KITAEV CHAIN ⟷ 1D TOPOLOGICAL SC

p-wave SC

2 phases { Topological, Trivial } ⟷ { Fermi, Fermi } of the Ising 1D quantum model

Topological Phase $|\mu| < |t|$

→ Appearance of 2 zero-energy localized edge modes

Γ^A, Γ^C $[\Gamma^\cdot, H] = 0$

$(\Gamma^A)^2 = (\Gamma^C)^2 = 1$

$\{\Gamma^A, \Gamma^C\} = 0$

2 fold degeneracy $|GS_+\rangle$, $|GS_-\rangle$

Fermionic parity $P = i\Gamma^A\Gamma^C$ → non local

$d_L = (\Gamma^A + i\Gamma^C)/2$

$d_L^\dagger = (\Gamma^A - i\Gamma^C)/2$

$d_L^\dagger d_L |GS_\pm\rangle = \begin{cases} 0|GS_+\rangle \\ 1|GS_-\rangle \end{cases}$

2 Majorana modes ⟷ Non-local d.o.f. 0, 1

[Zero-energy, localized fermionic mode] → Non local qubit

REALIZATION

$H_0 = \int dx \left[i\psi_x \partial_x \psi_x - i\psi_x^\dagger \partial_x \psi_x^\dagger \right] - \Delta \int dx \left(\psi_x^\dagger \psi_x^\dagger + H.c. \right)$

p-wave for polarized systems : hard to find in Nature

s-wave SC?

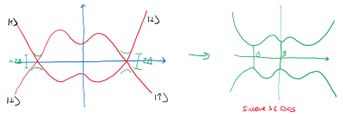
- Single band which interacts with itself through s-wave
  - Spin-orbit
  - Zeeman term
  - s-wave SC

  } Non-commuting → "effective" p-wave [Zero-energy]

$H = \frac{p^2}{2m} + \mu$ → $H = \frac{p^2}{2m} + \mu p + G$ → $H = \frac{p^2}{2m} + \frac{\alpha}{2} p + D\sigma_x$

s-wave

$H = \sum_k \frac{p^2}{2m} c_k^\dagger c_k + \sum_k c_k^\dagger \frac{\alpha}{2} \sigma_x c_k + \mu \sum_k c_k^\dagger c_k + B \sum_{k,\alpha\beta} c_{k\alpha}^\dagger \sigma_{\alpha\beta}^x c_{k\beta}$
$+ \Delta \left(c_{-k\uparrow}^\dagger c_{k\downarrow}^\dagger + H.c. \right)$



2012 : KOUWENHOVEN [DELFT TU]

InAs wires

Al SC

$\xrightarrow{\text{Strong Rashba Spin-orbit}}$ I_{sp} $I = 100\,nm$

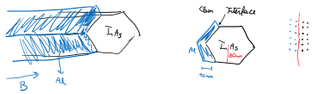
$B = 0.3\,T$ → is needed

V vs I : you have current

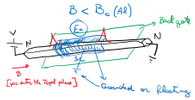
SHARP SC DOS

2015-16 : QDEV [Marcus, J. Nygård, P. Krogstrup...]

Epitaxial growth of Al on In As



$B < B_c(Al)$



1st generation of hybrid Semi/SC devices for Majoranas

- Not very scalable
- It is hard to prove that the zero-energy state come from Majoranas

2016 - - -

2D systems

Transport from N lead to a SC lead

↳ $\frac{dI}{dV}\Big|_{V=0} = \frac{2e^2}{h}$ [perfect Andreev] [Majorana] → 0

Majorana teleportation

Majorana Vs Andreev State

$\Gamma = \Gamma^\dagger$ → You need 2 Majoranas to get a state $d^\dagger d$ → $\Gamma^A + i\Gamma^C$

$\boxed{\text{Majoranas come in pairs}}$

$\Gamma^A \neq \Gamma^\dagger$ → $\Gamma^{A\dagger} = 0, 1$ ↳ Local fermion [Bound state]

To distinguish them : FRACTIONAL JOSEPHSON EFFECT

FRACTIONAL JOSEPHSON

Φ : magnetic flux

$H = H_{Kitaev\,L}(\varphi_L) + H_{Kitaev\,R}(\varphi_R) - J\left(c_L^\dagger d_L + d_L^\dagger c_L\right)$

$c \propto e^{i\varphi_L} c + \partial_x \partial_x \ldots$ → Topol [Energy transf.] → Topol → J is small compared to Δ

$c_L^\dagger = \frac{e^{-i\varphi_L}}{2}\left(\gamma_{2i-1} + i\gamma_{2i}\right)$ phase mode $\quad d_L = \frac{e^{-i\varphi_L/2}}{2}\left(\gamma_{2i} - i\gamma_{2i-1}\right)$ zero-mode

If we are deep in the topol. phases [$\mu \to 0$]

Pert. theory in $\frac{J}{t}, \frac{J}{\Delta} \ll 1$

$J c_L^\dagger d_L + J d_L^\dagger c_L \approx \frac{J}{2} e^{(\varphi_L - \varphi_R)/2} \left(i\gamma_{2i}\gamma_{2i}'\right) + H.c.$

$c_L^\dagger = \frac{e^{-i\varphi_L}}{2} i\gamma_{2i}$, $d_L = \frac{e^{-i\varphi_L/2}}{2}\gamma_{2i}$

At low energy → $\boxed{\left(J\cos\frac{\varphi_L - \varphi_R}{2}\right)\left(i\gamma_{2i}\gamma_{2i}'\right)}$

$H_0^{eff} = -\frac{J}{2} \boxed{\cos\frac{S\varphi_R}{2}} \left(i\gamma_{2i}\gamma_{2i}'\right)$ $\quad \Gamma^A \Gamma^C$ fermionic parity of the junction

if $\mu \neq 1$: $i\gamma_{2i}\gamma_{2i}'$ is a symmetry for H

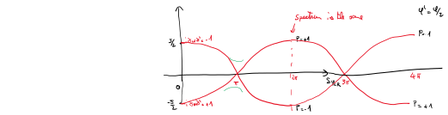
↳ periodic in $2\pi\varphi_R$

Standard Josephson $\quad H \approx -J\cos S\varphi_R$ $\quad e^{iS\varphi_R} = \boxed{e^{i\varphi_R}}$ order

↳ Cooper pairs are hopping

Fractional Josephson $\quad H^{eff} \approx -\frac{J}{2}\cos\frac{S\varphi_R}{2}\left(i\gamma_{2i}\gamma_{2i}'\right)$ $\quad e^{i\varphi_R/2} = \boxed{e^{i\Gamma^A}}$

↳ Fermion

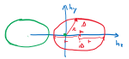


Spectrum is the same

TOPOLOGICAL PHASE for the KITAEV chain

- Phase with a non-trivial Winding [momentum space]

$H_{BdG} = \vec{h}(k) \cdot \vec{\sigma}$ $\quad \vec{h} = (h_x, h_z)$



$\epsilon = \epsilon(k) = \sqrt{h_x^2 + h_y^2}$

$\begin{pmatrix} 0 & h_x - ih_y \\ h_x + ih_y & 0 \end{pmatrix}$ $\quad h_x - ih_y = e^{i\theta(k)}|h(k)|$

$\int_{BZ} \partial_k \theta(k) = 2\pi n$ ↳ Winding number

If $n = 0$: trivial

If $n = \pm 1$: topol.

- Finite system [OBC]

$|\mu| > |t|$ → a single GS [No degeneracy] trivial phase [fully gapped]

$|\mu| < |t|$ → 2 GSs [almost degenerate]

2 zero energy modes Γ^A, Γ^C

→ Γ^A decays in the bulk of the system

$\{\Gamma^A, \Gamma^{A\dagger}\} \subset 0$ $\quad \Gamma^{A\dagger} = 1$

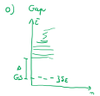
$\boxed{P = i\Gamma^A\Gamma^{A\dagger}}$

$|GS, P = +1\rangle$

$|GS, P = -1\rangle$

TRUE TOPOLOGICAL ORDER [2D T.O.]

0) Gap



$\Delta > 0$ in the thermodynamic limit

↳ Gap that separates GS from ES

$GS_\pm$ are "quasi-degenerate"

$\delta E \propto e^{-\frac{L}{\xi}}$ $\quad L$: system size

$V = L \times L$

1) The number of the GSs varies with the real space topology of the system

Sphere : $g = 0$ $\boxed{1\ GS}$

$g = 1$ $\quad \#GS_g = D^g$

[Toric code $D = 4$]

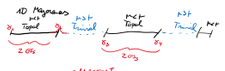
closed surfaces in real space

open surfaces

holes in the system with OBC

$\#GS = D^{\#holes}$

[Surface code $D = 2$]



$\#GSs$: $2^{\#\text{of topo segment}}$

2) For any local operator V_r

$\langle \psi_{0i} | V_r | \psi_{0j} \rangle = \bar{V} \delta_{ij} + o(1/N)$ ← matrix correction

With $\boxed{o \sim e^{-\frac{L}{\xi}}}$ → it gets negligible

- All expectation values are the same for all GSs
  → LOCAL INDISTINGUISHABILITY [for NO DEPHASING]
- For any local operator $\langle \psi_{0i} | V_r | \psi_{0i} \rangle = 0 + o(1/N)$
  → ROBUSTNESS AGAINST local operators [↳ NO DECOHERENCE]

0 + 1 + 2 ⟺ TOPOLOGICAL ORDER $\boxed{T.O.}$

3) → $\boxed{\text{Anyons}}$ excitations : { mass gap, localized } Neither fermions nor bosons $\boxed{\text{only in 2D}}$

Anyons are "emergent" : fluxes or vortices in the system

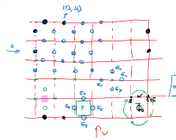
- TORIC CODE
- SURFACE CODES

} 2D systems of Spin-½ d = 2

→ • Many body systems with Topological order → H

- Error correcting quantum codes [quantum memories]
- LGT [deconfined phases]

• Many body Hamiltonian ⟺ self-correcting quantum memory at finite T → $\boxed{T = 0}$

↳ • Series of measurements to detect errors without projecting away info

TORIC code on SQUARE LATTICE