ADVANCED TOPICS in CMT

- RENORMALIZATION GROUP
- 1D MODELS → BOSONIZATION
  BKT
- TOPOLOGICAL PHASES
  - BERRY PHASES, CHERN NUMBER
  - TOP. INS.

RG } Classic, QFT or PATH INTEGRAL

Majorana Modes ↔ 1D TOP. SC

TOPOLOGICAL ORDER TO
- Anyons
- TORIC CODE
- { LGT $\mathbb{Z}_2$; ERROR CORR. ; CM — 50 qubits

RG
- General feat.
- Real Space / KADANOFF / Transf. Matrix

UNIVERSALITY

$H = \ldots$

$\Delta = \Delta_0\left(\frac{T_c - T}{T_c}\right)^{1/2}$

$F = \int d^Dx \; a|\psi|^2 + b|\psi|^4 + \frac{1}{2m^*}\left|(-i\vec\nabla + \vec A)\psi\right|^2 + \ldots + \psi^\dagger\psi(x)\,\psi^\dagger\psi(x+a)$

$a = -\frac{T_c - T}{T_c}$ $\quad b > 0$

$|\psi| \to \left(\frac{T_c - T}{T_c}\right)^{1/2}$

$M = M_0\left(\frac{T_c - T}{T_c}\right)^{1/2}$ [Mean Field]

- MICROSCOPIC DOF : fast oscillating — K.E. large
- MACROSCOPIC DOF : slow oscillating — K.E. small ← Universal features

$Z = \sum_{DOF} e^{-\beta H} = \sum_{slow}\left[\sum_{fast} e^{-\beta H(slow, fast)}\right] = \sum_{slow} e^{-\beta H'(slow)}$

RG : $H(fast, slow) \longrightarrow H'(slow)$

1. COARSE GRAINING : I integrate out the fast d.o.f → I lose info
2. RESCALING : Rescaling of the coordinates to compare H and H'

$x \to x' = \frac{x}{b}$, $k \to k' = bk$, $\quad b > 1$

$e^{ikx} \to e^{ik'x'}$

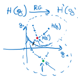
$H(fast, slow) \xrightarrow{CG} H'(slow) \xrightarrow[Resc.]{2} H'(rescaled\ slow)$

$H = \int d^Dx \; a|\psi|^2 + b|\psi|^4 + c(\nabla\psi)^2 + d|\psi|^6 \ldots$

$= \int d^Dx \sum_i g_i O_i(\psi)$ $\qquad \vec g = (a, b, c, d)$

$H' = \int d^Dx \; a'|\psi|^2 + b'|\psi|^4 \ldots + e'|\nabla\psi|^6$

$H(g) \xrightarrow{RG} H'(g') = H(g')$

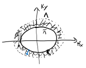


$H(g^*) \xrightarrow{RG} H(g^*)$: FIXED POINTS

STABLE : Uniform ⇔ gapped

UNSTABLE : Phase transitions (II order) ⇔ gapless systems

$b > 0$



Wilson RG

Λ is the largest momentum scale

$\Lambda = \frac{2\pi}{a}$

$\tilde\Lambda = \frac{\Lambda}{b}$ $\qquad b = 1 + dl = e^{dl}$

$\Lambda' = b\tilde\Lambda = \Lambda$

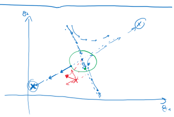
$H = \sum_i g_i \phi_i$

$g_i' = \tilde R_{bi}(\vec g)$ $\qquad b = 1 + dl$ $\qquad l = \ln b$

$\vec g' \simeq \vec g + dl \cdot \partial_l \vec R_b(\vec g)$ — Taylor in l

$\vec R(\vec g) \; \partial_l \vec R(g)$

g^* : $R(g^*) = 0$ $\qquad \vec g - \vec g^* = \delta\vec g$



$\vec g' - \vec g^* = dl\, R(\vec g - \vec g^*)$

$g'_a - g^*_a = dl\, W_{ab}(g_b - g^*_b)$

- 1st order exp of R in g — Taylor in g
- 2nd order $W^{(2)}_{abc}(g_b - g^*_b)(g_c - g^*_c)$

$R(g - g^*) = W_{ab}(g_b - g^*_b) + W^{(2)}_{abc}(g_b - g^*_b)(g_c - g^*_c) + W^{(3)} \ldots$

$\vec\phi_\alpha$ / $\phi_{\alpha a} W_{ab} = \phi_{\alpha b} \cdot \lambda_\alpha$ ←

$\vec\phi_\alpha W = \vec\phi_\alpha \lambda_\alpha$

$v_\alpha = \sum_a \phi_{\alpha a}(g_a - g^*_a)$ $\qquad \vec R(\vec g - \vec g^*)$

$\frac{\partial v_\alpha}{\partial l} = \sum_a \phi_{\alpha a}\,\partial_l(g_a - g^*_a) = \sum_a \phi_{\alpha a} R_{ab}(g_b - g^*_b) = \sum_a \phi_{\alpha a} W_{ab}(g_b - g^*_b) =$

$= \sum_b \phi_{\alpha b}\lambda_\alpha (g_b - g^*_b) = \lambda_\alpha v_\alpha$

v_α : primary operators in CFT

$\frac{\partial v_\alpha}{\partial l} = \lambda_\alpha v_\alpha$ $\qquad v_\alpha(l) = e^{\lambda_\alpha l}\, v_\alpha(l=0)$

$\lambda_\alpha = D - D_\alpha \Leftrightarrow \langle v_\alpha(x) v_\alpha(x')\rangle = \frac{1}{|x - x'|^{2D_\alpha}}$

- $\lambda_\alpha > 0$: v_α is exploding : away from the fixed point; v_α is RELEVANT in the RG SENSE
- $\lambda_\alpha < 0$: v_α is IRRELEVANT in the RG SENSE

H, $\tilde H = H + v_\alpha$

- $\lambda_\alpha = 0$: v_α is MARGINAL → 2nd order in g

$g^* = R(g^*)$

⇓

$\xi \to \frac{\xi}{b}$

$\Rightarrow \xi = \frac{\xi}{b}$ $\quad b > 1$

$\xi = 0$ → Massive : Stable

$\xi \to \infty$ → Self similar : Unstable points : Critical points — No length scale — Third order phase Trans.

Correlations decay as a power law

$\langle O(x) O(x')\rangle \begin{cases} \sim e^{-\frac{|x - x'|}{\xi}} & : \text{gapped phases} \quad \xi < \frac{v}{\Delta_{gap}} \\ \sim \frac{1}{|x - x'|^\beta} & : \text{gapless} \end{cases}$

Gapless system: all corr. functions decay as power law

Gapped system : at least one corr. function decay as exp.

REAL SPACE RG

→ Stat. phys.

KADANOFF

Many body System → lattice

$S_x = +1, -1$

Classical ISING :

$H = -J\sum_{\langle x', x\rangle} S_x S_{x'}$

a: original lattice const

$a^{(1)} = ba$: new lattice const

D space dim. $\qquad b = 2$

b^D : spins

For each block : $S^{(1)}_{x'} = \frac{1}{b^D}\sum_{x \in x'} S_x$

$H(s) \to H'(s')$ very difficult

Z : must be invariant under this coarse graining

$Z(s) = \sum_{\{s\}} e^{-\beta H(s)} \text{ "=" } \int \prod_x Ds_x\, e^{-\beta H(s)} =$

$= \int \prod_x Ds_x \prod_{x'} DS^{(1)}_{x'}\, e^{-\beta H(s)} \prod_{x'} \delta\left(S^{(1)}_{x'} - \frac{1}{b^D}\sum_{x \in x'} S_x\right) =$

$= \int \prod_{x'} D^{(1)}_{x'}\left[\int \prod_x DS_x \prod_{x'} \delta\left(S^{(1)}_{x'} - \frac{1}{b^D}\sum_{x \in x'} S_x\right) e^{-\beta H(s)}\right] = Z(S^{(1)})$

(the bracket is $e^{-\beta H^{(1)}(S^{(1)})}$) $\qquad V = L^D$

Formal coarse graining on the lattice — KADANOFF BLOCKING

$\langle S^{(1)}\rangle = \frac{1}{Z}\int \prod_{x'} D^{(1)}_{x'}\left[\prod_x DS_x \prod_{x'} \delta\left(S^{(1)}_{x'} - \frac{1}{b^D}\sum_{x \in x'} S_x\right) e^{-\beta H(s)}\right] \sum_{x'} S^{(1)}_{x'} \cdot \frac{b^D}{L^D}$ (rescaling)

of $S^{(1)}$ is $\frac{L^D}{b^D}$ $\qquad V \to \frac{V}{b^D}$

$= \frac{1}{Z}\int \prod_x Ds_x\, e^{-\beta H(s)} \sum_{x'}\sum_{x \in x'} \frac{S_x}{b^D} \cdot \frac{b^D}{L^D} = \langle S\rangle$

($\sum_{x'}\sum_{x \in x'} = \sum_x$)

1D Classical ISING Model

- TRANSFER MATRIX

$S_i = \pm 1$

$H_c = -J\sum_i S_i S_{i+1} - h\sum_i S_i$

$Z = \sum_{\{S_i = \pm 1\}} e^{-\beta H_c\{S_i\}} = \sum_{\{S_i = \pm 1\}} e^{\beta J\sum_i S_i S_{i+1} + \beta h\sum_i S_i}$

$h\sum_i S_i = h\sum_{i=1}^{L-1}\frac{S_i + S_{i+1}}{2} + h\frac{S_1}{2} + h\frac{S_L}{2}$

$|S_i\rangle = \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}$ $\qquad T_{S_{i+1}, S_i} = \langle S_{i+1}|T|S_i\rangle$

$Z = \sum \langle S_N|O(S_N)|S_N\rangle \prod_i T_{S_{i+1}, S_i} \langle S_1|O(S_1)|S_1\rangle$

$= \sum_{|S_i\rangle} \langle S_N|O(S_N)|S_{N}\rangle\langle S_{N}|T|S_{N-1}\rangle\langle S_{N-1}|T\ldots\rangle\langle S_2|T|S_1\rangle\langle S_1|O|S_1\rangle$

$T_{\uparrow\uparrow} = e^{\beta J + h\beta}$

$T = \begin{pmatrix} e^{\beta J + \beta h} & e^{-\beta J} \\ e^{-\beta J} & e^{\beta J - \beta h}\end{pmatrix} = \ldots + e^{\beta J}\sinh\beta h\,\sigma_z + e^{-\beta J}\sigma_x$

$e^{\beta J}\cosh\beta\, \mathbb{1}$

Open: $Z = \sum_{S_1 S_N}\langle S_N|e^{\beta h\frac{S_N}{2}} \cdot T^{N-1} e^{\beta h\frac{S_1}{2}}|S_1\rangle$ (Transfer matrix)

PBC

$Z = \mathrm{Tr}\, T^N = \lambda_+^N + \lambda_-^N$

$\lambda_\pm = e^{\beta J}\cosh\beta b \pm \sqrt{e^{2\beta J}\sinh^2\beta h + e^{-2\beta J}}$

$F = -\frac{1}{\beta}\ln Z \overset{N\to\infty}{\simeq} -\frac{1}{\beta}N\ln\lambda_+ = -N \quad -NT\ln\left[\cosh h\beta + \sqrt{\sinh^2\beta h + e^{-4\beta J}}\right]$

$M = \frac{\partial F}{\partial h}$

$\chi = \frac{\partial^2 F}{\partial h^2} \ldots$

$H \to H'$, $T \to T'$

$Z = \mathrm{Tr}\, T^N = \mathrm{Tr}\left(T^b\right)^{\frac{N}{b}} = \mathrm{Tr}\left(T'\right)^{\frac{N}{b}}$

$h = 0$ $\qquad v = \tanh\beta J$

$T_{S_i S_{i+1}} = e^{\beta J S_i S_{i+1}} = \cosh\beta J\left(1 + S_i S_{i+1}\tanh\beta J\right) = c\begin{pmatrix}1+v & 1-v\\ 1-v & 1+v\end{pmatrix}$ (with $c = \cosh\beta J$, $v = \tanh\beta J$)

$T^2 = 2c^2\begin{pmatrix}1+v^2 & 1-v^2\\ 1-v^2 & 1+v^2\end{pmatrix}$

$T^3 = 4c^3\begin{pmatrix}1+v^3 & 1-v^3\\ 1-v^3 & 1+v^3\end{pmatrix}$

$\vdots$

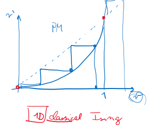
$T^b = 2^{b-1}c^b\begin{pmatrix}1+v^b & 1-v^b\\ 1-v^b & 1+v^b\end{pmatrix} = T'$

$c \to 2^{b-1}c^b$ $\qquad v \to v^b$ $\qquad v' = v^b$

$e^{\beta J S_i S_{i+1}} \to e^{\beta' A' + \beta' J' S'_i S'_{i+1}}$

$\tanh\beta' J' = (\tanh\beta J)^b$

$b > 1$ $\qquad b = 2$ $\qquad v' = v^2$



$v < 1$

→ $v = 0$ $\qquad v = 0$ is a fixed point

$v \to \infty$

1D classical Ising

Stable FP at $v = 0 = \tanh\beta J \Rightarrow J = 0$ or $T \to \infty$ — Paramagnetic

Unstable FP at $v = 1 \Rightarrow J \to \infty$ or $T \to 0$

$Z = \mathrm{Tr}\, T^N$?

$\ldots \langle S_{i+1}|T|S_i\rangle\langle S_i|T|S_{i-1}\rangle \ldots$

$T = e^{-\delta H_q}$ $\qquad H_q$ (2x2) $\qquad \delta$: $\delta\tau$

Classical model in 1+0 dimension
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Quantum model in 0+1 dimension

1+1 quantum
2+0 classical
(D-1) + 1 quantum
D classical model