}$$

This also implies that ξ diverges only if λ_- and λ_+ get degenerate, which is typically a signature of a phase transition.

VI. RENORMALIZATION GROUP IN MOMENTUM SPACE: WILSON RG

The most standard technique to derive the renormalization group equations of a quantum field theory relies on expressing the field theory in momentum space and it is usually referred to as Wilsonian RG. The starting point of this technique is to separate slow and fast degrees of freedom based on their energy or momentum. In particular, we will always consider a theory which displays a momentum cutoff $\Lambda_{\max}$. If our model is originally a lattice model, $\Lambda_{\max}$ is given by the inverse of the lattice spacing.

A. Coarse Graining

The **coarse graining** step can be schematically represented as follows:

- We model the system based on a bare action. This action is integrated over momenta such that $|k| < \Lambda_{\max}$. In most cases we will consider Lorentz invariant theories such that k has both an energy component ω/v and a spacial momentum component $\vec{k}$. In this case the modulo $|k|$ is the modulo of the $D+1$ momentum.
- We introduce a new cutoff $\tilde{\Lambda} < \Lambda_{\max}$. In particular we set:

$$\frac{\tilde{\Lambda}}{\Lambda_{\max}} = \frac{1}{b} = 1 - dl \tag{47}$$

  where we assume that $\tilde{\Lambda}$ is only infinitesimally smaller than $\Lambda_{\max}$.
- $\tilde{\Lambda}$ separates slow ($|k| < \tilde{\Lambda}$) and fast ($\tilde{\Lambda} < |k| < \Lambda_{\max}$) degrees of freedom.
- We integrate the partition function over the fast modes [to a certain level of perturbation theory]
- We derive a new "slow" action, as a function of modes with $|k| < \tilde{\Lambda}$ only.

More formally, let us introduce slow φ_s and fast φ_f modes. The initial partition function of the system will be, in Euclidean time:

$$Z = \int D_{\varphi_f} D_{\varphi_s} e^{-S[\varphi_f, \varphi_s]} \,. \tag{48}$$

The related action assumes a general form:

$$S[\varphi_f, \varphi_s] = S_s[\varphi_s] + S_f[\varphi_f] + S_I[\varphi_s, \varphi_f] \,, \tag{49}$$

where the three contributions represent the action for the slow fields only, the fast fields only, and their interaction. This implies:

$$Z = \int D\varphi_f D\varphi_s e^{-S_s[\varphi_s] - S_f[\varphi_f] - S_I[\varphi_s, \varphi_f]} = \\ \int D\varphi_s e^{-S_s[\varphi_s]} \int D\varphi_f e^{-S_f[\varphi_f] - S_I[\varphi_s, \varphi_f]} = \int D\varphi_s e^{-S_s[\varphi_s]} \left\langle e^{-S_I[\varphi_s, \varphi_f]} \right\rangle_f \tag{50}$$

where $\langle \ldots \rangle_f$ represents the average over the fast modes only. Therefore we get:

$$e^{-S'[\varphi_s]} = e^{-S_s[\varphi_s]} \left\langle e^{-S_I[\varphi_s, \varphi_f]} \right\rangle_f \,, \tag{51}$$

which implies that the coarse grained action reads:

$$S'[\varphi_s] = S_s[\varphi_s] - \ln \left\langle e^{-S_I[\varphi_s, \varphi_f]} \right\rangle_f \approx S_s[\varphi_s] - \ln \left\langle 1 - S_I + \frac{S_I^2}{2} \right\rangle_f \approx S_s[\varphi_s] + \langle S_I \rangle_f - \frac{1}{2} \left[\langle S_I^2 \rangle_f - \langle S_I \rangle_f^2 \right] \tag{52}$$

The last approximations hold when S_I can be considered "small" and it is often not very rigorous. In particular we are demanding that S_I can be considered a small correction with respect to S_s and S_f, which may be justified when the coupling constants in S_I are much smaller than the energy scales appearing in $S_s + S_f$. Eq. (52) provides the result of the coarse graining procedure at second order in the interactions. After the coarse graining, we must complete the RG step by applying the **rescaling**.

B. Rescaling and scaling dimensions of operators

The rescaling is aimed at bringing back the cutoff to its original value, such that $\tilde{\Lambda} \to \Lambda_{\max}$ and $L/b \to L$. There are two crucial elements involved in the rescaling.

- First, we rescale coordinates and momenta. In particular we set $\tilde{\Lambda}/\Lambda_{\max} = 1 - dl$ and we replace:

$$x = (1 + dl)x' \,, \qquad k = -i\partial_x = (1 - dl)k' \,. \tag{53}$$

  Here x' and k' are the new rescaled coordinates and momenta. In particular, here, I am assuming that we have Lorentz invariance as in most of the examples in this course [otherwise time and space scale differently and one needs to introduce a dynamical scaling exponent z such that $t = (1 + zdl)t'$].

- It is not sufficient to rescale coordinates and momenta. We must also rescale the fields. We need to replace $\varphi_s(x)$ with a suitable function of the rescaled $\varphi'(x')$. Therefore we must find a suitable function f such that $\varphi'(x') = f(\varphi_s(x))$.
- This replacement of the fields will allow us to compare the original action $S[\varphi(x)]$ with a new action $S_s[\varphi'(x')]$.

Next, we formalize this required **rescaling of the fields**. The starting point to determine the relation between $\varphi_s(x)$ and the rescaled $\varphi'(x')$ is the following assumption: *a massless Gaussian model is scale invariant*. This assumption corresponds to the statement that a Gaussian massless free theory (which is always characterized by $S_I = 0$) has no scale dependence, and it is consistent with the direct calculations of its two-point correlation functions, which typically display a power law decay [you will see a few examples in the exercise sets and other exercises]. Therefore, we will always consider any massless Gaussian theory as a scale-invariant fixed point, which means that, for these theories, the coarse graining has no effect, and, more importantly, the rescaling must leave both the partition function and the action invariant. Therefore, for Gaussian massless theories we demand that:

$$S = \int d^D x \, \mathcal{L}(x) = \int d^D x' \, \mathcal{L}'(x') \, . \tag{54}$$

This relation is indeed valid only for scale invariant models. Here $\mathcal{L}$ is the Lagrangian density, the integral is taken on both space and time coordinates (think about x as a four-vector), and primes indicate rescaled quantities. Let us consider a general bosonic theory. The previous relation implies:

$$S = \int d^D x \, (\nabla\varphi(x))^2 = \int d^D x' \, (\nabla'\varphi'(x'))^2 \, . \tag{55}$$

From Eq. (53) we obtain that, for an infinitesimal rescaling, $d^D x = (1+dl)^D \, d^D x' = (1+Ddl) \, d^D x'$, and $\nabla = \nabla'/(1+dl)$. Therefore:

$$\int d^D x \, (\nabla\varphi(x))^2 = \int (1+Ddl) \, d^D x' \left(\frac{\nabla'\varphi(x)}{1+dl}\right)^2 = \int [1+(D-2)\,dl] \, d^D x' \, (\nabla'\varphi(x))^2 \, . \tag{56}$$

To fulfill Eq. (55) we must impose:

$$(1+dl)^{D-2} \, (\nabla'\varphi(x))^2 = (\nabla'\varphi'(x'))^2 \, . \tag{57}$$

This gives us the rescaling of the free massless bosonic field:

$$(1+dl)^{\frac{D-2}{2}} \, \varphi(x) = \varphi'(x') \, . \tag{58}$$

This is the relation we were searching for, and it provides the naive or canonical or engineering **scaling dimension** of the field φ. We label this scaling dimension with $D_\varphi = \frac{D-2}{2}$. In full generality, we will always associate to any local primary operator $O_g(x)$ its canonical scaling dimension via the definition:

$$O_g(x) = (1+dl)^{-D_g} \, O'_g(x') \, . \tag{59}$$

In our previous example $O_g(x) \to \varphi(x)$ and $D_g \to D_\varphi$. Eq. (59) constitutes the definition we will adopt of the canonical scaling dimension. During the rescaling step we apply indeed general replacements of the kind in Eq. (59) for all fields and their combination appearing in the Lagrangian. This means that we consider the operator $O_g(x)$ to behave as the power k^{D_g} of the momentum, as you can see by comparing Eq. (59) with Eq. (53).

Hereafter, I will refer to the D_g simply as scaling dimension. These scaling dimensions are always consistent with the renormalization of the fields one derives based on the solely first order in the action expansion in Eq. (52). The second and higher orders, typically provide corrections to the scaling dimensions which are called anomalous dimensions. In this course we will typically not consider the anomalous scaling dimensions. Or better, we will be mostly interested in describing interacting $1+1$D theory through a technique, called bosonization, which will systematically account for anomalous scaling dimensions, in such a way that, for us, the distinction between canonical and anomalous will not be so relevant...

The scaling dimension of any local operator is intrinsically related to its two-point correlation functions. In particular, let us consider again a massless and Gaussian action as in the previous equations. In this case the system is scale invariant, and the decay of correlation functions can be only algebraic (no exponentials are allowed, because

they would introduce a finite correlation length, which constitutes a non-invariant length scale). Therefore, for these critical theories, we have:

$$\left\langle O_g^\dagger(x) O_g(y) \right\rangle_0 = b^{-2D_g} \left\langle O_g'^\dagger(x') O_g'(y') \right\rangle_{0'} , \tag{60}$$

where the subcript 0 indicates that the average is calculated based on the bare critical action (as a function of the bare operators), whereas the subscript $0'$ indicates that the average is takes based on the rescaled action (as a function of the rescaled operators). For a critical system, furthermore, we should typically assume a power law behavior:

$$\left\langle O_g^\dagger(x) O_g(y) \right\rangle_0 \propto (x-y)^{-\lambda} = b^{-\lambda} (x'-y')^{-\lambda} . \tag{61}$$

Finally if the action is scale invariant, as in Eq. (55), we also know that:

$$\left\langle O_g'^\dagger(x') O_g'(y') \right\rangle_{0'} \propto (x'-y')^{-\lambda} . \tag{62}$$

This is due to the fact that the Lagrangian of a fixed point assumes the exact same form (and same coupling constants) when expressed as a function of the fields before and after the rescaling; this observation implies that the exponent λ must be the same before and after rescaling, since, formally, the calculation to derive the correlation functions in Eqs. (61) and (62) from the related actions is exactly the same.

Comparing the previous relations, we conclude that, for any scale invariant action, the two-point correlation function of a primary operator are of the form:

$$\left\langle O_g^\dagger(x) O_g(y) \right\rangle_0 \propto (x-y)^{-2D_g} . \tag{63}$$

This equation is quite important because it often allows us to obtain the scaling dimension D_g of a primary operator O_g from its correlation function calculated based on a massless (scale-invariant) Gaussian action.

C. A simple example: Gaussian theory with a mass in 1+1D

In the previous subsections, we defined the coarse graining and rescaling steps in the Wilson RG procedure. In particular, the rescaling of the fields was based on the behavior of a massless Gaussian model. We consider here, as a simple example of the full procedure, the case of a massive Gaussian model in $1+1D$:

$$\begin{aligned} S = \frac{1}{2\pi} \int d^2x \frac{g}{v} (\partial_t \varphi)^2 + gv (\partial_x \varphi)^2 + m^2 \varphi^2 = \\ = \frac{1}{2\pi} \int d^2k \varphi(-k) G^{-1}(k) \varphi(k) = \frac{1}{2\pi} \int dq d\omega \varphi(-q,-\omega) \left(\frac{g\omega^2}{v} + gq^2 v + m^2 \right) \varphi(q,\omega) , \end{aligned} \tag{64}$$

where $k = (\omega/v, q)$ is the energy-momentum vector.

We distinguish slow and fast modes by writing:

$$\varphi(x,t) = \varphi_s(x,t) + \varphi_f(x,t) . \tag{65}$$

where the slow term includes the modes with $|k| < \tilde{\Lambda}$ and the fast the ones with $\tilde{\Lambda} < |k| < \Lambda_{\max}$. For Gaussian theories the interaction part of the action is not present (which makes things much easier than usual):

$$S = S_s[\varphi_s] + S_f[\varphi_f] . \tag{66}$$

Given the non-interacting structure, the coarse graining step is indeed trivial, and we get:

$$Z = \int D\varphi_f D\varphi_s e^{-S_s[\varphi_s] - S_f[\varphi_f]} = \int D\varphi_s e^{-S_s[\varphi_s]} . \tag{67}$$

In this particular case we get:

$$S_s'[\varphi_s] = S_s[\varphi_s] = \frac{1}{2\pi} \int d^2x \frac{g}{v} (\partial_t \varphi_s)^2 + gv (\partial_x \varphi_s)^2 + m^2 \varphi_s^2 . \tag{68}$$

We now apply the rescaling:

$$dx = (1+dl)dx' \,,\; dt = (1+dl)dt' \,,\; \partial_x = \frac{\partial_{x'}}{1+dl} \,,\; \partial_t = \frac{\partial_{t'}}{1+dl} \,; \tag{69}$$

we get:

$$S'_s[\varphi_s] = \frac{1}{2\pi}\int d^2x' \frac{g}{v}\left(\partial_{t'}\varphi_s\right)^2 + gv\left(\partial_{x'}\varphi_s\right)^2 + (1+dl)^2 m^2 \varphi_s^2 \,. \tag{70}$$

From Eq. (58) we know that $\varphi_s(x) = \varphi'_s(x')$ (which is a special case for $D = 2$, namely $D_\varphi = 0$; in general D_φ must be taken into account). Hence:

$$S'[\varphi_s] = \frac{1}{2\pi}\int d^2x' \frac{g}{v}\left(\partial_{t'}\varphi'_s\right)^2 + gv\left(\partial_{x'}\varphi'_s\right)^2 + (1+dl)^2 m^2 \varphi_s'^2 \,. \tag{71}$$

By comparing the initial and the rescaled action we get:

$$g' = g \,, \quad v' = v \,, \quad m'^2 = (1+dl)^2 m^2 \,. \tag{72}$$

The last relation can be reformulated as:

$$\frac{dm'}{dl} = m \,. \tag{73}$$

This is the RG equation for the coupling constant m. Its solution is clearly:

$$m(l) = e^l m(0) \,. \tag{74}$$

We can associate *a scaling dimension to the coupling constant* m, and in this case, it is 1. The previous equation indicates that m becomes larger and larger under the RG flow. It is therefore a relevant perturbation of the gapless system at $m = 0$. It drives the system towards a fixed point where $m \to \infty$. By calculating the spectrum of the theory, you clearly see that m constitutes indeed a gap in the theory, therefore it breaks its property of being scale invariant.

D. Dimensional analysis

From the previous example, we understood that if the theory is Gaussian, the coarse graining is trivial and we are left with the rescaling only.

When you consider an interacting system, in general, things are different, and you should properly deal with the coarse graining as well. However, a preliminary and brutal understanding of the RG flow can be obtained by the simple rescaling and the analysis of the (canonical) scaling dimensions. Therefore, in the next example, we implement the rescaling step only, even though we are considering an interacting theory.

We consider the bosonic field action:

$$S = \int d^D x \, (\nabla\varphi)^2 + h\varphi + g\varphi^4 \,. \tag{75}$$

Based on Eq. (53) and the scaling of the bosonic field (58) we get the following rescaling:

$$\int d^D x \, h\varphi = \int d^D x' \, (1+dl)^D \, h \, (1-dl)^{\frac{D-2}{2}} \, \varphi' = \int d^D x' \, (1+dl)^{\frac{D}{2}+1} \, h\varphi' \,. \tag{76}$$

Therefore $h' = (1+dl)^{\frac{D}{2}+1} h$. This implies that h grows under the RG flow for each D, and this term is always a relevant perturbation of the Gaussian massless model.

Here, the operator coupled to the constant h is the field φ itself, which displays a scaling dimension D_φ. The associated coupling constant h flows with the exponent $(D - D_\varphi) > 0$. Since this exponent is indeed always positive, we conclude that the operator φ is relevant for any D.

We can repeat the same analysis for the g term. In this case, the scaling dimension of the operator φ^4 is $4D_\varphi$:

$$D_{\varphi^4} = 4D_\varphi = 2D - 4 \,, \tag{77}$$

which implies the relation $g' = g(1+dl)^{D-D_{\varphi^4}} = g(1+dl)^{4-D}$. This result implies in turn that:

- For $D < 4$, the interaction $g\varphi^4$ is relevant, because g grows with dl.
- For $D = 4$, $g\varphi^4$ is marginal, and additional analysis is required.
- For $D > 4$ (in case you have some weird theory in which this makes sense), $g\varphi^4$ is irrelevant. g decreases with the RG flow and it is supposed to be unimportant for the long-distance properties of the system.

This conclusions are only preliminary though, and a more accurate coarse graining, with higher-order perturbative terms would provide a more precise understanding of the system behavior.

The general situation is the following. We consider an interacting action term:

$$\int d^D x \, g O_g(x) \,, \tag{78}$$

where O_g is a suitable operator. The scaling dimension D_g of this operator is such that, at the leading level, Eq. (63) is fulfilled.

At first order, g scales with the exponent $D - D_g$, therefore:

- If $D - D_g > 0$, then $g' > g$ and the operator O_g is **relevant**.
- If $D - D_g < 0$, then $g' < g$ and the operator O_g is **irrelevant**.
- If $D - D_g = 0$, O_g is **marginal** and we need a higher order analysis to understand the general behavior.

This analysis has several implications. For example when you consider the operator ∇O_g, its scaling dimension is:

$$D\left(\nabla O_g\right) = D_g + 1 \,. \tag{79}$$

This implies that **$\nabla \mathbf{O_g}$ is always less relevant than $\mathbf{O_g}$** and, for understanding the long-distance / infrared behavior of a model, it can be neglected if also the interaction O_g is present.

In general we consider:

$$S = \int d^D x \sum_j g_j O_j \,. \tag{80}$$

In this case, the operator with the smallest scaling dimension D_{g_j} dominates and typically defines the thermodynamic properties of the system [if it is relevant!]. Sometimes, if several O_j are relevant and commute, they concur in determining the properties of the system. If they do not commute, they compete and the most relevant terms dominates.