Topological Order and Quantum Computation
Toric code

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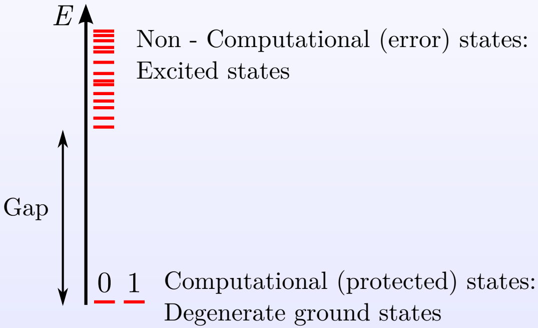
February 12, 2014

1. Toric (surface) codes and Abelian anyons
2. Non - Abelian anyons
3. Majorana modes in topological superconductors

The first necessary ingredient of quantum computation is the possibility to store, as reliably as we can, quantum information. We need a **protected** “portion” of Hilbert space.



- Noise, Temperature, ... $<$ Gap.
- $H_{\text{eff}} \approx 0$ describes the time evolution of the ground states.
- This degeneracy may be provided by **symmetry**... or **topology**.

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The previous conditions are necessary for:

3. Anyons appear among the excitations.

Toric (surface) code

The main toy-model for topological order is the toric code.

There are two possible approaches to toric (surface) codes:

1. - Is it possible to build a self-correcting quantum memory?
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There are two possible approaches to toric (surface) codes:

1. - Is it possible to build a self-correcting quantum memory?
   - Toric code as a **physical system** (Hamiltonian approach)

2. - Can we build efficient quantum correction protocols based on local operators?
   - Surface codes as **error correction schemes**.

Spin $1/2$ model on the square lattice (periodic boundary conditions).

Hamiltonian:

$$H = -\sum_v A_v - \sum_p B_p$$

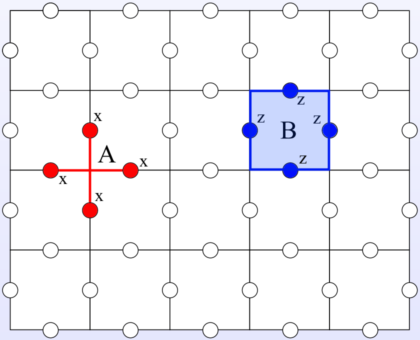
The Hamiltonian is the sum of two kind of terms (stabilizers):

$$A_v = \prod_{i \in v} \sigma_{x,i} \,, \qquad B_p = \prod_{i \in p} \sigma_{z,i}$$

All these terms commute:

$$[A_i, A_j] = [B_i, B_j] = [A_i, B_j] = 0$$

Spins sit on the **edges**:



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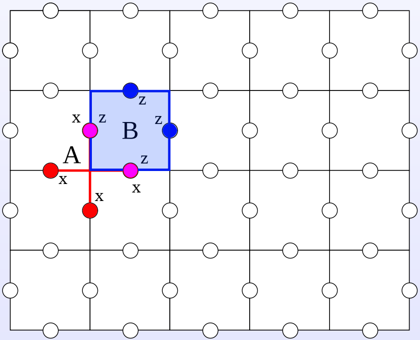

$$H = -\sum A_v - \sum B_p$$

Since all the stabilizers A and B commute, a GS is identified by:

$$A_v = \prod \sigma_{x,i} = 1\,, \qquad B_p = \prod \sigma_{z,i} = 1.$$

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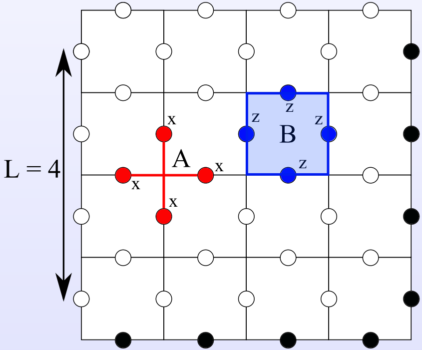
- Number of physical spins: $N = 2L^2$
- Number of stabilizers: $N_A = L^2$, $N_B = L^2$
- 2 constraints:

$$\prod A_v = 1\,, \quad \prod B_p = 1.$$

- Number of ground states:

$$2^{N-(N_A+N_B-2)} = 4.$$

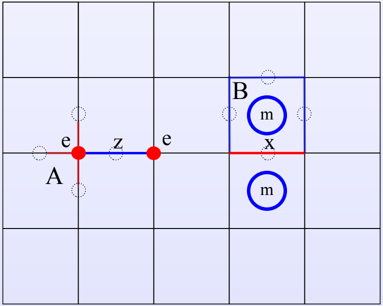


$$H = -\sum_v A_v - \sum_p B_p \, ; \qquad A_v = \prod_{i \in v} \sigma_{x,i} \, , \quad B_p = \prod_{i \in p} \sigma_{z,i} \, .$$

If $A_v = -1$ or $B_p = -1$, a localized excitation appears with energy 2.

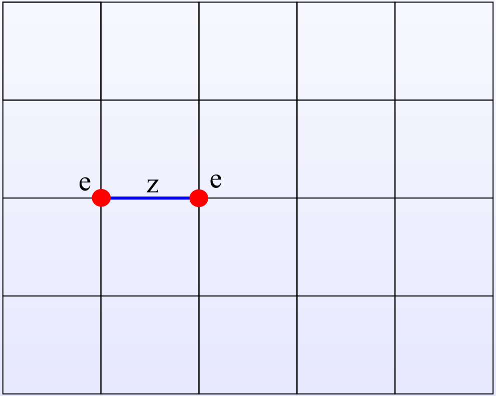
- $A = -1$: electric charge e.
- $B = -1$: magnetic vortex m.

Local operators σ_z or σ_x create pairs of excitations:



String operators

String operators $\prod_i \sigma_{z,i}$ and $\prod_i \sigma_{x,i}$ create and move excitations:



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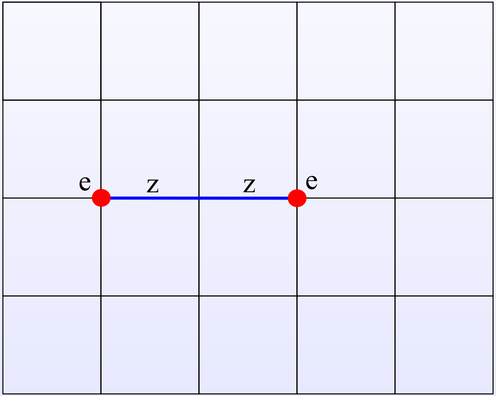

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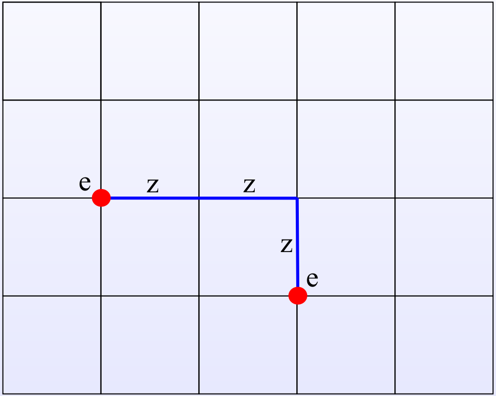

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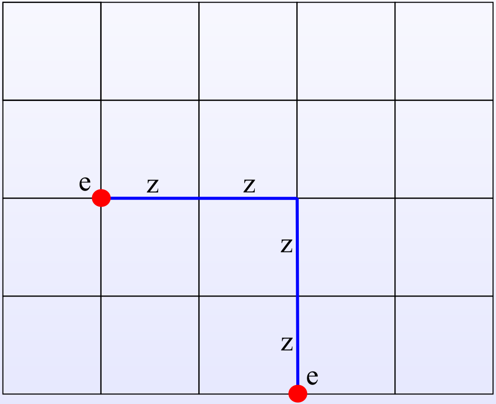

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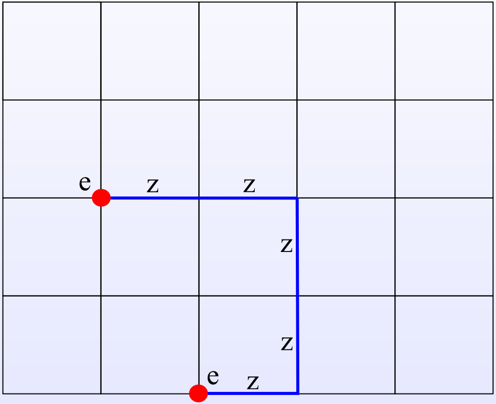

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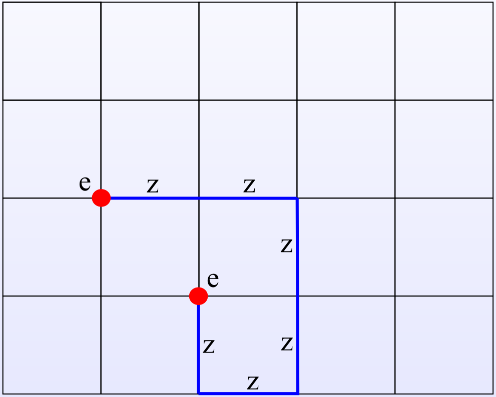

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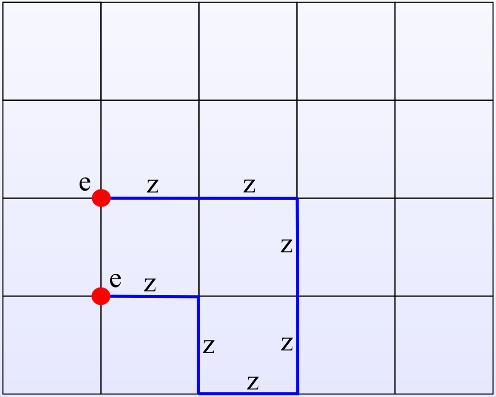

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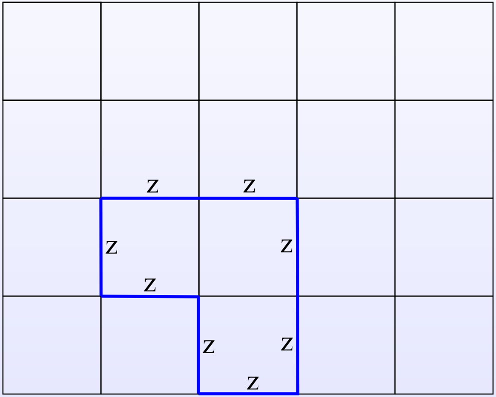

- A string of σ_z creates and moves a pair of electric defects.
- A closed string commutes with the Hamiltonian: it creates, moves and annihilates the electric defects.

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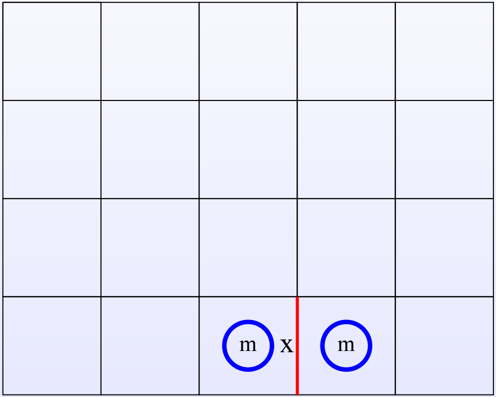

- A string of σ_x on the dual lattice creates and moves a pair of magnetic vortices.
- Also in this case closed strings commute with the Hamiltonian.

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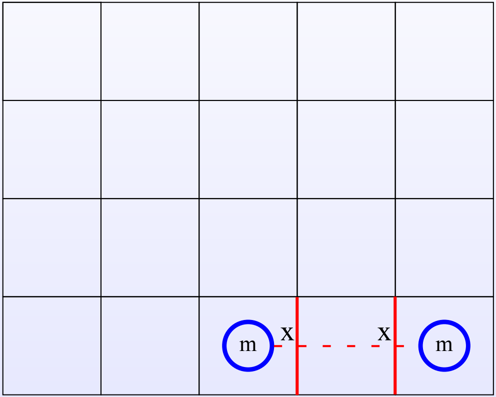

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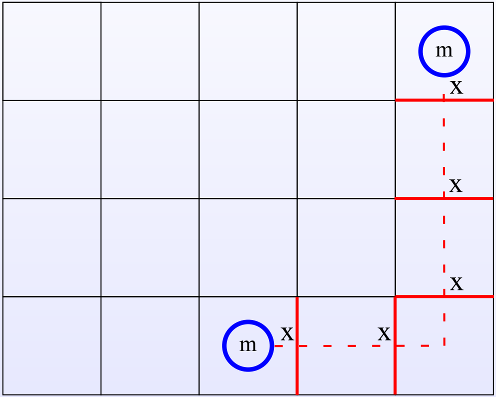

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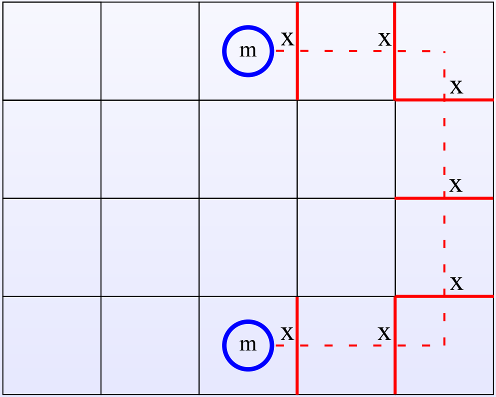

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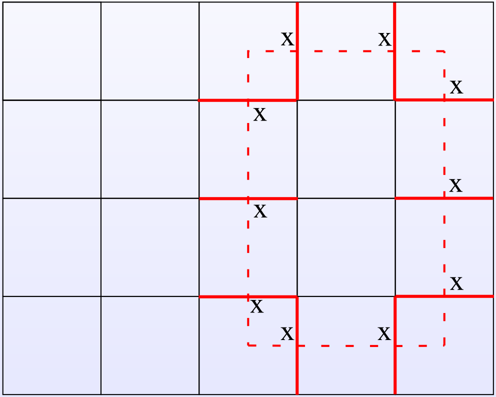

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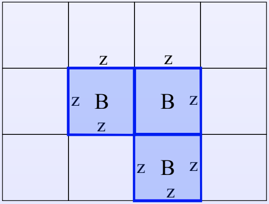
Symmetries

- All the closed strings of σ_z operators on the lattice are symmetries.
- All the closed strings of σ_x operators on the dual lattice are symmetries.
- They correspond to create a pair of excitations, move them, and annihilate them, leaving the energy invariant.

There are two kinds of these symmetries:

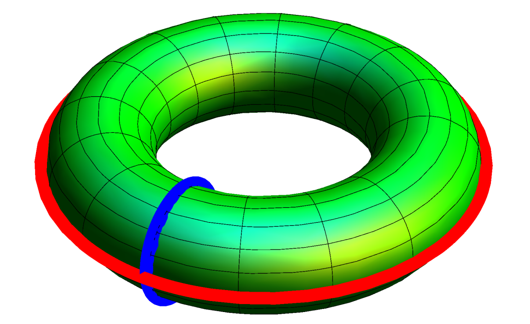
1. **Trivial symmetry (stabilizers)**: it is the product of stabilizers A or B, thus it is the identity over the ground states.



2. **Non-trivial symmetry** (not a product of stabilizers). It is a string with non-trivial homology and its value is not fixed.

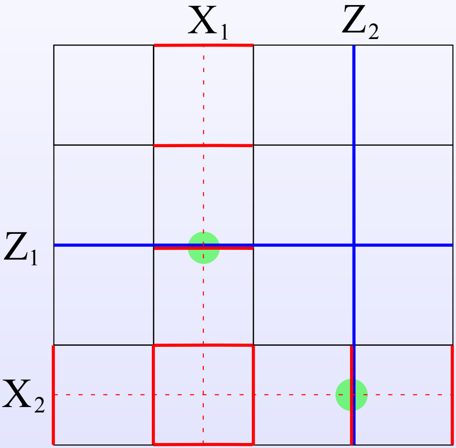
Non-trivial String Symmetries

Non-trivial string symmetries correspond to **non-contractible loops** on the torus either of the X or of the Z kind:



The **contractible** string symmetries instead can always be reduced to the product of local stabilizers.

There are **four independent symmetries** and they correspond to the following **non-contractible loop operators**:



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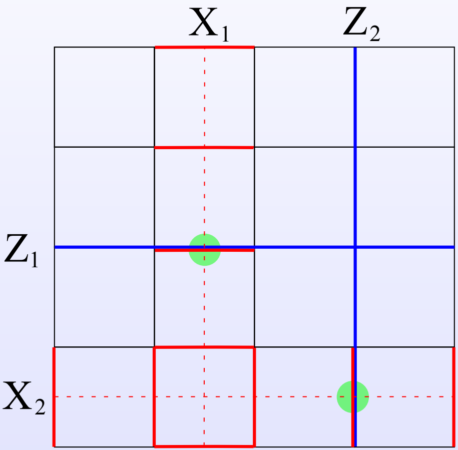

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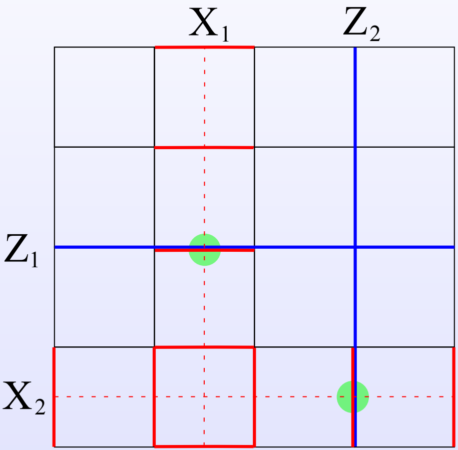
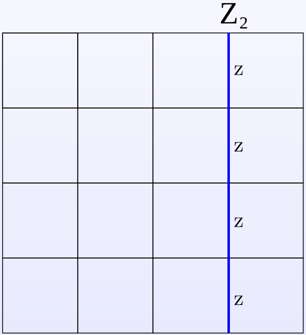


- The 4 string-symmetries with non-trivial homology do not commute with each other.
- There are 4 degenerate ground states which encodes 2 logical qubits.
- These string-operators commute with the stabilizers but are not stabilizers.

String operators with the same homology and of the same kind are equivalent through the multiplication by stabilizers:



Z_2

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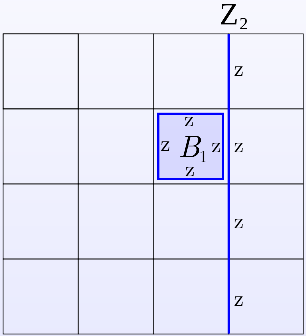

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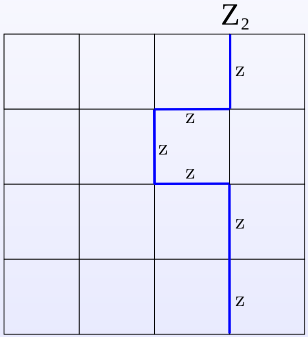

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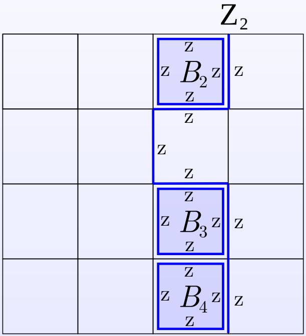

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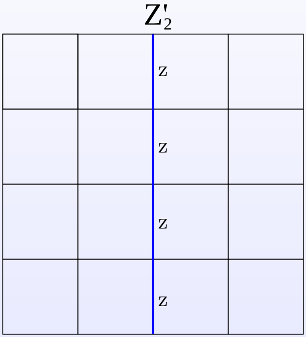

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Localized excitations: anyons

- e - excitations are created and moved by z-strings.
- Since all the z-strings commute with each other, e - excitations obey a bosonic statistics.
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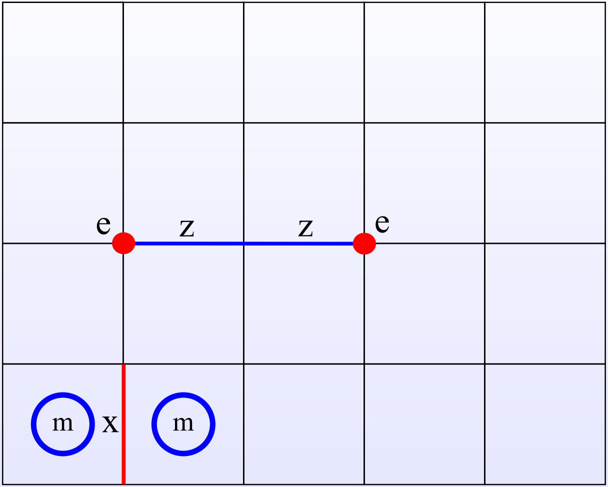
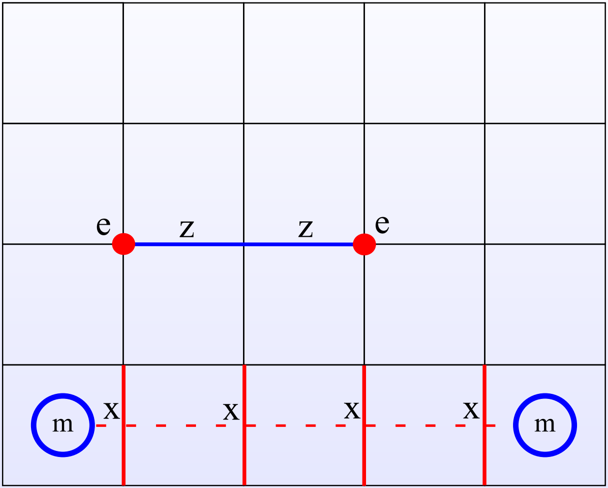
- e - excitations are created and moved by z-strings.
- Since all the z-strings commute with each other, e - excitations obey a bosonic statistics.
- The same is true for m - excitations which are driven by x-strings.
- When a charge e is moved around a vortex m, however, a non-trivial phase appears.



Moving m around e, the wavefunction acquires a phase -1.



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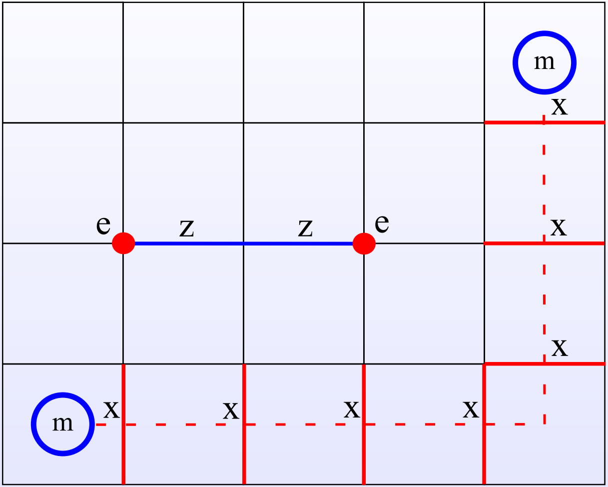

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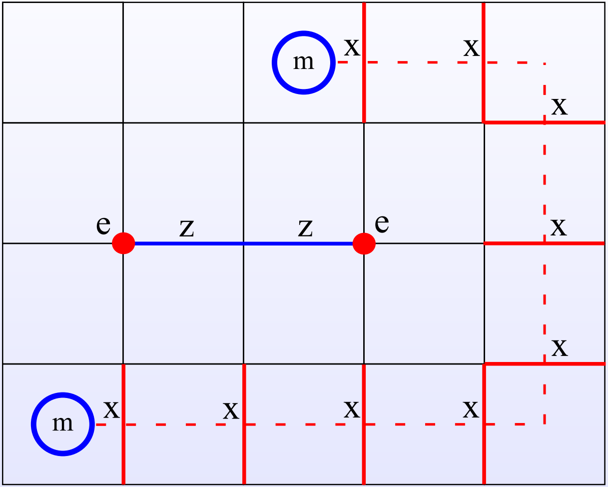

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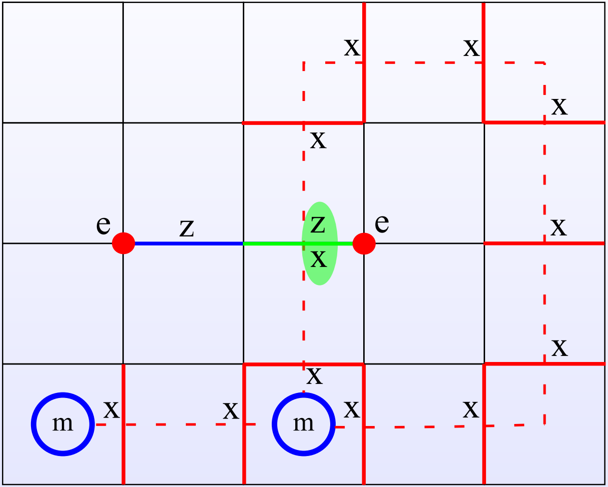

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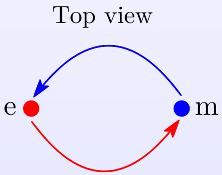
- **Abelian Anyons** are localized excitations, living in 2D systems, whose statistics is neither bosonic nor fermionic.
- Their exchange is defined by a non-trivial phase different from ± 1.
- In particular if we consider the wavefunction $\Psi(r_e, r_m)$ with a pair of e and m excitations, by winding e around m we obtain:

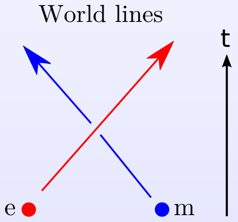
$$\Psi(r_e, r_m) \to R_{em}^2 \Psi(r_e, r_m) = -\Psi(r_e, r_m)$$

- The process is topologically equivalent to two counterclockwise exchanges of the positions r_e and r_m.

- Moving a particle around another is topologically equivalent to two exchanges of their position.
- These exchanges are also called **braidings**.
- The mutual statistics of e and m is described by a phase:

$$R_{em} = e^{i\frac{\pi}{2}}$$

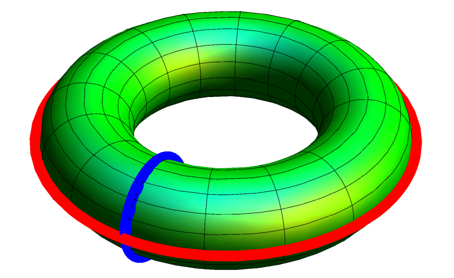




- In particular we demonstrated $R_{em}^2 = -1$.

Braiding and degeneracy of the ground states

- The braiding statistics and the degeneracy of the ground states are related.
- In particular the logical operator X_1 corresponds to wind a pair of m along one of the loops of the torus and annihilate them.
- Z_1 instead winds a pair of e in the other direction.



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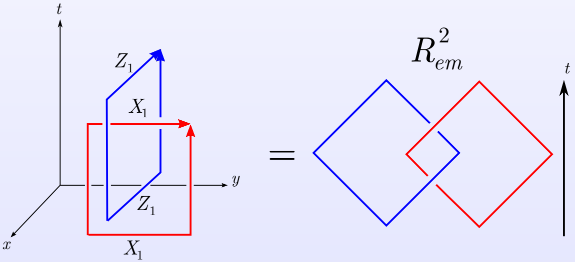
$X_1 Z_1 X_1^{-1} Z_1^{-1} =$



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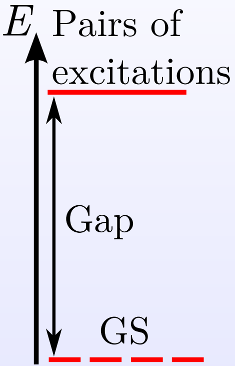
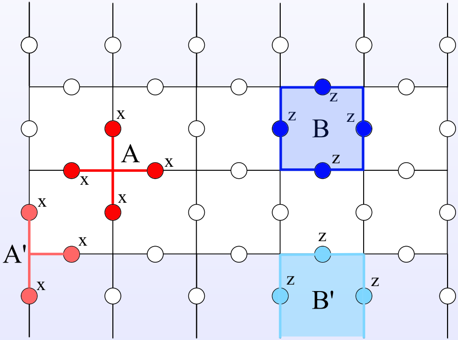


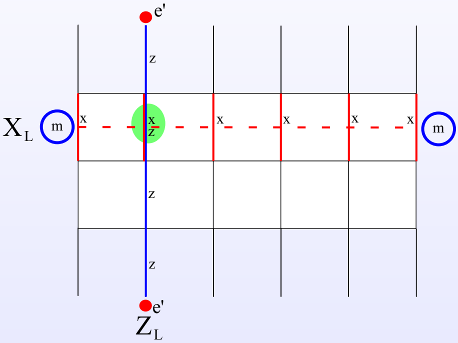
0. The toric code has 4 degenerate GS protected by a gap;
1. Their degeneracy depends on non-contractible string operators: it has a topological nature;
2.
   - No local operator allows transition among the GS.
   - The local operators are either stabilizers $(= 1)$ or create pairs of excitations.
3. The excitations are (mutually) anyons.

The torus is an unphysical system. Let us change BC:



There are two possible **boundary conditions** defined by two different 3-qubit stabilizer code elements, A' and B'

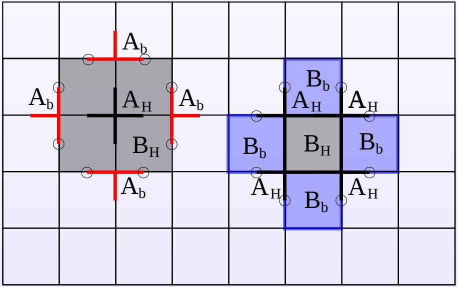
The degeneracy now is different due to the different geometry and the absence of constraints.



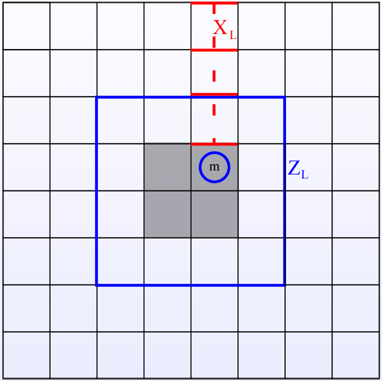
- The boundary terms imply that now we have only two non-commuting string-simmetry operators which commute with the Hamiltonian.
- Thus a rectangular system of this kind has a twofold degeneracy.
- We store one qubit.

Holes in the surface code

To store more than one qubit we may consider a “hole geometry”:



Some of the stabilizers are excluded from the Hamiltonian.
Due to the two possible boundaries there are two different kinds of holes which can host a magnetic or an electric degree of freedom.



- Two states are distinguished by the presence of a magnetic flux in the hole.
- A loop of Z detects the state of the system. A cut of X changes the state.
- The degeneracy of the ground state doubles for each hole and is proportional to 2^g.

Thermal Fragility

Nussinov and Ortiz, Ann. Phys. 324 (2009)

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- However, once a pair of anyons is created, they can freely propagate without paying any kinetic or confinement energy.
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- At thermic equilibrium for $T > 0$ there is no topological protection.
- One possible way around it could be disorder to localize the excitations.

Error Correction

- To overcome thermal fragility we adopt a different strategy: **active error correction**.
- The stabilizers are essentially local projectors:

$$\frac{1 + A_v}{2}, \quad \frac{1 + B_p}{2};$$

- The set of all the stabilizers project on the ground states of the Hamiltonian, which correspond to the subspace of **protected states**.
- A failure in one projector implies the presence of excitations, therefore **errors**.
- Error correction, in this case, can be seen as an artificial dynamics: in each time step we **measure** all the local stabilizers to detect whether an error occured.

Error Correction

$$A_v = \prod \sigma_{x,i} = 1\,, \qquad B_p = \prod \sigma_{z,i} = 1.$$

- Vertex and plaquette operators can be considered as commuting 4-qubit measurements.
- Since all the stabilizers commute with the ground-logical states, they give no information about the logical qubits.

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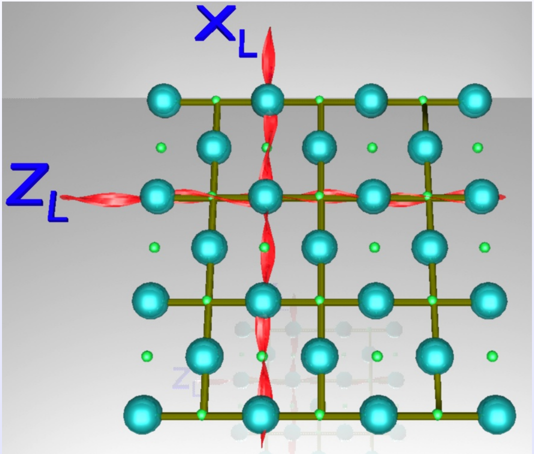
- Since all the stabilizers commute with the ground-logical states, they give no information about the logical qubits.
- To do error correction we discretize time and, at each time step, measure (syndrome measurements) alternatively all the As and all the Bs.
- If a measurement results in -1, we detect and localize an error (excitation of the toric code): we know that a noise operator acted on one qubit.
- By applying a suitable string operator we can correct the error (or we can simply keep track of them).

To perform the measurements, one strategy is to double the number of qubits. We distinguish **physical qubits** and **ancillary qubits**.

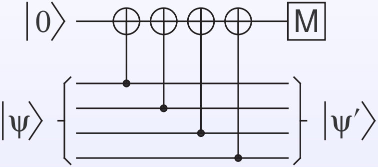
Physical qubits are the ones of the previous Hamiltonian.

We add one **ancilla** for each plaquette and vertex to perform the syndrome measures.

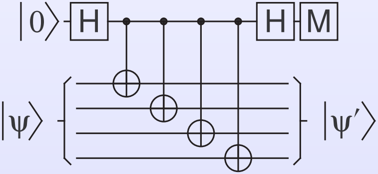


The stabilizer operators can be actively implemented through the following circuits involving CNOTs, single-qubit operators and measurements.

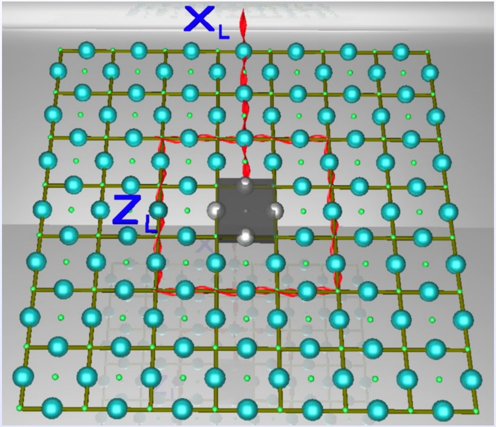
B=



A=



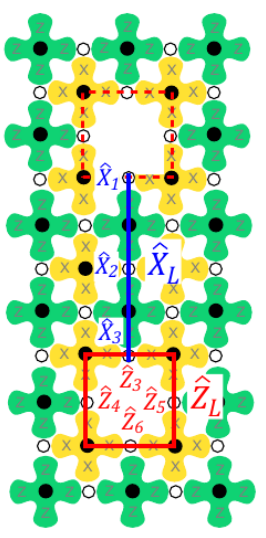
Single Defect Qubit



By excluding some of the syndrome measures we can store information in the holes (either magnetic or electric).

Double Defect Qubit

See, for example, Fowler *et al.*, Phys. Rev. A **86**, 032324 (2012)



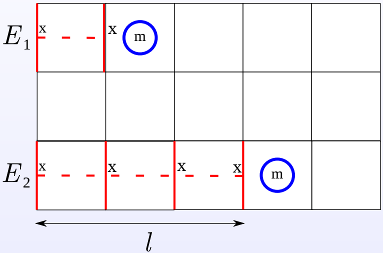
- Let us consider two magnetic holes which may be empty $|0\rangle$ or host a vortex $|1\rangle$.
- It is convenient to define the following logical states in a space with no overall vortices:

$$|0\rangle_L = |0\rangle_1 |0\rangle_2 \,, \quad |1\rangle_L = |1\rangle_1 |1\rangle_2$$

- The operators X_L and Z_L are single-qubit logical operators.
- Exploiting both magnetic and electric pairs of holes (and the anyonic statistics!) one can engineer two-qubit gates.

Very rough estimate of logical errors

A logical error occurs when if there appear a chain of errors in the logical qubit whose length is greater than $L/2$:



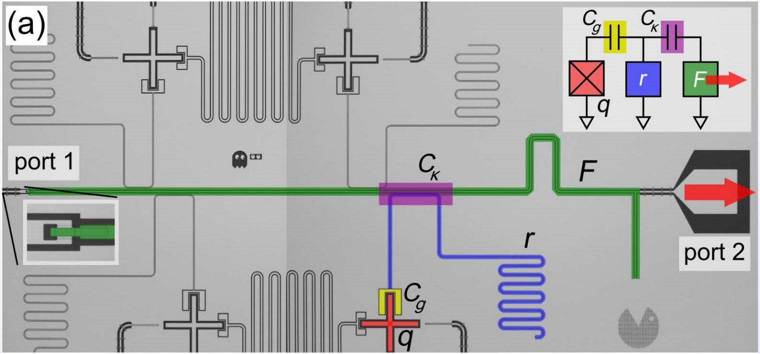
- E_1 can be efficiently corrected;
- E_2 cannot be corrected since $l > L/2$;

Estimate of the probability of logical errors:

$$P_L \approx \sum_{l>L/2} L \frac{L!}{(L-l)!l!} p_e^l \approx \sqrt{\frac{2L}{\pi}} (4p_e)^{L/2}$$

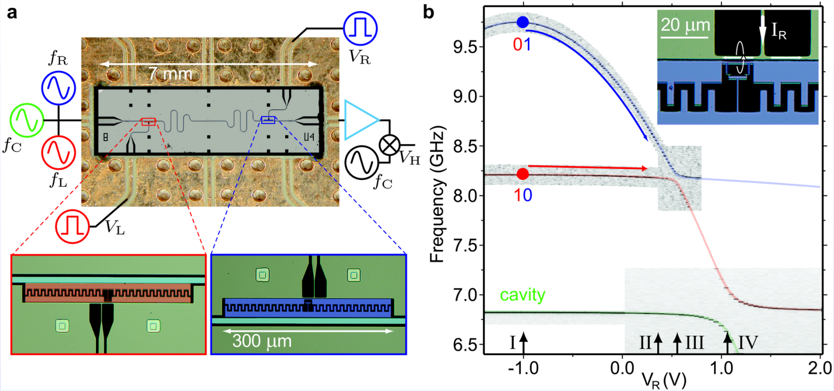
where the total number of qubits scales as $2L^2$.
Error treshold with perfect ancillas $\sim 15\%$.
Real error treshold ~ 0.008.

What about experiments?
Martinis *et al.*, arXiv:1401.0257v3



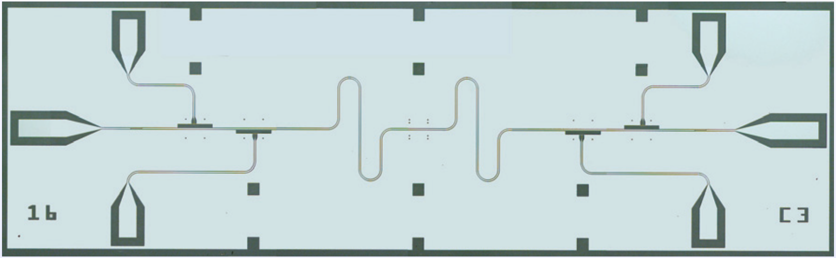
- 4 simultaneous single-qubit measurements with fidelity of 99% in less than 200ns.
- Coherence time $> 10\mu$s (they claim that 100μs is reachable).

DiCarlo *et al.*, Nature **460** (2009)



A Control Phase gate is obtained with fidelity $\gtrsim 0.90$.
The system can be modelled through a double Jaynes-Cummings:

$$H = \omega_r a^\dagger a + \frac{\Omega_L}{2}\sigma_{z,L} + \frac{\Omega_R}{2}\sigma_{z,R} + \sum_{k=L,R} g\left(a^\dagger \sigma_i^- + a\sigma_i^+\right)$$

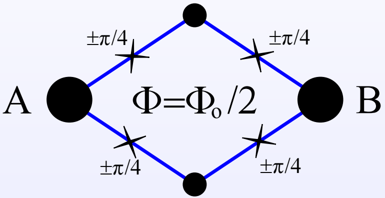


A 3-qubit Toffoli CCNOT gate is obtained with fidelity ~ 0.80.
The architecture could be used for 4-qubit gates.

Suggested references

- General review about Kitaev's model (Toric code, Majorana chain):
  A. Kitaev and C. Laumann, *Topological phases and quantum computation*, arXiv:0904.2771 (2008 Les Houches summer school).
- Original paper on the toric code:
  A. Kitaev, *Fault-tolerant quantum computation by anyons*, Ann. Phys. **303** (2003), arXiv:quant-ph/9707021.
- Original work on the surface code:
  S. Bravyi and A. Kitaev, *Quantum codes on a lattice with boundary*, arXiv:quant-ph/9811052 (1998).
- Reviews about surface codes:
  A. G. Fowler *et al.*, *Surface codes: Towards practical large-scale quantum computation*, Phys. Rev. A **86**, 032324 (2012), arXiv:1208.0928.
  D. DiVincenzo, Fault tolerant architectures for superconducting qubits, arXiv:0905.4839 (2009).

The elementary qubit is obtained from a flux qubit with 4 Josephson junctions E_J:



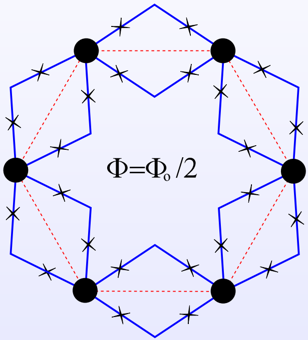
$$U = -2E_J\left(\left|\cos\frac{\varphi_{AB}}{2}\right| + \left|\sin\frac{\varphi_{AB}}{2}\right|\right)$$

- Two (semiclassical) degenerate ground states:

$$|\uparrow\rangle : \ \varphi_{AB} = \frac{\pi}{2}\,; \qquad |\downarrow\rangle : \ \varphi_{AB} = -\frac{\pi}{2}.$$

- In the regime $E_J \gg E_C$ the amplitude of the spin flip σ_x is:

$$r \approx E_J^{3/4} E_C^{1/4} e^{-1.61\sqrt{EJ/EC}}$$



Plaquette constraint for $\Phi = \Phi_0/2$:

$$\sum_{\langle ij \rangle} \varphi_{ij} = \pi$$

$$B_p = \prod_{\langle ij \rangle} \sigma_z^{ij} = 1$$