BOSONIZATION

- Mapping from int. bosons or fermions to 1D gapless fields

$$\underline{H = -t\sum_r c^\dagger_{r+1}c_r + h.c. + H_{int}} \xrightarrow{c \to \boxed{\psi \to e^{i(\varphi \pm \theta)}}} H = \frac{1}{2\pi}\int dx \; v K(\partial_x\varphi)^2 + \frac{v}{K}(\partial_x\theta)^2 + U\cos 4\theta$$

$$\left\langle \left[\psi^\dagger_L \psi_R(x)\right]\left[\psi^\dagger_R \psi_L(y)\right]\right\rangle = \langle e^{i2\theta(x)} e^{-i2\theta(y)}\rangle = \frac{1}{(x-y)^{2K}}$$

$\psi^\dagger\psi \quad \psi^\dagger\psi$

- Spin 1/2 fermions $\psi_s \to \underbrace{\varphi_s, \theta_s}_{\text{Bosons}}, \underbrace{K_s}_{\text{Fermions}}$

Recipes to describe fermions + int and $\underline{\text{bosons + interactions}}$ $\boxed{n_r = b^\dagger_r b_r = 0, 1, 2 \ldots}$

not "infinite"

onsite $U\, n_r(n_r - 1) \xrightarrow{U \to \infty} n_r = 0, 1$

BOSONS with infinite onsite interactions [lattice]

→ Hard-core bosons : $n_r = 0, 1$ $\quad [b_x, b_y] = 0$

→ Spin 1/2 systems

$\boxed{\text{SPIN 1/2 CHAINS}}$

• • • • • • At each site $\{|\uparrow\rangle_r, |\downarrow\rangle_r\}$

$\mathcal{H} = \bigotimes_r \{|\uparrow\rangle_r, |\downarrow\rangle_r\}$

Pauli matrices $\sigma_z|\uparrow\rangle = |\uparrow\rangle \qquad \sigma_z|\downarrow\rangle = -|\downarrow\rangle$

[Mott insulators]

Magnetic Materials

- Mapping [Jordan - Wigner] from spin 1/2 dof into fermions
- Also Spin 1/2 systems [with suitable sym : $\mathbb{Z}_2$] can be mapped into φ, θ

Spin 1/2 chain ⟶ Hard-core bosons $\xrightarrow{\boxed{JW}}$ $c_r, c^\dagger_r$ $\xrightarrow{\text{bosonization}}$ φ, θ

XXZ $\{\sigma_z, \sigma_y, \sigma_x\}$ $\qquad b_r, b^\dagger_r$ $\qquad$ Unitary and non-local $\qquad$ ↓ Fermions

$|\uparrow\rangle_r, |\downarrow\rangle_r \longrightarrow |\uparrow\rangle_r = |0\rangle_r, \; |\downarrow\rangle_r = |1\rangle_r \longrightarrow$

XXZ Model : $\boxed{a = 1}$

$$H = \sum_r -J_{xy}\left[\sigma_{x,r}\sigma_{x,r+1} + \sigma_{y,r}\sigma_{y,r+1}\right] + J_z \sum_r \sigma_{z,r}\sigma_{z,r+1}$$

xx+yy, zz

$J_{xy} = J_z$: Heisenberg model $\qquad J_z = 0$: spin 1/2 xy model

$\qquad J_{xy} = 0$: Ising

XXZ Model

- good playground to learn JW
- it is maped into fermionic Hubbard model
- the first example of a new kind of phase tr.
  Berezinskii - Kosterlitz - Thouless phase transition
  Phase tr. between a gapless phase [Luttinger] and a gapped phase.
  BKT has no local order parameter : topological phase tr. | Vortices
- XXZ is maped into the sine-Gordon model
- A good playground to see RG in action
- XXZ is also integrable : it can be exactly solved

H_{xxz}

$$H = -J_{xy}\sum_r \left(\sigma_{x,r}\sigma_{x,r+1} + \sigma_{y,r}\sigma_{y,r+1}\right) + J_z \sum_r \sigma_{z,r}\sigma_{z,r+1}$$

$P : \prod_r \sigma_{z,r} \qquad [\sigma_{z,r}\sigma_{z,r+1}, \sigma_{x,r}\sigma_{x,r+1}]$

$P^2 = \mathbb{1} \Rightarrow P = \pm 1$ [eigenvalues] $\quad \mathbb{Z}_2$ sym "-1"

$Q_z = \sum_r \sigma_{z,r} \qquad [Q_z, H] = 0 \Rightarrow [e^{i\alpha Q_z}, H] = 0 \;\forall\alpha$ $\quad U(1)$

$[Q_z, \sigma_{z,r}\sigma_{z,r+1}] = 0$

$$\boxed{[Q_z, \sigma_{x,r}\sigma_{x,r+1} + \sigma_{y,r}\sigma_{y,r+1}] = 0}$$

$\sigma^+ = \begin{pmatrix}0&1\\0&0\end{pmatrix} = \frac{\sigma_x + i\sigma_y}{2} \quad ; \quad \sigma^- = \begin{pmatrix}0&0\\1&0\end{pmatrix} = \frac{\sigma_x - i\sigma_y}{2}$

$$[\sigma_z, \sigma^+] = \underbrace{\begin{pmatrix}1&\\&-1\end{pmatrix}\begin{pmatrix}0&1\\0&0\end{pmatrix}}_{\begin{pmatrix}0&1\\0&0\end{pmatrix}} - \underbrace{\begin{pmatrix}0&1\\0&0\end{pmatrix}\begin{pmatrix}1&\\&-1\end{pmatrix}}_{\begin{pmatrix}0&-1\\0&0\end{pmatrix}} = \begin{pmatrix}0&2\\0&0\end{pmatrix} = 2\sigma^+$$

$[\sigma_z, \sigma^-] = -2\sigma^- \qquad \sigma^x = \sigma^+ + \sigma^- \qquad \sigma^y = -i\sigma^+ + i\sigma^-$

$$\sigma^x_r\sigma^x_{r+1} + \sigma^y_r\sigma^y_{r+1} = (\sigma^+_r + \sigma^-_r)(\sigma^+_{r+1} + \sigma^-_{r+1}) + \left[-i(\sigma^+_r - \sigma^-_r)(-i)(\sigma^+_{r+1} - \sigma^-_{r+1})\right] =$$
$$= \cancel{\sigma^+_r\sigma^+_{r+1}} + \cancel{\sigma^-_r\sigma^-_{r+1}} + \sigma^+_r\sigma^-_{r+1} + \sigma^-_r\sigma^+_{r+1} =$$
$$- \cancel{\sigma^+_r\sigma^+_{r+1}} - \cancel{\sigma^-_r\sigma^-_{r+1}} + \sigma^+_r\sigma^-_{r+1} + \sigma^-_r\sigma^-_{r+1}$$
$$= 2\left[\sigma^+_r\sigma^-_{r+1} + \sigma^-_r\sigma^+_{r+1}\right]$$

$$[\sigma_{z,r} + \sigma_{z,r+1}, \sigma^x_r\sigma^x_{r+1} + \sigma^y_r\sigma^y_{r+1}] = 0$$

$\Rightarrow [Q_z, H] = 0$

φ, θ

- $|J_{xy}| = |J_z|$: Heisenberg model

→ $\boxed{\boxed{P_y = \prod \sigma_{y,r} \quad P_x = \prod_r \sigma_{x,r}} \quad [P_i, H] = 0 \;\forall \frac{J_{xy}}{J_z}}$

• → $Q_y = \sum_r \sigma_{y,r}$ $\quad$ • → $Q_{x,r} = \sum_r \sigma_{x,r}$ $\qquad J_z > 0$

$\Rightarrow SU(2) : e^{i\vec{J}\cdot\vec{Q}}$ $\qquad +XX +YY +ZZ$

$\qquad -XX -YY -ZZ$

$U^\dagger Q_y U, \quad U^\dagger Q_z U$

y: − − +

U: + + −

$J_{xy} \leftrightarrow -J_{xy}$: unitary mapping

$U = \prod_j \sigma_{z,2j-1}$

$U^\dagger H(J_{xy}, J_z) U = H(-J_{xy}, J_z)$

Limits $J_z \gg J_{xy}$: $H \approx J_z \sum_r \sigma^z_r \sigma^z_{r+1}$

→ Antiferromagnetic chain $|\uparrow\downarrow\uparrow\downarrow\ldots\rangle$, $|\downarrow\uparrow\downarrow\uparrow\downarrow\uparrow\ldots\rangle$

$J_z \to -\infty$, $\frac{|J_z|}{J_{xy}} \gg 1$ $\qquad H \approx -|J_z| \sum \sigma^z_r\sigma^z_{r+1}$ $\quad |\uparrow\uparrow\uparrow\uparrow\ldots\rangle$, $|\downarrow\downarrow\downarrow\downarrow\ldots\rangle$

'ordered' [~~spontaneously~~ breaking P_x or P_y] | $h\sigma_z$

Q_z is continuous

Mermin Wagner

Mapping from Spin 1/2 to HC bosons

$|\uparrow\rangle = |0\rangle \qquad |\downarrow\rangle = |1\rangle$

$n_r = b^\dagger_r b_r$, $n_r|0\rangle = 0 \qquad n_r|1\rangle = |1\rangle$

$\sigma^+_r = b_r \qquad \sigma^-_r = b^\dagger_r$

$n_r = b^\dagger_r b_r = \sigma^-_r\sigma^+_r = \frac{1 - \sigma_{z,r}}{2}$

$\sigma_{z,r} = -2n_r + 1 = (-1)^{n_r}$

$P_z = \prod_r \sigma_{z,r} = \prod_r (-1)^{n_r} = (-1)^{\sum_r n_r}$: parity of the particle number $N = \sum_r n_r$

$Q_z = \sum_r \sigma_{z,r} = \sum_{r=1}^{L}(1 - 2n_r) = L - 2N$

$H_{xxz} = \begin{pmatrix} Q_z = 0 & & \\ & Q_z = +1 & \\ & & Q_z = -1 \end{pmatrix}$

$Q_z = 0$ sector ⇓ $Q_z|\psi\rangle = 0 \Rightarrow (L - 2N)|\psi\rangle = 0$

$N|\psi\rangle = \frac{L}{2}|\psi\rangle$

In this sector $\boxed{\rho = \frac{1}{2}}$

$[b_r, b_{r'}] = 0 = [b^\dagger_r, b^\dagger_{r'}]$ ← bosonic when $r \neq r'$ $\quad [b_r, b^\dagger_{r'}] = 0$

$b^\dagger_r b_r + b_r b^\dagger_r = \mathbb{1}$ ← fermionic !!? HC bosons

Fermionic operators $c_r, c^\dagger_r$ s.t. $\boxed{c^\dagger_r c_r = n_r = b^\dagger_r b_r = \frac{1 - \sigma_{z,r}}{2}}$

JW string :

$\boxed{c^\dagger_r = \sigma^-_r \prod_{j<r} \sigma_{z,j}} \quad ; \quad \boxed{c_r = \sigma^+_r \prod_{j<r} \sigma_{z,j}}$

JW String : non local

Classically '$S_r = \tau_0\tau_1\tau_2 \ldots \tau_{r-1}$' ($\sigma_z$ σ_z)

Based on these def. $\{c_r, c^\dagger_{r'}\} = \delta_{rr'}$

$\{\sigma^\mp_r, \sigma_{z,r}\} = 0$ → verify this

c_j: $\sigma_z\;\sigma_z\;\sigma_z\;\sigma_z\;\sigma_z\;\sigma^+_j$

c_k: $1\;1\;1\;\sigma^+_k\;1\;1$

JW :

- Unitary : Energy spectrum is preserved
- Non local
- $\mathbb{Z}_2$ sym : $P_z = \prod_r \sigma_{z,r} = (-1)^N$
- if O is local and $[O, P_z] = 0$, O it is local for both fermions and bosons

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$\sigma^-_r = c^\dagger_r \prod_{j<r}(-1)^{c^\dagger_j c_j}$

$\sigma^+_r = c_r \prod_{j<r}(-1)^{c^\dagger_j c_j}$ $\qquad \sigma_{z,r} = 1 - 2n_r$

$$H_{xxz} = -2J_{xy}\sum_r\left(\sigma^+_{r+1}\sigma^-_r + \sigma^-_{r+1}\sigma^+_r\right) + J_z\sum_r \sigma_{z,r}\sigma_{z,r+1}$$

=

$$\sigma^+_{r+1}\sigma^-_r = c_{r+1}\boxed{\prod_{j<r+1}(-1)^{c^\dagger_j c_j}}\, c^\dagger_r \prod_{j<r}(-1)^{c^\dagger_j c_j}$$

$(-1)^{c^\dagger_r c_r} \cdot \prod_{j<r}(-1)^{c^\dagger_j c_j}$

$$= \underbrace{c_{r+1}(-1)^{c^\dagger_r c_r} c^\dagger_r}\; \cancel{\prod_{j<r}(-1)^{c^\dagger_j c_j}}\;\cancel{\prod_{j<r}(-1)^{c^\dagger_j c_j}}$$

$c_{r+1}(-1)^{c^\dagger_r c_r} c^\dagger_r \, |1\rangle_r = 0$

$c_{r+1}(-1)^{c^\dagger_r c_r} c^\dagger_r \, |0\rangle_r = c_{r+1}(-1)^{c^\dagger_r c_r}|1\rangle_r = -c_{r+1}|1\rangle_r$

$\boxed{\sigma^+_{r+1}\sigma^-_r = -c_{r+1}c^\dagger_r} = \boxed{c^\dagger_r c_{r+1}}$

$[\sigma^+_{r+1}\sigma^-_r + H.c.] = c^\dagger_r c_{r+1} + c^\dagger_{r+1}c_r$

$J_z\sum_r \sigma^z_r\sigma^z_{r+1}$

$$H_{xxz} = -2J_{xy}\sum_r c^\dagger_r c_{r+1} + c^\dagger_{r+1}c_r + J_z\sum_r (1 - 2n_r)(1 - 2n_{r+1})$$

$-2t\cos k$

$$= \boxed{-\underbrace{2J_{xy}}_{t}\sum_r(c^\dagger_r c_{r+1} + H.c.) + \underbrace{4J_z}_{U}\sum_r n_r n_{r+1} - 4J_z\sum_r n_r}$$

$\boxed{Q_z \leftrightarrow L - 2N}$ $\qquad \rho$ is related to $\frac{Q_z}{L}$

for $\boxed{Q_z = 0} \Rightarrow \rho = \frac{1}{2}$ Half-filling $\boxed{K_F = \frac{\pi}{2a}}$

$\psi^\dagger(r+\varepsilon)\psi(r)$

$$\rho = \underbrace{\rho_0 - \frac{\partial_x\theta}{\pi}}_{S.O.} + \left[\underbrace{A e^{i2K_F r} e^{-i2\theta}}_{F.O.} + H.c.\right]$$

UMKLAPP

$$\rho(r)\rho(r+a) \to \left[A e^{i2K_F r}e^{-i2\theta(r)} \cdot A e^{i2K_F(r+a)}e^{-i2\theta(r+a)} + \ldots\right]$$

$$A^2 \underbrace{e^{i4K_F r + 2K_F a}}_{F.O.} e^{-i2\theta} + \ldots$$

$K_F = \frac{\pi}{2a} \Rightarrow 4K_F r \Rightarrow \frac{4\pi}{2a}\cdot r = 2\pi\cdot\frac{r}{a} \Rightarrow 2\pi p$

$\boxed{e^{i4K_F r} = 1}$

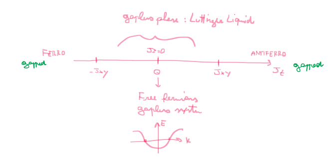
$K_F = \frac{\pi}{2a}$

$$\boxed{H_{xxz} = \frac{v}{2\pi}\int dx\left[(\partial_x\varphi)^2 K + \frac{1}{K}(\partial_x\theta)^2\right] - \frac{J_z}{2\pi^2 a^2}\int dx \cos 4\theta}$$

Taylor of H_{xxz} ⇒ $\underbrace{K = \sqrt{\frac{\pi J_{xy}}{\pi J_{xy} + 4J_z}}}_{\text{Appr. solution}}$ $\qquad K = 1$ for $J_z = 0$ ↓ $K = 1$ for the non-int. case

There exists an exact solution

$$\boxed{\frac{J_z}{J_{xy}} = -\cos\frac{\pi}{2K}} \leftarrow K$$



FERRO $\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow$, $\uparrow\uparrow\uparrow\downarrow\downarrow\downarrow$ $\; 2|J_z|$ $\} |J_z| \gg J_{xy}$

$\cos 4\theta \to e^{i4\theta} + e^{-i4\theta}$

$$\langle e^{i4\theta(r)}e^{-i4\theta(0)}\rangle \sim e^{16\langle\theta\theta\rangle} = e^{-16\cdot\frac{K}{2}\ln|r|} \sim \frac{1}{r^{8K}} = \frac{1}{r^{2D}}$$

$D = 4K \Rightarrow 2 - 4K \begin{cases} > 0 : \text{relevant} \\ < 0 : \text{irrelevant}\end{cases}$

large J_z/J_{xy} ← $K < 1/2$: $\cos 4\theta$ is relevant ⇒ it opens a gap

$\boxed{J_z = J_{xy}}$ $\quad K = 1/2$: phase tr. [BKT, Heisenberg point]

$\frac{J_z}{J_{xy}} \to 0$ $\quad K > 1/2$: $\cos 4\theta$ is irrelevant : Luttinger

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Sine - Gordon model

- $\boxed{\cos\beta\theta}$: $K < \frac{4}{\beta^2}$ relevant
- $\boxed{\cos\beta\varphi}$ $\quad K > \frac{\beta^2}{4}$ relevant

$\frac{K\beta^2}{2} = D$

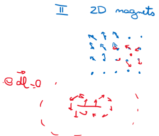
$\boxed{KT}$ $\quad$ I : ^{4}He Superfluid $\quad \psi(x,y) = \sqrt{\rho}\, e^{i\theta(x,y)}$

$\theta \in [0, 2\pi]$

$(\partial_x\theta)^2 + (\partial_y\theta)^2 \quad + (e^{i\varphi} + e^{-i\varphi})$

$$\sim \quad H_{xy} = \sum_{\langle\vec{r},\vec{r}'\rangle} -J\cos[\theta(\vec{r}) - \theta(\vec{r}')]$$

II 2D magnets or antiferro magnets $\quad \theta \to \theta + \alpha$ $\quad \boxed{U(1)}$



$\langle e^{i\theta(x)}e^{-i\theta(y)}\rangle \to 0$ for $|x - y| \to \infty$

quasi long range

$\langle e^{i\theta}e^{-i\theta}\rangle \sim \frac{1}{r^{2D}}$

$\oint \vec{\nabla}_r\theta \cdot d\vec{l} = 0$