# Introduction to Anyon models

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# Chapter 1

# Introduction

*We shall not cease from exploration*
*And the end of all our exploring*
*Will be to arrive where we started*
*And know the place for the first time.*

T. S. Eliot, Four Quartets, Little Gidding **V**

## 1.1 Topological order

The fascinating phenomenon of the Quantum Hall Effect, discovered by von Klitzing, Dorda and Pepper in 1980 [1], constitutes one of the most remarkable and unexpected development of condensed matter physics in the last decades. Its importance is not only related to the incredible accuracy of the quantization of the Hall conductivity, which is a striking manifestation of quantum phenomena at the mesoscopic scale, but it relies also on its connection with fundamental principles of physics. The theoretical challenge imposed by the understanding of the Quantum Hall Effect required to build a new paradigm in our knowledge of condensed matter systems: since the works by Laughlin [2] and Thouless [3] it was clear that a satisfactory explanation of such a robust and universal phenomenon had to be built on as much solid theoretical bases, as gauge invariance [2] and topological invariance [3, 4].

The topological properties of the quantum Hall conductivity became even more important after the experimental observation of the Fractional Quantum Hall Effect (FQHE), conducted by Tsui, Stormer and Gossard in 1982 [5]. This discovery made evident the existence of several quantum Hall states, characterized by different physical properties and labelled by their filling factor $\nu$, which can be regarded as different phases of a phase diagram. Nevertheless, their classification cannot be simply related to a local order parameter as prescribed by the usual Ginzburg-Landau approach, but requires the novel idea of a *topological order*, a long-range order of topological origin (see the reviews [6, 7]). Such condensed phases of matter are topological invariant, at least at small enough temperatures, and their properties are insensitive to local perturbation such as impurities or deformations. This behaviour, however, is not related to a given symmetry of the Hamiltonian describing the quantum Hall systems, but emerges as an effective symmetry at low energy. Therefore the order parameter classifying these phases doesn't arise from a broken symmetry of the Hamiltonian, as in the Ginzburg-Landau theory, but must be described in terms of non-local observables that define a new kind of ordering, universal and robust against arbitrary perturbation. To a certain extent we can thus consider the existence of a topological order as the contrary of a symmetry breaking: topological order does not require any preexisting symmetry of the Hamiltonian but it brings to new conservation laws of the system [8].

The topological nature of the quantum Hall states becomes explicit thanks to the description of such systems with an effective topological field theory in the low energy regime [9]. In particular quantum Hall systems can be modelled by Chern-Simons field theories (see [10] for a review), topological field theories of the kind first studied by Witten [11, 12]. One of the advantages of this approach is to allow for an abstract description of the different topological excitations of the quantum Hall systems, namely the anyonic quasiholes or quasiparticles we will discuss in the following, under the light of knot theory [12, 13, 14, 15] or tensor category theory [8, 16].

The most puzzling characteristic of FQHE is perhaps the fractional charge of the gapped excitations in these systems. This phenomenon, together with the electronic incompressibility that derives from the exceptional stability of the observed plateaux at fractional filling, cannot be explained in terms of non-interacting electrons. The most important contribution in the understanding of the structure of such states is due to Laughlin [17], who managed to describe the $\nu = 1/3$ plateau in terms of the well-known wavefunction named after him. After his seminal work it became clear that the charged excitations of the fractional quantum Hall states are not, in general, fermions or bosons, but obey more exotic anyonic statistics [18]. Moreover, with generalizations of the Laughlin wavefunction, it is possible to build hierarchies of states which describe the quantum Hall plateaux with an odd denominator filling [19].

Yet another experimental milestone of the study of the Quantum Hall Effect gave rise to new and deeper theoretical investigations. In 1987 Willett *et al.* [20] observed a quantum Hall plateau having an even denominator filling, the celebrated $\nu = 5/2$ quantum Hall state. This state cannot be directly related to the previously studied wavefunctions and brought to the necessity of finding new tools in the descriptions of such systems. Even if, so far, there is no completely unambiguous description of the $\nu = 5/2$ state, the main proposals done are based on the idea by Moore and Read [21] of writing the quantum Hall wavefunctions as correlation functions of Conformal Field Theories (CFTs). In particular they proposed a new wavefunction, the Pfaffian state, adopting the Ising minimal model to describe the

$\nu = 5/2$ state. This brings to the revolutionary notion of non-Abelian anyons that we will analyze in the following chapter.

CFTs [22] are a useful tool to analyze quantum Hall states, and, in general, to investigate systems showing a topological order. Their relation with the Quantum Hall Effect was already implicit in the Chern-Simons description of these systems, since it can be shown that such topological field theories can be mapped in CFTs [12]. After the work by Moore and Read, CFTs had been successfully used to describe both quantum Hall states and their edge modes, and their study can be put at the basis of most of the anyonic models (see [16] and references therein).

The connection between CFTs and topological order gave rise to many models showing topological properties beside the Quantum Hall Effect. The simplest examples are constituted by the celebrated Kitaev's models: the toric code giving rise to Abelian anyons [23], and the honeycomb lattice model characterized by non-Abelian Ising anyons [8]. Both of them are spin models on a lattice whose topological structure emerges studying the characteristics of the ground state and its low energy excitations. More abstract and general models featuring a topological order can be built starting from loop or RSOS models [24, 25, 26], and, finally, their most general formulation can be accomplished through the so called string-net models [14, 15] that emphasize the mathematical framework underlying topological phases and allow us to realize a Hamiltonian for every tensor category theory. Moreover it has been shown that such structures can be generalized also in three dimensions and their partition function can be expressed in terms of knot invariants [27].

Since the works by Kitaev, Freedman and Wang [28, 29, 30, 23] it was realized by several authors [31, 32, 33, 34, 35, 36, 37, 38] that topological order can be regarded also as a resource. Quantum systems showing topological properties can be exploited to overcome the decoherence phenomena and the intrinsic noise sources that affect all the main schemes to achieve quantum computation. This is the main idea at the origin of *Topological Quantum Computation*: to encode and manipulate information in a topologically protected way, insensitive to local sources of errors (see the reviews [16, 39]). In particular the possibility of storing information is related to the topological degeneracy of the ground states typical of systems characterized by Abelian anyonic excitation, as, for example, the toric code [23]; whereas the construction of quantum gates relies on the nontrivial braiding properties of non-Abelian anyons we will analyze in the following.

Topological quantum computation and the study of topological states of matter evolved together in the last decade. The appealing quantum computation schemes offered by the manipulation of non-Abelian anyons prompted the research of new physical systems suitable to present such quasi-particles as excitations above their ground state. Beside the previously mentioned studies on Quantum Hall Effect, based on two-dimensional electronic gas in high-mobility semiconductor structures, in the last years new theoretical and experimental proposals were considered, in order to obtain physical systems with nontrivial topological properties. Among them I would like to mention the studies on quantum Hall regime in cold atomic gases (see [40] for an extensive review) and the works on p-wave superconductors [41] that are strongly related to the $\nu = 5/2$ Hall state [42]. Finally the new research field on topological insulators allowed to extend the notion of topological order to several physical systems and models characterized by different symmetries (see [43, 44] and references therein) and it seems to offer the possibility of realizing Majorana fermions (and therefore non-Abelian Ising anyons) in one-dimensional systems characterized by both an s-wave superconductor pairing and a strong spin-orbit interaction [45, 46, 47].

Anyons are the main feature linking these systems and their existence is intrinsically related to a non trivial topological order. Moreover, they are the key to encode and manipulate information in topological quantum computation; therefore, in the next section, I will introduce their main characteristics and the mathematical tools to describe them.

## 1.2 Anyons and the braid group

A main feature of quantum theories is the idea of indistinguishable particles, which implies that the exchange of two identical particles in a system is a symmetry which can be related to a unitary operator on the Hilbert space describing that system. In a three-dimensional space the well known spin-statistics theorem states that particles must be bosons or fermions, and their exchange operator is described respectively by the identity or by the multiplication of a factor -1. In two spacial dimensions, however, the spin-statistics theorem does not hold and this opens the possibility of having a much wider variety of particle statistics: indistinguishable particles that are neither bosons nor fermions are called *anyons*.

In general we can state that, for fermions or bosons, a system of identical particles can be described by states which depend only on the permutation of these particles. The wavefunctions describing these states are independent on the past configurations of the particles and, in particular, on the worldlines describing the past evolution of the system. The permutation group, in this case, is enough to identify every possible configuration of the system and the main feature that characterizes the behaviour of bosons and fermions is that the operator $\sigma$ associated to the exchange of two particles is its own inverse: $\sigma^2 = 1$.

Anyons are described, instead, by the more general *braid group*. The time evolution of a two-dimensional system of identical particles is defined by the pattern of the worldlines of these particles which constitute the strands of a braid (see Fig. 1.1). Since the worldlines are forbidden to cross, such braids fall into distinct topological classes that cannot be smoothly deformed one to another. In this case the exchange of two particles can happen counterclockwise, $\sigma$, or clockwise, $\sigma^{-1}$, and, in general, $\sigma^2 \neq 1$. Moreover the temporal order of the exchanges is important because different orders bring to non-equivalent braids, and this is the basis of the topological degeneracy of anyonic systems.

Let us briefly summarize the main characteristics of the braid group, which will be useful to the purpose of topological quantum computation. A more extensive introduction to the subject can be found in [48]. The braid group is generated by the counterclockwise exchange operators $\sigma_i$ between the pair $(i, i{+}1)$ of neighboring particles. If we consider the possible braids of $n$ strands, corresponding to the

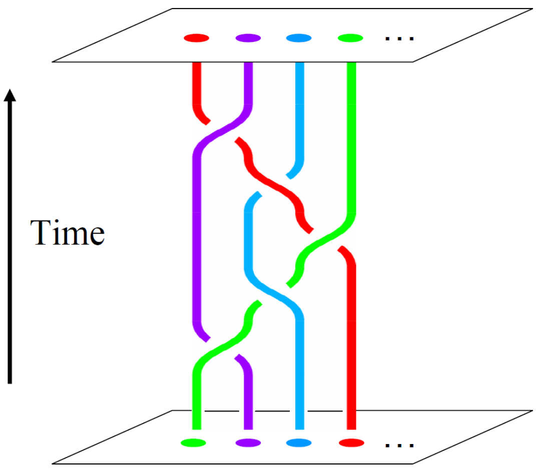



Figure 1.1: The worldlines of anyons in a two dimensional space are represented by braids constituted by the ordered exchanges $\sigma$ of the particles.

worldlines of $n$ anyons, this group is characterized by the following relations:

$$\sigma_i\sigma_j = \sigma_j\sigma_i \qquad \text{if} \quad |i-j|>1\,, \tag{1.1}$$
$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \qquad \text{for} \quad i=1,...,n-2 \tag{1.2}$$

The first equation just states that the exchanges of disjoint pairs of anyons commute, whereas (1.2) is the Yang-Baxter relation which can be easily verified observing Fig. 1.2: both the terms in the equation represent two particles exchanging their position by encircling a third one.

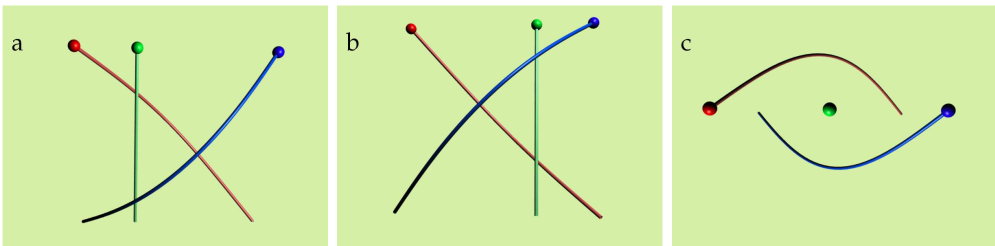



Figure 1.2: The Yang-Baxter relation 1.2 is illustrated: **a)** represents the anyonic worldlines in 2+1 dimensions corresponding to the left hand side of Eq. 1.2, **b)** corresponds to the right hand side and **c)** depicts the trajectories of the anyons on the plane.

The braid group admits an infinite number of unitary irreducible representations: the Abelian anyons are indistinguishable particle that transform as a one-dimensional representation of the braid group; in this case each generator of the braid group is associated with the same phase $e^{i\theta}$, in particular the case $\theta=0$ corresponds to bosons whereas $\theta=\pi$ corresponds to fermions. The Abelian anyons, first studied by Wilczek [49], are present as localized and gapped quasiholes and quasiparticles in the Abelian states of FQHE and, in general, are characterized by a fractional charge, as first observed in 1995 [50].

The peculiar statistics of Abelian anyons can be understood once we consider them as composite particles constituted by a unitary flux $\Phi_0 = hc/e$ and a fractional charge, which, in the most common case of a Laughlin state at filling $\nu$, corresponds to $e^* = \nu e$. This description allows us to associate an Aharonov-Bohm phase to their exchange: when an Abelian anyons moves around another quasihole corresponding to a flux quantum, the resulting Aharonov-Bohm phase acquired by their wavefunction is $e^* \Phi_0/\hbar c = 2\pi\nu$. Therefore the exchange of two Abelian anyons is characterized by a phase $\nu\pi$ and the fractional charge determines the statistics of such particles (see [51, 52, 53] for extensive introductions about anyons in FQH systems). There are cases, however, in which the statistics cannot be easily deduced only from the Aharonov-Bohm effect and must be calculated explicitly through the interplay between Berry phases and the monodromy of the wavefunctions involved [18].

The existence of quasiholes with a fractional charge can be easily related to the quantum Hall conductance through a simple gedanken experiment due to Laughlin: let us consider a sample of quantum Hall liquid at filling factor $\nu$ and imagine to pierce it with an infinitely thin solenoid as in Fig. 1.3, creating an annulus. Adiabatically increasing the magnetic flux $\Phi_B(t)$ inside the solenoid from 0 to $\Phi_0$ causes, by Maxwell's laws, an azimuthal electric field $E \propto \frac{d\Phi_B}{dt}$ to arise. This electric field generates, in turn, a radial current whose intensity $j = \sigma_H E = \nu E e^2/h$ is determined by the fractional Hall conductance. Once the flux reaches the value $\Phi_0$, the total amount of charge transferred from the inner edge of the annulus to the outer one is $e^* = \nu e^2 \Phi_0/hc = \nu e$ and this amount of charge can be considered as the charge of the excitation created through the insertion of a flux quantum. This simple argument shows that an excitation characterized by a flux quantum has a fractional charge proportional to the filling factor of the quantum Hall liquid considered. Therefore the existence of quasiholes with a fractional charge is intrinsically related to the fractional Hall conductance simply by Maxwell's equations.

The last property of Abelian anyons I would like to mention is the topological degeneracy that, in principle, characterizes systems with a nontrivial topology. In every anyonic model there is a ground state corresponding to the absence of anyons. On the plane such state is unique, but on two-dimensional surfaces with nontrivial topology, such as the torus, this vacuum state becomes degenerate [48]. Such degeneracy depends on the existence of non-local operators that commute with the Hamiltonian but do not commute with each other. The most celebrated example of this degeneracy is found in the so called toric code [23]: in this case the Abelian anyons are constituted by electric and magnetic charges, whereas the non-local operators correspond to drag different Abelian anyons along the two possible inequivalent circumferences of the torus. These operators do not affect the energy but allow to divide the ground state space into four topological sectors that cannot be locally distinguished. In this way one can store information in a topologically protected way inside such a system, however the Abelian nature of the system makes it unsuitable to manipulate this information.

So far we discussed the characteristics of Abelian anyons and we mentioned that, from a mathematical point of view, their behaviour corresponds to a one-

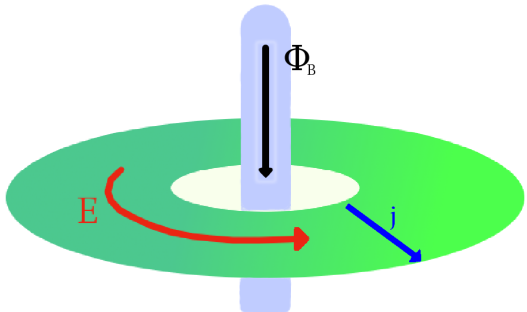



Figure 1.3: The adiabatic insertion of a magnetic flux in a quantum Hall liquid induces an azimuthal electric field which, in turn, implies a radial flow of a charge $\nu e$ due to the quantum Hall conductance.

dimensional representation of the braid group, whereas from the physical point of view, their statistics is usually described through suitable wavefunctions related to different Abelian quantum Hall states. There are, however, also higher-dimensional representations of the braid group that give rise to non-Abelian anyons: particles whose exchanges are defined by nontrivial unitary operators, corresponding to the braid generators $\sigma_i$, that, in general, do not commute with each other.

In particular the non-Abelian states of matter are characterized by a novel topological degeneracy: a collection of non-Abelian anyonic excitations with fixed positions spans a multi-dimensional Hilbert space and, in such a space, the quantum evolution of the multi-component wavefunction of the anyons is realized by braiding them. As we will discuss in the next chapter, the quantum dimension of this space is determined by peculiar characteristics of the considered anyonic model; and the operators defined by the exchange of these particles can be naturally represented by unitary matrices and constitute the building blocks to realize a topological quantum computation [16].

The signature of the anyonic properties of the quantum Hall excitations is their nontrivial evolution under braiding; therefore it is natural to probe this behaviour via interference measurements (see [16, 52, 54] and references therein for detailed reviews on the subject). Interferometry allows not only to detect the charge value of the excitations through the Aharonov-Bohm effect, but also to distinguish whether their nature is Abelian or non-Abelian. Concerning the charge measurements, for the $\nu = 1/3$ quantum Hall state the charge $e^* = e/3$ was first observed in 1995 [50]; after that, more precise observations were achieved by using measurements of quantum shot noise [55]. Regarding the $\nu = 5/2$ state there are several measurements of the fractional charge $e/4$ of the elementary excitations based both on shot noise and interferometry (see, for example, [56, 57]); and recently, some convincing indication about its non-Abelian nature was obtained [58], coherently with the Moore and Read description of the bulk and edge states. From the quantum computation point of view, interference could provide a useful tool to distinguish the different states obtained after the braidings of several non-Abelian anyons, therefore it corresponds to the state measurement required for computation and it constitutes the main instrument to investigate anyons.

The following step towards the creation of a topological quantum computer would be the experimental realization of a system that allows us not only to detect non-Abelian anyons, but also to manipulate their positions in order to physically implement topologically protected quantum gates through their ordered braidings. Unfortunately, gaining such a control in the semiconductor quantum Hall systems we mentioned seems to be a prohibitive task; therefore we are justified in the search for new alternative systems characterized by non-Abelian anyons.

# Chapter 2

# Anyon Models

In this chapter we will investigate the theoretical structure of anyon models emphasizing the elements that are essential to the purpose of topological quantum computation. In the last section we will describe, as an example, the model of Fibonacci anyons, which is not only the simplest non-Abelian anyon model, but also the main example of anyons supporting fully-protected universal quantum computation.

The mathematical structure underlying anyon models is extremely rich and it is related to the study of unitary tensor categories, whose discussion is far beyond the purpose of this thesis. Here we will analyze only the main feature of anyon models; broader introductions to the subject can be found in [8, 16, 48, 54].

The structure of anyon models originated from CFT [22] and can be understood under the light of conformal models. The connection between FQHE and CFT is very strong: as we already mentioned trial wavefunctions for the bulk of quantum Hall liquids can be retrieved by two-dimensional correlation functions in CFTs; moreover, the edge states in QHE are successfully described by 1+1 dimensional CFTs. CFTs are therefore powerful tools to investigate the topological order of anyon models and provide the basic instruments that are necessary for their complete description. In particular the definition of an arbitrary anyon model relies on the following elements which are strictly related to several features of CFTs:

- **Topological superselection sectors**: each anyon model is defined starting from the set of all the possible anyons in the model. Every different anyon is characterized by a topological charge (or spin) and corresponds to a primary field (in the holomorphic sector);

- **Fusion Rules**: the outcome of the fusion between two anyons with arbitrary topological charges can be described by the fusion rules of the corresponding primary fields;

- **Associativity Rules** ($F$ matrices): the fusion process of 3 anyons can be related to a 4-point correlation function in a CFT. The different fusion channels in the anyon model correspond to different expansions of the correlation function in the conformal blocks: the $F$ matrices allow in both cases to describe the process choosing different fusion channels and thus are related to the crossing symmetry in conformal models;

- **Braiding Rules** ($R$ matrices): the braiding rules for the anyons are strictly connected to the monodromy of the corresponding primary fields in the holomorphic sector.

## 2.1 Fusion rules and quantum dimensions

The first ingredient to define an anyon model is a finite set $\mathcal{C}$ of superselection sectors which are labelled by conserved topological charges and can be related to the primary fields (in particular to their holomorphic components) in a given CFT. Among them, every anyon theory presents the vacuum (or identity) sector, $1 \in \mathcal{C}$, corresponding to the absence of a topological charge. These topological sectors characterize the properties of the anyons included in the model and must obey a commutative and associative fusion algebra that states the possible topological charges obtained considering a pair of anyons:

$$a \times b = \sum_{c \in \mathcal{C}} N_{ab}^{c} c \tag{2.1}$$

where, in general, the fusion multiplicities $N_{ab}^{c}$ are non-negative integers and specify the number of different ways the charges $a$ and $b$ can give the charge $c$ as an outcome. However, to the purpose of this thesis, it will be sufficient considering these coefficient to be 0 or 1. The identity sector 1 is defined by the relation $N_{a1}^{c} = \delta_{ac}$ and, in the models we will consider, every topological sector is its own conjugate (*self-duality*): $N_{aa}^{1} = 1$ (even if, in general, for every charge $a$ there exists an anti-charge $\bar{a}$).

The commutativity and associativity of the fusion algebra imply that:

$$N_{ab}^{c} = N_{ba}^{c} \tag{2.2}$$

$$\sum_{e} N_{ab}^{e} N_{ec}^{d} = \sum_{f} N_{af}^{d} N_{bc}^{f}, \tag{2.3}$$

moreover, in the case of self-duality of all the charges, $N_{ab}^{c} = N_{ac}^{b}$ and, for each charge $a$, we can define a symmetric fusion matrix $N_a \equiv N_{ab}^{c}$ that describes the fusion outcome of an arbitrary state with the charge $a$.

The theory is non-Abelian if there is at least a pair of charges $a$ and $b$ such that:

$$\sum_{c} N_{ab}^{c} > 1 \tag{2.4}$$

and the charge $a$ corresponds to a non-Abelian anyon if $\sum_c N_{aa}^{c} > 1$, whereas $a$ is Abelian if $\sum_c N_{ab}^{c} = 1$ for every $b$.

For every outcome $c$ of the fusion $a \times b$ we can associate a fusion space $V_{ab}^{c}$ whose dimension is given by $N_{ab}^{c}$ (1 in the models we will examine). When we consider a set of different anyons $a_1, ..., a_n$ whose total charge is $c$, the Hilbert space of the states describing this system can be decomposed in the following form:

$$V^c_{a_1 \dots a_n} = \bigoplus_{b_1,\dots,b_{n-1}} V^{b_1}_{a_1 a_2} \otimes V^{b_2}_{b_1 a_3} \otimes V^{b_3}_{b_2 a_4} \otimes \dots \otimes V^c_{b_{n-2} a_n}. \tag{2.5}$$

This Hilbert space cannot be decomposed in the tensor product of subsystems corresponding to the initial anyons $a_i$ but rather it is described in terms of a direct sum of tensor products of the fusion spaces. The dimension of the Hilbert space $V^c_{a_1 \dots a_n}$ can be expressed in terms of the fusion matrices $N_{a_i}$:

$$\dim \left(V^c_{a_1 \dots a_n}\right) = \left(N_{a_2} N_{a_3} \dots N_{a_n}\right)^c_{a_1} \tag{2.6}$$

If we assume that all the anyons $a_i$ are of the same sector $a$, we obtain an expression for the dimension of the Hilbert space describing $n$ non-Abelian anyons with charge $a$:

$$D_{a,n} = \sum_c \left(N_a^{n-1}\right)^c_a \simeq d_a^n \tag{2.7}$$

where we introduced the *quantum dimension* $d_a$ associated to anyons with a charge $a$. Such quantum dimension can be defined as the Perron-Frobenius eigenvalue of the matrix $N_a$ and, in general, it is not an integer number. For non-Abelian anyons $d_a > 1$ and it is important to notice that from the fusion relations one can derive

$$d_a d_b = \sum_c N^c_{ab} d_c \tag{2.8}$$

that generalizes the usual relations of the dimensions of the irreducible representations of the unitary groups. The previous relation can be written also as $N_a \vec{d} = d_a \vec{d}$ where $\vec{d} = d_a, d_b, ...$ with $a, b, ... \in \mathcal{C}$ is the eigenvector associated to the eigenvalue $d_a$ of the fusion matrix $N_a$. The norm of the vector $\vec{d}$

$$\mathcal{D} = \sqrt{\sum_{i \in \mathcal{C}} d_i^2} \tag{2.9}$$

is the total quantum dimension of the anyon model.

The equation (2.7) relates the dimension of the Hilbert space of $n$ equal anyons with their quantum dimension: to the purpose of topological quantum computation we can therefore state that the quantum dimension of an anyon refers to the quantity of information that a system of such anyons can store; besides, it is interesting to notice that such information is encoded in a non-local and topological protected way: in fact, to label each state in the Hilbert space, it is necessary to consider the sequence $b_i$ of the fusion outcomes in (2.5), that, involving all the anyons in the system, is a non-local observable. In order to understand better the structure of this kind of Hilbert space it is useful to introduce the Brattelli diagrams which allow to easily count all the orthogonal states in the system. Let us consider as an example the model of *Ising anyons*, corresponding to the $\mathcal{M}_{4,3}$ conformal minimal model [22, 59]. The superselection sectors are the vacuum

1, the spin sector $\sigma$ (non-Abelian Ising anyon) and the fermionic sector $\varepsilon$. The corresponding (nontrivial) fusion rules are:

$$\sigma \times \sigma = 1 + \varepsilon\,, \qquad \sigma \times \varepsilon = \sigma\,, \qquad \varepsilon \times \varepsilon = 1. \tag{2.10}$$

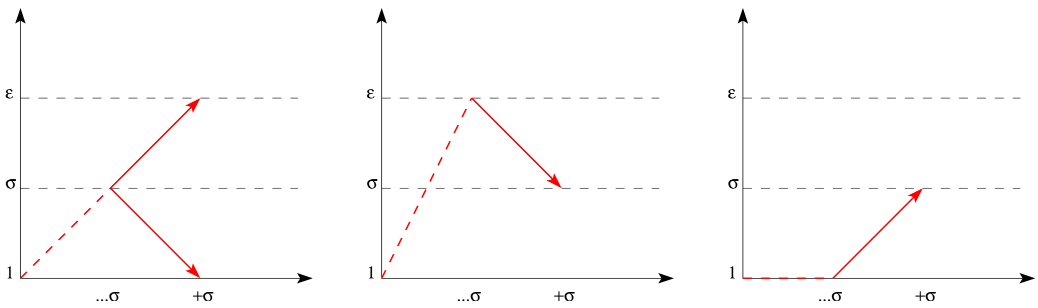
From these rules it is easy to calculate the quantum dimensions exploiting the equation (2.8):

$$d_\varepsilon = 1\,, \qquad d_\sigma^2 = 1 + 1 \quad \Rightarrow \quad d_\sigma = \sqrt{2} \tag{2.11}$$

In the basis $(1, \varepsilon, \sigma)$ the fusion matrix $N_\sigma$ reads:

$$N_\sigma = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \tag{2.12}$$

This non-Abelian fusion matrix can be represented through the diagrams in Fig. 2.1.




$\sigma \times \sigma = 1 + \varepsilon$ $\qquad$ $\varepsilon \times \sigma = \sigma$ $\qquad$ $1 \times \sigma = \sigma$

Figure 2.1: The Brattelli diagrams for the fusion rules of the non-Abelian Ising anyon $\sigma$ are shown.

The Brattelli diagrams help us to describe the Hilbert space of a set of non-Abelian anyons. Let us consider a chain of $\sigma$ anyons (see Fig. 2.2): the total charge of the first two of them can be either 1 or $\varepsilon$, therefore the first pair of Ising anyons is correctly described by a two-state Hilbert space, that highlights the quantum dimension $d_\sigma = \sqrt{2}$ of a single Ising anyon. Adding a third $\sigma$ to the first pair, the total charge must be $\sigma$, since the Ising anyons are characterized by an odd parity unlike the vacuum and fermionic sectors, as shown by the fusion rules (2.10). Therefore, in this case, the third anyon doesn't add new states to the Hilbert space. After the fusion of the fourth one, instead, there is again a double possibility: the total charge of $4\sigma$ can be either 1 or $\varepsilon$ and, therefore, a system with 4 Ising anyons is described by 4 states. For each pair of anyons, the number of states in the Hilbert space is doubled since their total charge can be either 1 or $\sigma$.

The Brattelli digram in Fig. 2.2 illustrates the doubling of the number of states at each even anyon. Every point of the diagram represents the total charge of the system, $1, \varepsilon$ or $\sigma$, as a function of the number of non-Abelian anyons considered and

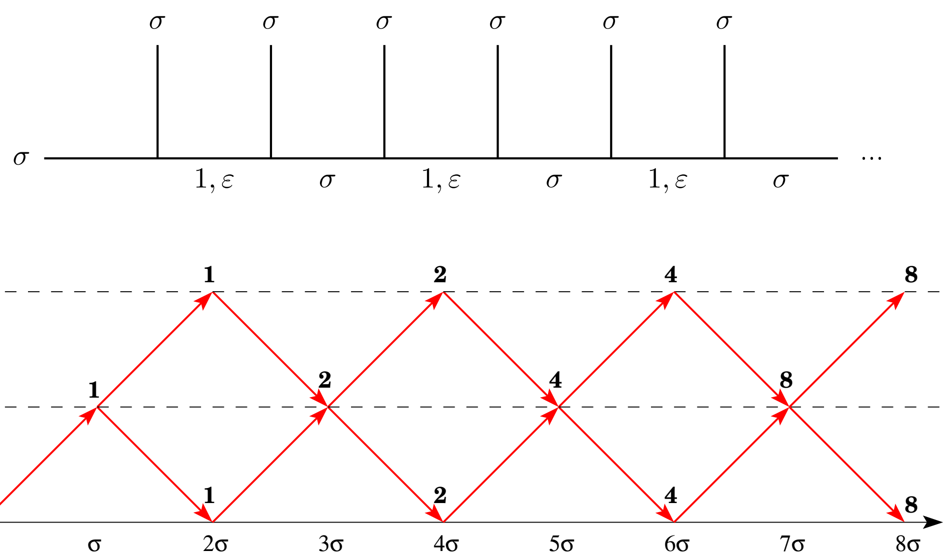



Figure 2.2: Fusion graph and Brattelli diagram corresponding to a chain of non-Abelian Ising anyons. The chain is composed by $\sigma$ anyons and the possible fusion outcomes are indicated. Depending on the parity of the anyons considered, their total charge is given by $\sigma$ (odd number of anyons) or by the two possible charges 1 and $\varepsilon$ (even number of anyons). In the Brattelli diagram the number of independent states at each fusion is shown: they correspond to the different paths along the diagram.

it is labelled by the number of orthogonal states characterized by the corresponding charge. These states can be visualized as the different paths on the Brattelli diagram and their number grows as $2^{\frac{n}{2}}$ where $n$ is the number of Ising anyons considered.

The fusion rules of anyon models can be related to several physical processes; in particular they allow to describe the amplitudes of scattering processes among anyons since the probability $p(ab \to c)$ that the total charge of the anyons $a$ and $b$ is $c$ is given by [48]:

$$P\left(ab \to c\right) = \frac{N_{ab}^{c} d_c}{d_a d_b}. \tag{2.13}$$

In particular, if $a$ is self-dual, one obtains that the quantum dimension $d_a$ is linked to the probability that two equal anyons annihilate: $p(aa \to 1) = d_a^{-2}$. Finally, considering the steady state distribution of an hypothetical anyonic gas, the anyons with charge $a$ appear with a probability [48]:

$$p_a = \frac{d_a^2}{\mathcal{D}^2}. \tag{2.14}$$

Therefore, if anyons are created through a random process, those with a charge associated to a larger quantum dimension are more likely to be produced.

As we previously mentioned, anyon models can be strictly related to conformal models; under this point of view the fusion rules of topological superselection sectors can be considered as the operator product expansions of primary fields:

$$\phi_a(z_1)\phi_b(z_2) = \sum_c N_{ab}^c \frac{\phi_c(z_2)}{(z_1 - z_2)^{\Delta_a + \Delta_b - \Delta_c}} \tag{2.15}$$

where $z_1$ and $z_2$ are the positions in complex coordinates of the anyons $a$ and $b$, and $\Delta_i$ is the conformal dimension of the field $\phi_i$ in the holomorphic sector. This is the key relation to map conformal models into anyon models and, as we will see in the next sections, the conformal weights play an important role in defining the monodromy matrices, and thus the braiding rules, characterizing anyon models.

## 2.2 Associativity rules and $F$-matrices

The fusion of topological sectors is associative:

$$(a \times b) \times c = a \times (b \times c)\,. \tag{2.16}$$

This simple mathematical requirement corresponds to the fact that the total topological charge of three anyons is an intrinsic property of the particles and must not depend on the fusion path one chooses to evaluate the final outcome. The associativity of the fusion rules implies, at the level of the fusion matrices $N$, the equation (2.3) which represents the fusion of three particles $(abc) \to d$; an analogous relation can be written for the fusion space $V_{abc}^d$ defined by equation (2.5):

$$V_{abc}^d = \bigoplus_x V_{ab}^x \otimes V_{xc}^d = \bigoplus_y V_{ay}^d \otimes V_{bc}^y, \tag{2.17}$$

where the two expressions correspond to different fusion decompositions as shown in Fig. (2.3).

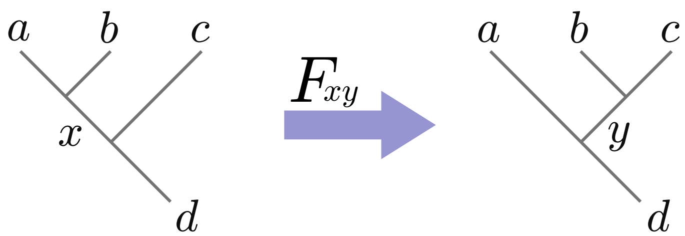



Figure 2.3: The $F$-matrix describes a change of basis between two equivalent representations of the same fusion space $V_{abc}^d$ (see eq.(2.18)).

Since there are two equivalent decompositions of the fusion space $V_{abc}^d$, it is natural to consider the isomorphism between the two spaces in equation (2.17). This isomorphism is usually called an $F$-move and corresponds to the following change of basis:

$$|ab \to x\rangle \otimes |xc \to d\rangle = \sum_y \left(F_d^{abc}\right)_{xy} |ay \to d\rangle \otimes |bc \to y\rangle \tag{2.18}$$

where the states on the left-hand side are elements of $V^x_{ab} \otimes V^d_{xc}$ and the ones on the right-hand side correspond to $V^d_{ay} \otimes V^y_{bc}$. The matrix $\left(F^{abc}_d\right)_{xy}$ depends on the charges $a, b, c$ and $d$ involved in the fusion process and its indices $x$ and $y$ run, in general, over all the possible sectors of the model. In the most common cases the $F$ matrix is a unitary matrix $\left(F^{abc}_d\right)^{-1} = \left(F^{abc}_d\right)^{\dagger}$ and the $F$-moves can be thought as a generalization of the $6j$-symbols arising in the tensor product of the group representations; however, starting from non-unitary conformal models, it is possible to build anyon models characterized by non-unitary $F$-matrices [60] and it is interesting to notice that systems having the same topological sectors and fusion rules may differ for the $F$ matrices.

In order to generalize the $F$-moves for every number of anyons, one must enforce a consistency relation which assures that the isomorphism obtained as a change of basis of the fusion spaces depends only on the initial and final decomposition of the space, and not by the particular sequence of moves from which the isomorphism is constructed. This consistency condition is called *pentagon equation* and it is represented in Fig. 2.4.

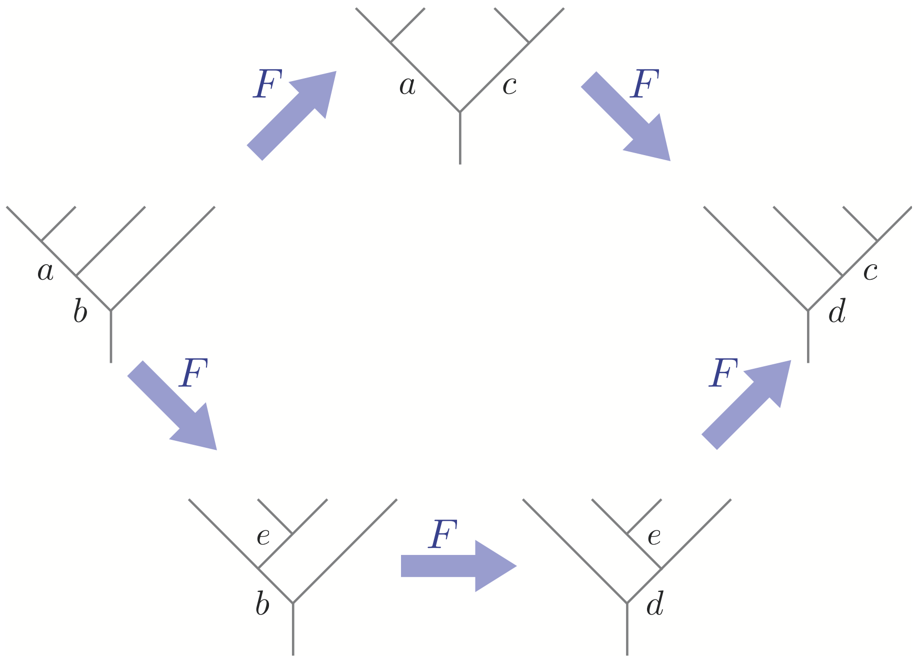



Figure 2.4: The same decomposition of the fusion space of four anyons can be obtained through different sequences of $F$-moves. The pentagon equation (2.19) assures the consistency of the two sequences of decompositions illustrated. (Taken from [61]).

In particular the pentagon equation relates two different sequences of $F$-moves for the fusion of four anyons, from the 'left-ordered basis' to the 'right-ordered basis' as shown in Fig. 2.4. If we label with $\alpha, \beta, \gamma$ and $\delta$ the four anyons and we assume that their total charge is $\tau$, the pentagon equation reads:

$$\sum_e \left(F_d^{\beta\gamma\delta}\right)_{ec} (F_\tau^{\alpha e\gamma})_{bd} \left(F_b^{\alpha\beta\gamma}\right)_{ae} = \left(F_\tau^{\alpha\beta c}\right)_{ad} \left(F_\tau^{a\gamma\delta}\right)_{bc} \tag{2.19}$$

This equation involves only four anyons, however a well-known result in tensor category theory, the MacLane coherence theorem, assures that this consistency equation is sufficient to enforce the consistency of all the possible sequences of $F$-moves for every number of anyons (see [8] and references therein). Besides the pentagon equation provides a first constraint to explicitly determine the expression for the $F$ matrices; as we will see in the next section, the other main constraint is the hexagon equation, involving the braidings matrices and the Yang-Baxter relations. Moreover, in general, the unitary $F$-matrices are also linked to the quantum dimensions of the anyons by the following relation:

$$(F_a^{aaa})_{11} = \frac{1}{d_a} \tag{2.20}$$

which is related to the possibility of describing anyon models in terms of loop or string-net models (see, for example, [26, 15]) and of calculating their partition function.

The physical meaning of the $F$-moves can be related to an elastic scattering process (see Fig. 2.5) and to the corresponding crossing symmetry. Let us consider a system of four anyons, $a, b, c$ and $d$, whose total topological charge is trivial. This neutrality constraint imposes the relation $a \times b = c \times d$ but also $a \times d = b \times c$ (under the hypothesis of self-duality); thus the fusion of the four anyons can be described in two different ways that correspond to the $s$ and $t$ channels of an elastic scattering process of two particles.

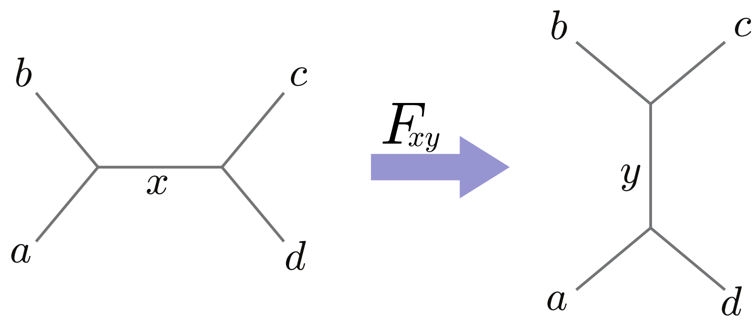



Figure 2.5: The $F$-moves allow to describe the crossing symmetry of the fusion of four anyons. The graph in this figure is topologically equivalent to the one in Fig. 2.3.

The $F$-moves allow to relate the $s$ and $t$ channels and define the change of basis between the two fusion spaces. We can consider the example of four Ising anyons ($a = b = c = d = \sigma$). Knowing that the total charge of the Ising anyons $a$ and $b$ is

$\varepsilon$ the probability of the fusion outcome between the anyons $b$ and $c$ is given by:

$$P(\sigma\sigma \to 1) = |(F_{\sigma}^{\sigma\sigma\sigma})_{\varepsilon\,1}|^2 = \frac{1}{2} \tag{2.21}$$

$$P(\sigma\sigma \to \varepsilon) = |(F_{\sigma}^{\sigma\sigma\sigma})_{\varepsilon\,\varepsilon}|^2 = \frac{1}{2} \tag{2.22}$$

since the $F$ matrix of the Ising model reads (see, for example, [8, 54]):

$$F_{\sigma}^{\sigma\sigma\sigma} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \tag{2.23}$$

Under the point of view of conformal field theory, the $F$-matrices describe the crossing transformation of the four-point correlation functions [22]; they can be interpreted as a change of basis of the conformal blocks. Adopting the Dotsenko and Fateev approach [62], the four-point function of a primary field in a minimal model can be expressed in two equivalent forms:

$$\langle \phi(0)\phi(1)\phi(z)\phi(\infty)\rangle = \alpha I_1(z) + \beta I_2(z) = \alpha' I_1'(1-z) + \beta' I_2'(1-z) \tag{2.24}$$

where the $I_i$ are hypergeometric functions [22, 59, 62]. The two expressions correspond to different decompositions in conformal blocks: in the first case the two-point function of the fields in $z$ and in 0 is considered, in the second one, instead, the correlation between the fields in $z$ and 1 is calculated. Once the hypergeometric functions $I_i$ are properly normalized, the $F$-matrix allows to express $I_i(z)$ as a linear combination of $I_1'(1-z)$ and $I_2'(1-z)$ which correspond to a different integration contour in the expression of the four-point correlation function (see [22, 62] for more detail). The $F$-matrix can be therefore calculated by evaluating this change of basis between the conformal blocks and by imposing the unitarity constraint or other conditions.

## 2.3 Braiding rules

The exchange of two anyons on the plane does not affect their total charge $c = a \times b$. Therefore their counterclockwise braiding defines an isomorphism $R_{ab}^c$ between the fusion spaces $V_{ab}^c$ and $V_{ba}^c$. Since we are considering only models where the dimension of the fusion spaces $V_{ab}^c$ is 1, $R_{ab}^c$ is simply determined by a phase and its inverse $(R_{ab}^c)^{-1}$ corresponds to a clockwise exchange of the anyons $a$ and $b$. Starting from all the possible outcomes of the fusion $a \times b$ it is possible to build the *braiding operator* $R_{ab} = \bigoplus_c R_{ab}^c$, also known as $R$-move (see Fig. 2.6).

Applying twice the braiding operator $R$ we obtain the *monodromy operator*:

$$R_{ab}^2 : V_{ab}^c \to V_{ab}^c = \bigoplus_c (R_{ab}^c)^2 \tag{2.25}$$

This unitary operator corresponds to winding counterclockwise one anyon around the other and, similarly to $R_{ab}$, can be expressed in the basis given by the possible total charges $c$. The monodromy operator $R^2$ can be easily derived from the

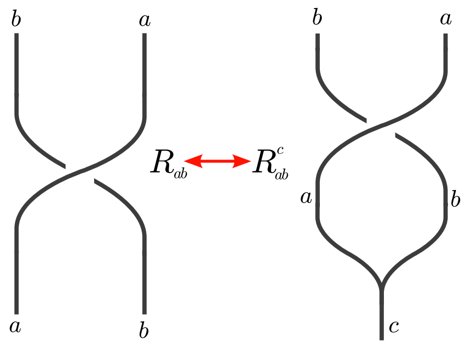



Figure 2.6: The counterclockwise exchange of two anyons leaves invariant their total charge. Therefore the operator $R_{ab}$ can be decomposed in its projections $R^c_{ab}$ on the fusion spaces $V^c_{ab}$ characterized by a definite total charge $c$.

conformal field theory underlying the anyon model: let us consider the pair of anyons $a \times b$ which obey a fusion rule related to the operator product expansion in equation (2.15); from this equation it is evident that the monodromy operator corresponds to:

$$\left(R^c_{ab}\right)^2 = e^{-2\pi i(\Delta_a+\Delta_b-\Delta_c)} \tag{2.26}$$

where $\Delta_j$ is the conformal weight in the holomorphic sector of the primary field $\phi_j$, corresponding to the topological charge $j$. Therefore the mapping between anyon and conformal models allows us to determine the unitary operator $R_{ab}$ up to the signs of the phases $R^c_{ab}$. Usually the phases $2\pi\Delta_j$ in (2.26) are also called *topological spins* of the corresponding anyons $j$ and it can be shown that these phases are related to the following braiding [8, 48]:

$$R^1_{aa} = e^{-i\theta_a}. \tag{2.27}$$

Therefore the topological spin describes both the rotation of $2\pi$ of a single anyon, meant as a charge-flux composite, and the exchange process of a pair of equal anyons that annihilate [8, 48]. Moreover, the expression (2.26) shows that the monodromy operator $\left(R^c_{ab}\right)^2$, expressed as a function of the topological spins, can be interpreted as a rotation of $c$ by $2\pi$ while rotating $a$ and $b$ by $-2\pi$.

The conformal weights (or the topological spins) that appear in (2.26) are not enough to fully determine the braiding operators $R_{ab}$. However, such operators must provide a representation of the braid group, and, therefore, are constrained to fulfill other conditions, such as the Yang-Baxter equations (1.2). In particular, similarly to the $F$-moves, also the $R$-moves must satisfy a consistency relation that guarantees that the isomorphism between equivalent fusion spaces obtained by a sequence of $F$ and $R$-moves depends only on the initial and final decomposition of the space. This constraint is the *hexagon equation* (see Fig. 2.7):

$$R^c_{\gamma\beta}\left(F^{\alpha\gamma\beta}_{\delta}\right)_{ac} R^a_{\alpha\gamma} = \left(F^{\alpha\beta\gamma}_{\delta}\right)_{bc} R^\delta_{\gamma b}\left(F^{\gamma\alpha\beta}_{\delta}\right)_{ab} \tag{2.28}$$

The hexagon equation describes two equivalent sequences of $F$ and $R$-matrices involving the fusion of three anyons $\alpha, \beta$ and $\gamma$ giving a total charge $\delta$. The

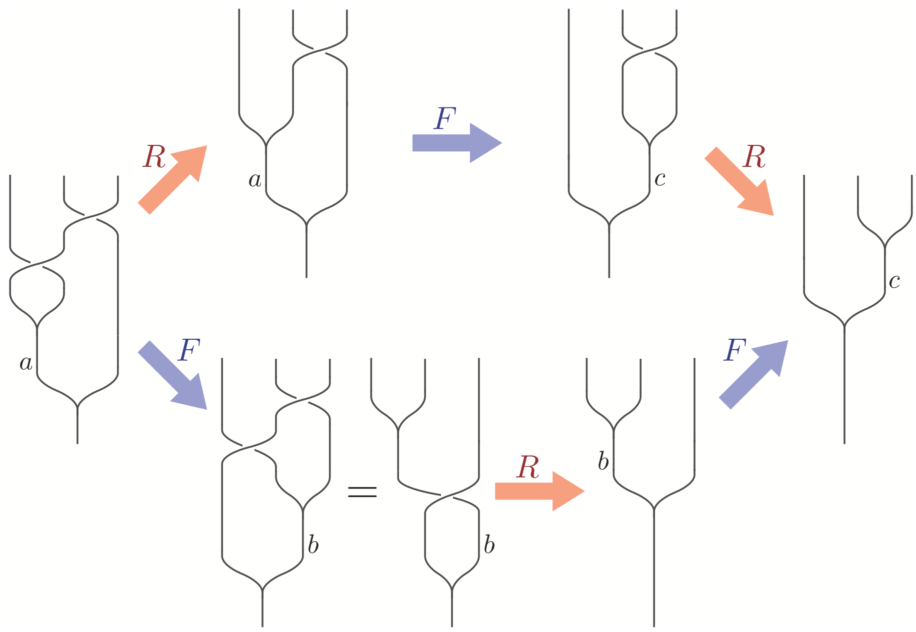



Figure 2.7: The $F$-moves and the $R$-moves must obey the hexagon equation (2.28). This constraint enforces the consistency of two different sequences of braidings and fusions for three anyons and guarantees that the isomorphism between two different fusion bases depends only on the bases and not by the moves applied to obtain them. The hexagon equation implies also that the Yang-Baxter equations (1.2) are fulfilled. (Taken from [61].)

equation (2.28) imposes, essentially, the property that a worldline may be passed over or under the fusion vertices, which, in the language of knot theory, constitutes one of the Reidemeister move (see, for example, [13]) and it is equivalent to the Yang-Baxter relations (1.2). Therefore braidings and fusions must commute, as shown in the second step of the lower path in Fig. 2.7.

The pentagon (2.19) and hexagon (2.28) equations, sometimes called Moore-Seiberg polynomial equations, together with the unitarity of the $F$-matrices, completely specify an anyon model; a fundamental result in tensor category theory, the MacLane theorem, assures that no further consistency relations are required beyond these equations (see [8, 54] and references therein for further detail).

Let us consider now a system of many anyons: as previously seen, the corresponding Hilbert space can be represented as a direct sum (2.5) where the sequences of the total charges of the first $k$ anyons, $b_k$, label every possible state. In the basis described by the sequences of $b_k$, usually called the standard basis, the charge of two subsequent anyons $a_i \times a_{i+1}$ is not a diagonal observable, therefore also the generic braiding $R_{a_i a_{i+1}}$ has not a diagonal expression. Thus, to represent the braid group generators $\sigma_i$, one has to express the braidings $R$ in the standard basis through appropriate $F$-moves (see Fig. 2.8). Hence, the generators $\sigma$ can be expressed in the following form:

$$(\sigma_i)_{b_i b_i'} = \left(F^{b_{i-1}a_i a_{i+1}}_{b_{i+1}}\right)^{-1}_{c_i b_i'} R^{a_i a_{i+1}}_{c_i} \left(F^{b_{i-1}a_i a_{i+1}}_{b_{i+1}}\right)_{b_i c_i} \tag{2.29}$$

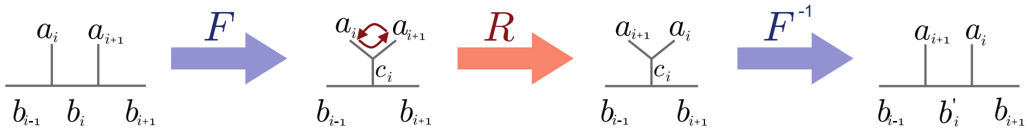



Figure 2.8: The generators $\sigma$ of the braid group can be represented in the standard basis $\{b_k\}$ by transforming the matrices $R$ with appropriate $F$-moves.

The $F$-matrices are therefore essential in defining proper representations of the braid group. The matrices $\sigma_i$ obtained in this way are the building blocks for topological quantum computation and, under certain hypothesis, can be considered as a computational basis to obtain single-qubit gates. Let us consider, for example, the Hilbert space constituted by four Ising anyons, $\sigma_a, \sigma_b, \sigma_c$ and $\sigma_d$ having a trivial total charge. There are two possible states corresponding to the fusion outcomes:

$$|0\rangle: \quad \sigma_a \times \sigma_b = \sigma_c \times \sigma_d = 1, \tag{2.30}$$

$$|1\rangle: \quad \sigma_a \times \sigma_b = \sigma_c \times \sigma_d = \varepsilon; \tag{2.31}$$

these states are represented in Fig. 2.9. In the figure the Ising anyons are depicted as green dots: the red ellipses correspond to the fusion outcome of the first and second pair of anyons: $\sigma \times \sigma = 1 + \varepsilon$. The fusion outcomes must be equal for the two pairs since the total charge of the system (blue ellipse) is trivial, and they are 1 for the state $|0\rangle$ and $\varepsilon$ for the state $|1\rangle$.

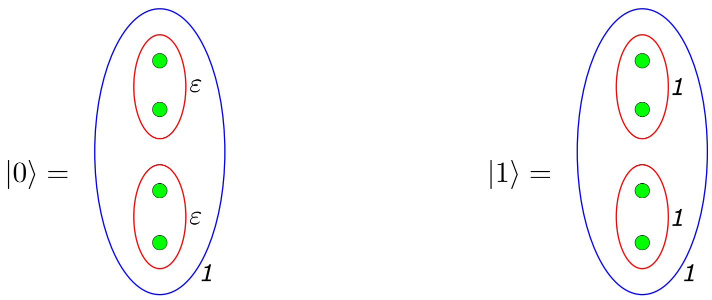
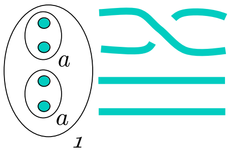



Figure 2.9: A system of four Ising anyons with a trivial total charge is characterized by two possible states. Each pair of anyons must have the same total charge. Depending on the charge of the pairs, the states are labelled as $|0\rangle$ or $|1\rangle$

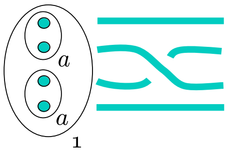
The braiding matrix $R_{\sigma\sigma}$ of a pair of Ising anyons in the basis $(1, \varepsilon)$ reads:

$$R_{\sigma\sigma} = \begin{pmatrix} e^{-i\frac{\pi}{8}} & 0 \\ 0 & e^{i\frac{3\pi}{8}} \end{pmatrix} \tag{2.32}$$

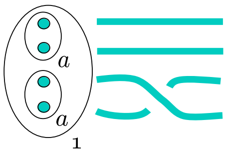
according to the conformal weights $\Delta_\sigma = 1/16$ and $\Delta_\varepsilon = 1/2$. The system of four Ising anyons shown in Fig. 2.9 presents three possible braid generators representing the exchanges of subsequent anyons. To calculate the corresponding matrices one has to consider the associativity matrix (2.23) $F_\sigma^{\sigma\sigma\sigma} = (F_\sigma^{\sigma\sigma\sigma})^{-1}$:




$$\sigma_1 = R_{\sigma\sigma} = e^{-i\frac{\pi}{8}} \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \tag{2.33}$$




$$\sigma_2 = F_\sigma^{\sigma\sigma\sigma} R_{\sigma\sigma} F_\sigma^{\sigma\sigma\sigma} = \frac{e^{i\frac{\pi}{8}}}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} \tag{2.34}$$




$$\sigma_3 = R_{\sigma\sigma} = e^{-i\frac{\pi}{8}} \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \tag{2.35}$$

The generators $\sigma_1$ and $\sigma_3$ involve anyons in the same pair, therefore they are diagonal in the chosen basis, which depends only on the pairs total charge, and they both correspond to the matrix $R_{\sigma\sigma}$. The braiding $\sigma_2$, instead, represents the exchange of two anyons in different pairs and thus it presents also off-diagonal terms which allow the transition from a state to the other. As we will see in the next chapter, these braidings generate only a finite subgroup of SU(2), therefore Ising anyons are not suitable to implement a universal topological quantum computation. In the next section we will analyze another non-Abelian model, the Fibonacci anyons, whose braiding rules allow, on the contrary, to cover in a dense way the whole SU(2) space.

## 2.4 Fibonacci anyons

The simplest example of non-Abelian anyon model is the one of Fibonacci anyons; this model is characterized by only two topological charges: the vacuum sector 1 and the topological sector corresponding to the presence of a single Fibonacci anyon, hereafter labelled as $\tau$. These two charges are related by the following fusion rules

$$1 \times \tau = \tau\,, \tag{2.36}$$
$$\tau \times \tau = 1 + \tau\,; \tag{2.37}$$

which highlight that $\tau$ is a non-Abelian anyon since the fusion of two of them can result either in an annihilation or in the presence of a single anyon.

Despite the simplicity of this model, Fibonacci anyons provide the main example of a universal topological quantum computation [29, 30] and they show intriguing connections with CFTs and RSOS models. The purpose of this section is to describe the main characteristics of Fibonacci anyons that are useful for quantum computation and to provide an example of the non-Abelian anyon models described above. Therefore we will deal only with the case of non-interacting anyons, even if Fibonacci anyons are also a natural playground to study the effects of ferromagnetic of antiferromagnetic interactions between non-Abelian anyons, through the so-called *golden chain* model [63, 64]. Further details on the general theory of Fibonacci anyons can be found in [35, 61] and references therein.

From an abstract point of view, the Fibonacci model corresponds to the $SU(2)_3$ algebra (and the related Chern-Simons theory), up to Abelian phases that are irrelevant to the purpose of quantum computation. Its fusion and braiding rules characterize several quantum Hall states that are supposed to describe different quantum Hall plateaux, in particular the $SU(2)_3$ Read-Rezayi state [65] and the $SU(3)_2$ non-Abelian spin-singlet state at $\nu = 4/7$ [66].

The Read-Rezayi states are a family of wavefunctions which generalize the Moore-Read Pfaffian state [21, 67] to the $SU(2)_k$ algebras and correspond to the correlation functions of $\mathbb{Z}_k$ parafermions in CFT [68]. In particular the $SU(2)_3$ Read-Rezayi wavefunction describes a quantum Hall state with filling factor $\nu = 3/5$ whose particle-hole conjugate is the main candidate description for the observed $\nu = 12/5$ plateau [69] that, therefore, could be suitable to have Fibonacci anyons as gapped excitations. Moreover, explicit quasihole wavefunctions have been worked out for the $k = 3$ Read-Rezayi state using quantum group techniques, with results consistent with the predicted $SU(2)_3$ braiding properties [70].

The non-Abelian spin-singlet states [66, 71] are instead a generalization of the Abelian Halperin wavefunctions [72] which extend the non-Abelian statistics to multi-component quantum Hall liquids considering also the spin degree of freedom of the electrons.

For a review of all the mentioned wavefunctions and their role in topological quantum computation see [73].

In order to better analyze the Hilbert space characterizing systems of Fibonacci anyons it is useful to introduce the Brattelli diagrams as seen in section 2.1:

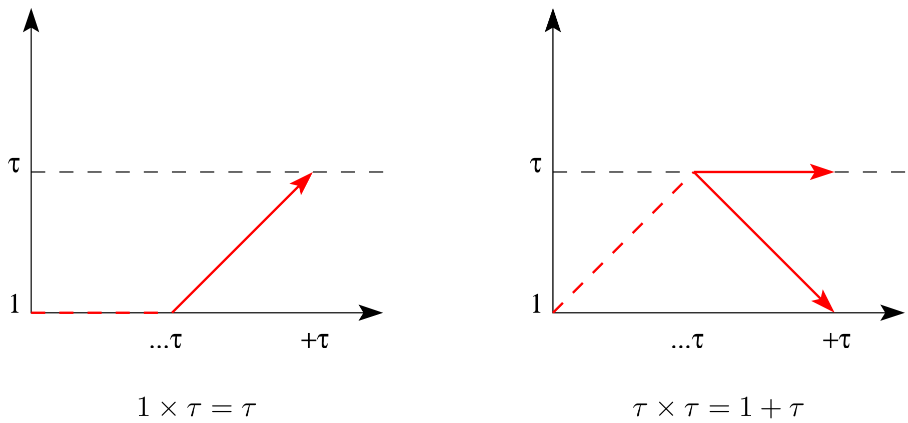



Figure 2.10: Brattelli diagrams for the fusion rules of the non-Abelian Fibonacci anyons (2.36,2.37).

From the fusion rule (2.37) one can easily obtain that the quantum dimension associated with a Fibonacci anyons is the well-known golden ration $d_\tau = \varphi = \frac{1}{2}\left(\sqrt{5}+1\right) \simeq 1.618$. To understand the meaning of this particular value (and to understand why the Fibonacci anyons are named after an Italian mathematician of the XIII century) we must consider a chain composed of $n$ Fibonacci anyons (see Fig. 2.11).

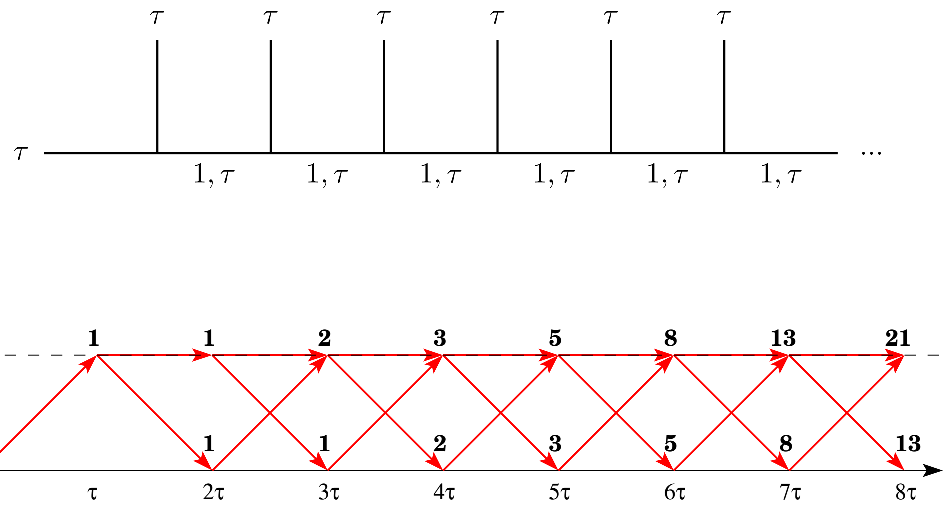



Figure 2.11: Fusion graph and Brattelli diagram corresponding to a chain of non-Abelian Fibonacci anyons. The Fibonacci numbers determine the number of orthogonal states for a system of $k$ Fibonacci anyons at a given topological charge.

As we already saw, the states in the fusion space of $n$ anyons can be identified by the sequence of the intermediate charges $b_k$. After the fusion of the first two anyons, the intermediate charge $b_k$ of the first $k$ Fibonacci anyons may assume both the values 1 and $\tau$. However, since $1 \times \tau = \tau$, there is one constraint: if $b_{k-1} = 1$ then $b_k = \tau$ and therefore, a charge 1 cannot be followed by another intermediate charge 1. This condition implies that the quantum dimension of a chain can be calculated by recursion. If the total charge of $n$ Fibonacci anyons is $c = \tau$, then the first $n-1$ anyons can fuse giving either the vacuum or $\tau$, and there is a one to one correspondence of the states of $n-1$ anyons and the states with $n$ anyons but fixed total charge $\tau$; therefore $D_n^{c=\tau} = D_{n-1}$. Otherwise, if the total charge of $n$ Fibonacci anyons is $c = 1$, then the first $n-1$ anyons must fuse in $\tau$ and one has $D_n^{c=1} = D_{n-1}^{c=\tau} = D_{n-2}$. Therefore the total dimension of the fusion space of $n$ Fibonacci anyons is given by:

$$D_n = D_n^{c=\tau} + D_n^{c=1} = D_{n-1} + D_{n-2}; \tag{2.38}$$

this recursion relation generates the Fibonacci numbers $F_n$, thus the quantum dimension of a system of $n$ Fibonacci anyons is the Fibonacci number $F_{n+1}$. Since these numbers grow asymptotically as a power of the golden ratio $\varphi^n$, one recovers the quantum dimension $d_\tau = \varphi$.

So far we considered only the fusion rules (2.36,2.37) of the model. It is interesting to notice that such fusion rules characterize both the non-unitary Yang-Lee conformal model (the minimal model $\mathcal{M}_{5,2}$) and the $\mathbb{Z}_3$ parafermions related to the $\mathrm{SU}(2)_3$ algebra. To understand the difference between the two conformal theories one must consider the associativity matrices of Fibonacci anyons. In particular the Fibonacci model has only one nontrivial matrix, $F_\tau^{\tau\tau\tau} \equiv F$ (we drop the $\tau$ indices for the sake of simplicity), whereas all the other $F$-matrices, involving at least one vacuum, are trivially the identity. To find the $F$ matrix one must consider the pentagon equation (2.19). Such equation imposes several conditions and, in particular, one finds:

$$\begin{aligned} F_{1,\tau}F_{\tau,1} &= F_{11}\,, \qquad \text{for} \quad b=c=1\,; & (2.39)\\ F_{11}+(F_{\tau\tau})^2 &= 1\,, \qquad \text{for} \quad c=b=d=\tau \quad \text{and} \quad a=1 & (2.40)\end{aligned}$$

Imposing also the unitarity of the $F$-matrix, the only possible solution, up to arbitrary phases of the off-diagonal terms, is [48, 64]:

$$F_F = \begin{pmatrix} \varphi^{-1} & \varphi^{-1/2} \\ \varphi^{-1/2} & -\varphi^{-1} \end{pmatrix} \tag{2.41}$$

where $\varphi$ is the golden ratio. One can notice that the unitary matrix $F_F$ has determinant $-1$ and $F_F^{-1} = F_F$. This associativity matrix corresponds to the unitary theory $\mathrm{SU}(2)_3$ (or $\mathrm{SO}(3)_3$ to be more precise [35]) and it will be used in the following to build a basis for universal quantum computation. It is possible, however, to relax the unitarity condition [60]; in this case there is another solution to the pentagon and hexagon equation that implies the following non-unitary $F$-matrix:

$$F_{YL} = \begin{pmatrix} -\varphi & -i\varphi^{1/2} \\ -i\varphi^{1/2} & \varphi \end{pmatrix}. \tag{2.42}$$

Such matrix corresponds to the Yang-Lee conformal model which brings to a non-unitary dynamics.

To find the only nontrivial braiding matrix $R_{\tau\tau}$ one has to solve the hexagon equation (or similarly the Yang-Baxter equations). The resulting $R$ matrix reads (in the basis $1,\tau$) [48, 64]:

$$R_{\tau\tau} = \begin{pmatrix} e^{-i\frac{4}{5}\pi} & 0 \\ 0 & -e^{-i\frac{2}{5}\pi} \end{pmatrix} \tag{2.43}$$

and the only other solution is its complex conjugate which corresponds to exchange clockwise and counterclockwise braidings. One can observe that $R_{\tau\tau}$ is compatible with the conformal weight $\Delta_\tau = -\frac{1}{5}$ of the Yang-Lee model [22] that defines the monodromy matrix $R^2_{\tau\tau}$.

Once we stated the fundamental characteristics of Fibonacci anyons, we can analyze a system of four anyons to find a suitable basis to encode the logical qubit [33, 35]. To this purpose it is conventional to use a notation which is slightly different from the one we adopted so far: since in the Fibonacci model only one nontrivial sector is present, we can label the topological charge of a set of anyons in terms of the corresponding number of Fibonacci anyons; therefore, the vacuum sector which topologically corresponds to the absence of a charge is labelled with 0, whereas the $\tau$ sector is labelled with a unitary charge 1.

To encode a single qubit we can consider the two-state system composed by four Fibonacci anyons with a trivial total charge. Similarly to the previously discussed case of Ising anyons (2.30,2.31), there are only two possible states characterized by a null total charge (see Fig. 2.12 (a)): in the state $|0\rangle$ both the first and second pair of anyons (represented by the small ellipses) show a total charge 0, whereas the state $|1\rangle$ is characterized by having both pair with total charge 1. The logical qubit can be encoded also in a system with three Fibonacci anyons (Fig. 2.12 (b)) with a total charge 1, which is ideally obtained from the previous case by removing one anyon. In this case the fusion outcome of the first pair, depicted as the smaller ellipse, determines the logical value of the qubit, whereas the total charge is constrainted to be 1; finally the third possible state of the system, $|NC\rangle$, characterized by a trivial total charge, is a non-computational state [33, 35] which cannot be obtained by simple braids of the three anyons once the system is initialized in the two logical states.

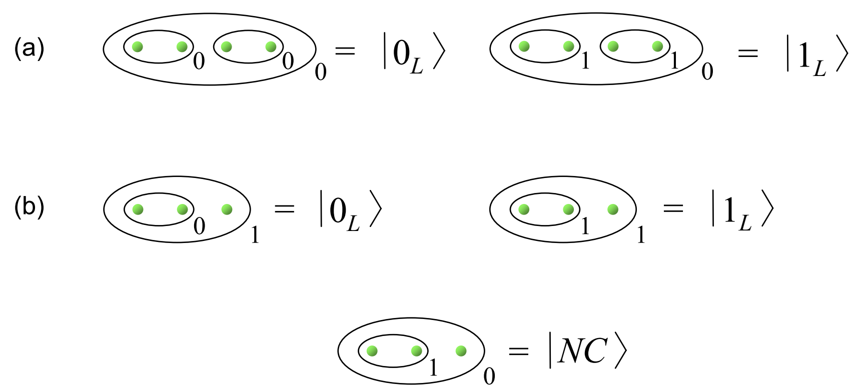
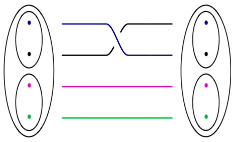
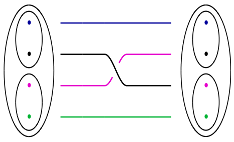



Figure 2.12: The logical qubit states $|0_L\rangle$ and $|1_L\rangle$ can be encoded either in a system with four Fibonacci anyons with a trivial topological charge (a) or in a system with three Fibonacci anyons and a total charge 1 (b). The state $|NC\rangle$ in (b) represents the only state with a charge 0 and constitutes a non-computational state. (Taken from [35]).

Having defined the logical basis to encode a qubit, we can now describe the effect of braidings on the four-anyon system represented in Fig. 2.12 (a). Like the case of Ising anyons, the exchanges of two Fibonacci anyons in the same pair, $\sigma_1$ and $\sigma_3$, are diagonal in the basis $|0\rangle, |1\rangle$ and they are represented by the same matrix $R_{\tau\tau}$ (2.43). Therefore, to the purpose of quantum computation, we will use only the braidings $\sigma_1$ and $\sigma_1^{-1}$. The braiding $\sigma_2$, which involves anyons of different pairs, is instead non-diagonal in the logical basis and must be obtained through the application of the $F$-matrix (2.41):



$$\sigma_1 = R_{\tau\tau} = \begin{pmatrix} e^{-i\frac{4}{5}\pi} & 0 \\ 0 & -e^{-i\frac{2}{5}\pi} \end{pmatrix} \tag{2.44}$$



$$\sigma_2 = F_F R_{\tau\tau} F_F = \begin{pmatrix} -\varphi^{-1} e^{-i\pi/5} & -\sqrt{\varphi^{-1}} e^{i2\pi/5} \\ -\sqrt{\varphi^{-1}} e^{i2\pi/5} & -\varphi^{-1} \end{pmatrix} \tag{2.45}$$

where $\varphi = \frac{1}{2}\left(1+\sqrt{5}\right)$ is the the golden mean. One can verify that the above braidings fulfill the Yang-Baxter equation; besides, the Fibonacci braidings are characterized by the relation $\sigma_1^{10} = \sigma_2^{10} = 1$ since $R_{\tau\tau} = e^{-i\frac{\pi}{10} - i\frac{7\pi}{10}\sigma_z}$ in the standard $U(2)$ representation (here $\sigma_z$ is the Pauli matrix).

The main property of the representation of the braid group provided by Fibonacci anyons is that $\sigma_1, \sigma_2$ and their inverses generate a dense group in SU(2), which brings to the possibility of approximating with a braid of Fibonacci anyons every single-qubit gate.

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