TOPOLOGICAL PHASES of MATTER

→ IQH [Class Insulators]

- Gapless and "protected" edge modes
  in the bulk we have a insulator
- Topological invariant → CHERN NUMBER
  by Gap
- Robustness against local perturbations
  $H(\lambda)$ → lowest band → CHERN NUMBER
  $H(\lambda+\delta\lambda)$
- Only by closing gaps you change the Chern number

TOPOLOGICAL PHASES

	Symmetry - Protected Top phases	IQH	Systems with TOPOLOGICAL ORDER
GAPPED	Protected edge or surface modes [gapless] → Topological features are stable as long as I do not break some symmetries → T, C, P,	KITAEV CHAIN	2D [FQH, Toric codes] • Degenerate GS • Without any symmetry • Anyons [gapped qp which are neither bosons nor fermions] • Long-range entanglement → Entropy • String operators [string ops]
	• Short range entanglement • Fermionic systems → Non-interacting models		
Gapless Topological models	Non-interacting [Weyl semimetals 3D]		Interacting

TOPOLOGICAL ORDER

- Degenerate GS

Normally, Deg GS are derived by symmetry or symmetry breaking

Symmetries ⇒ Conservation rules [Noether]

- TO: Degeneracy without symm. Conserved quantities without symm.

$[S,H]=0$

Symmetric $S|\psi\rangle = s|\psi\rangle$ [Paramagnet]

Ordered [symm broken] $S|\psi\rangle \neq s|\psi\rangle$ [Ferro]

$S|\uparrow\uparrow\uparrow\uparrow\rangle = |\downarrow\downarrow\downarrow\downarrow\rangle$

$S = \prod_i \sigma_i^x$

0) KITAEV CHAIN (1D) → Majorana modes [QDEV] → "Anyons"
1) Definition of T.O.? in 1D? Main characteristics of the Topol. phase of the Kitaev Chain
2) TORIC CODE → T. order → 2D → Anyons {LGT; Error correcting code → quantum memories; Many Body problem}
3) TOPOLOGICAL QUANTUM COMPUTATION → Theory of anyons
4) Majoranas as anyons

What is Topological Order good for?

→ Quantum memory

Degenerate GS ⇔ Logical states for quantum computer

$T \ll \Delta$

- Δ: suppresses thermal fluctuations $e^{-\frac{\Delta}{k_B T}}$
- Degeneracy $e^{-iHt}|0\rangle = e^{-iE_0 t}|0\rangle$, $e^{-iHt}|1\rangle = e^{-iE_0 t}|1\rangle$ } Avoid dephasing

$\alpha|0\rangle + \beta|1\rangle \xrightarrow{e^{-iHt}} \left(\alpha|0\rangle + \beta|1\rangle\right) e^{-iE_0 t}$

Degenerate States

Magnetic Dipoles: SSB ⇒ classical $|\uparrow\uparrow\uparrow\uparrow\rangle$ $|0\rangle$ ⇔ $|\uparrow\uparrow\downarrow\uparrow\uparrow\uparrow\rangle$ $|1\rangle$; $e^{-\frac{\Delta}{k_B T}}$

Quantum Qbits: $\alpha|0\rangle + \beta|1\rangle$: degeneracy is important to avoid dephasing

System ⊗ Environment

Random events $\Theta = \bigotimes_s |O_s| \otimes \Theta_E$

We desire:

$\langle 0|O_s|1\rangle = 0$ ⇒ No decoherence

$\langle 0|O_s|0\rangle = \langle 1|O_s|1\rangle$ ⇒ No dephasing

Typically if $|0\rangle$ and $|1\rangle$ are ordered symmetry broken states

$\langle 0|O_s|1\rangle \neq 0$ and $\langle 0|O_s|0\rangle \neq \langle 1|O_s|1\rangle$

Physically O_s is a local operator [or a sum of local operators]

$O_s = \sum_x \vec{h}\cdot\vec{\sigma}$

$\vec{\sigma}_x \vec{\sigma}_{x+s}$

What we want is that $\forall$ O_s local

- $\langle \Psi_g | O_s | \Psi_g \rangle = 0$ [Robustness]
- $\langle \Psi_g | O_s | \Psi_g \rangle = \bar{O}$ independent on g

[$\{\Psi_g\}$ are the ground states of a gapped system]

→ degeneracy

KITAEV CHAIN

$$H = \sum_x \mu\, c_x^\dagger c_x - \frac{t}{2}\sum_x \left(c_{x+1}^\dagger c_x + c_x^\dagger c_{x+1}\right) - \sum_x \left(\frac{\tilde\Delta}{2} c_x^\dagger c_{x+1}^\dagger + \frac{\tilde\Delta^*}{2} c_{x+1} c_x\right)$$

$\tilde\Delta = \langle c_x c_{x+1}\rangle$

$U(1)$ fa. : $c_x \to c e^{i\alpha}$

↳ Violated by $\tilde\Delta$: $\tilde\Delta \to \tilde\Delta e^{+i\alpha}$

$\tilde\Delta$ breaks $U(1)$ down to $\mathbb{Z}_2$: $\alpha = 0, \pi$:

Symmetry is $c \to -c$ $[c^\dagger \to -c^\dagger]$

$\mathbb{Z}_2$ symmetry

⇒ Conservation of $P = (-1)^{\sum_x c_x^\dagger c_x}$

Translational inv.

↳ does not depend on position (not necessary)

$\tilde\Delta = |\tilde\Delta| e^{i\phi}$ $c^\dagger \to c^\dagger e^{-i\phi/2}$ $c \to c e^{i\phi/2}$

I apply a unitary $U(1)$ fa :

$$H_\Delta = +\sum_x \mu\, c_x^\dagger c_x - \frac{t}{2}\sum_x \left(c_{x+1}^\dagger c_x + H.c.\right) - \frac{\Delta}{2}\sum_x \left(c_x^\dagger c_{x+1}^\dagger + c_{x+1} c_x\right)$$

I begin with infinite chains [PBC]

$$c_x = \int \frac{dk}{\sqrt{L}} e^{ikx} c_k \qquad a=1$$

$$H = \sum_k \mu\, c_k^\dagger c_k - t\sum_k \cos k\, c_k^\dagger c_k + H_0$$

$$H_0 = -\frac{\Delta}{2}\sum_x \int\frac{dk\,dk'}{L}\left[e^{-ikx} e^{-ik'(x+1)} c_k^\dagger c_{k'}^\dagger + H.c.\right] =$$

$$= -\frac{\Delta}{2}\int dk\,dk'\, \delta_{k,-k'}\, e^{-ikx} c_k^\dagger c_{k'}^\dagger + H.c. =$$

$$= -\frac{\Delta}{2}\int dk\, e^{ik} c_k^\dagger c_{-k}^\dagger + H.c. =$$

$$= -\frac{\Delta}{4}\int dk \left[e^{ik} c_k^\dagger c_{-k}^\dagger + e^{-ik} c_{-k}^\dagger c_k^\dagger\right] =$$

$$= -\frac{\Delta}{4}\int dk\, e^{ik} c_k^\dagger c_{-k}^\dagger - e^{-ik} c_{-k}^\dagger c_k^\dagger =$$

$$= -\frac{\Delta}{2}\int dk\, i\sin k\, c_k^\dagger c_{-k}^\dagger$$

$$H = \sum_k \mu\, c_k^\dagger c_k - t\sum_k \cos k\left(c_k^\dagger c_k + H.c.\right) - \frac{\Delta}{2}\sum_k \left[i\sin k\, c_k^\dagger c_{-k}^\dagger - i\sin k\, c_{-k} c_k\right]$$

Bogoliubov De Gennes

$$H = \frac{1}{2}\sum_{k\in B\tilde z} \begin{pmatrix} c_k^\dagger & c_{-k} \end{pmatrix} \begin{pmatrix} \mu - t\cos k & -i\Delta\sin k \\ i\Delta\sin k & -\mu + t\cos k \end{pmatrix} \begin{pmatrix} c_k \\ c_{-k}^\dagger \end{pmatrix}$$

$$= \frac{1}{2}\sum_{k\in B\tilde z} \begin{pmatrix} c_k^\dagger & c_{-k} \end{pmatrix} \left(\underbrace{(\mu - t\cos k)}_{h_x}\tau_z + \underbrace{\Delta\sin k}_{h_y}\tau_y \right) \begin{pmatrix} c_k \\ c_{-k}^\dagger \end{pmatrix}$$

$h_x(k)\tau_x + h_y(k)\tau_y$ $\quad \vec h = (h_x, h_y)$

$$E^\pm = \pm\sqrt{(\mu - t\cos k)^2 + \Delta^2 \sin^2 k}$$

Particle-hole symmetry $\quad H(k) = -\tau_x H^*(-k)\tau_x$

$H(k)|\psi(k)\rangle = E^+(k)|\psi(k)\rangle$

$\Rightarrow H(k)\tau_x|\psi^*(-k)\rangle = -E(k)\tau_x|\psi^*(-k)\rangle$

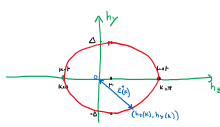
$\forall$ state $|\psi(k)\rangle$ there is a p-h conjugate $|\psi^*(-k)\rangle$ with the opposite energy

[Bdg says we doubled the degrees of freedom, $|\psi(k)\rangle \Leftrightarrow |\psi^*(-k)\rangle$ are the same]

TIME REVERSAL: $H(k) = H^*(-k)$ (not necessary) if $\Delta \in \mathbb{R}$ [ϕ does not depend on x]

Particle-hole symmetry: if $|\psi(k)\rangle = |\psi^*(-k)\rangle$ ⇒ $E(k) = 0$

$H = h_x\tau_x + h_y\tau_y$ $\quad \begin{cases} h_x(k) = \mu - t\cos k \\ h_y(k) = \Delta\sin k \end{cases}$



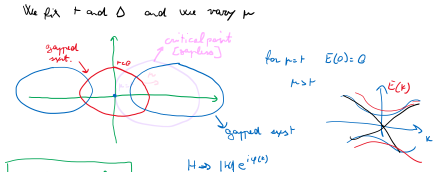
$E^\pm = \sqrt{h_x^2 + h_y^2}$

$k \in B\tilde z \in [0, 2\pi)$

The system is gapped if $E^\pm(k) \neq 0$ $\forall k$

The system is gapless $|\mu \pm t| = 0$ $\quad \Delta > 0$

We fix t and Δ and we vary μ



Winding number

$$W = \frac{1}{2\pi}\int_0^{2\pi} dk\, \partial_k \varphi$$

$H \propto |k| e^{i\varphi(k)}$

$= 0$ if $\mu > t$ or $\mu < -t$

$= \pm 1$ if $-t < \mu < t$

Apply a unitary tr. $\quad \tau_y \to \tau_y$, $\tau_x \to \tau_z$

Different basis : $H(k) = \begin{pmatrix} 0 & h_x - ih_y \\ h_x + ih_y & 0 \end{pmatrix} = \begin{pmatrix} 0 & E(k)e^{-i\varphi(k)} \\ E(k)e^{i\varphi(k)} & 0 \end{pmatrix}$

↓

Eigenbasis of τ_x ↓ PH symmetry

Trivial phase : $W = 0$

Topological phase : $W \neq 0 = \pm 1$

What are these phases?
What are their differences?

KITAEV CHAIN in a finite SYSTEM

$$H = \sum_x \mu\, c_x^\dagger c_x - \frac{t}{2}\sum_x \left[c_{x+1}^\dagger c_x + H.c.\right] - \frac{\Delta}{2}\sum_x \left[c_x^\dagger c_{x+1}^\dagger + H.c.\right]$$

$c_{x+1}^\dagger c_x$ $\quad c_x c_{x+1}$

I restrict to a special case $|t| = \Delta$

$\gamma_{2x-1} = c_x^\dagger + c_x$ $\quad \gamma_{2x} = i(c_x - c_x^\dagger)$

$c_x^\dagger = \frac{\gamma_{2x-1} + i\gamma_{2x}}{2}$ $\quad c_x = \frac{\gamma_{2x-1} - i\gamma_{2x}}{2}$

$$H = \sum_x \mu\, c_x^\dagger c_x - \frac{t}{2}\sum_x \left(c_{x+1}^\dagger + c_{x+1}\right)\left(c_x - c_x^\dagger\right) =$$

$$= \sum_x \frac{\mu}{2}\left(1 + i\gamma_{2x}\gamma_{2x-1}\right) + \frac{t}{2}\sum_x \gamma_{2x+1}\gamma_{2x}$$

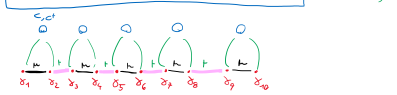
$\gamma_x^2 = 1$; $\{\gamma_x, \gamma_y\} = 2\delta_{xy}$ ⇒ Clifford Algebra

$\gamma_x = \gamma_x^\dagger = \gamma_x^{-1}$ → Majorana modes

$$c_x^\dagger c_x = \frac{1 + i\gamma_{2x}\gamma_{2x-1}}{2}$$

$$= \frac{\gamma_{2x-1} + i\gamma_{2x}}{2}\cdot\frac{\gamma_{2x-1} - i\gamma_{2x}}{2} = \frac{2 + \gamma_{2x}\gamma_{2x-1} - \gamma_{2x-1}\gamma_{2x}}{4}$$

$$H = \frac{\mu}{2}\sum_x \left(-i\gamma_{2x-1}\gamma_{2x}\right) + \frac{t}{2}\sum_x -i\gamma_{2x}\gamma_{2x+1} + \frac{N\mu}{2}$$



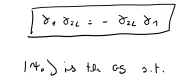
$i\gamma_{2x-1}\gamma_{2x} = 1$ for $c_x^\dagger c_x = 0$

$i\gamma_{2x-1}\gamma_{2x} = -1$ for $c_x^\dagger c_x = 1$

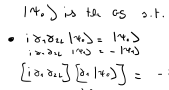
$P = (-1)^{\sum_x c_x^\dagger c_x} = \prod_x (i\gamma_{2x-1}\gamma_{2x})$

$i\gamma_{2x-1}\gamma_{2x} \cdot i\gamma_{2x+1}\gamma_{2x} = -\gamma_{2x-1}\gamma_{2x}\gamma_{2x+1}\gamma_{2x} = \gamma_{2x-1}^2\gamma_{2x}^2 = 1$

$\mu \gg t$ [$t = \Delta = 0$] [Trivial phase]



$\mu = 0$ [$t = \Delta > 0$] [TOPOLOGICAL PHASE]



GS: $i\gamma_{2x}\gamma_{2x+1}|GS\rangle = |GS\rangle$

1 + 1 MBK GS

$P = \gamma_1\gamma_2\gamma_3\gamma_4\gamma_5\gamma_6\gamma_7\gamma_8$

$P|GS\rangle = i\gamma_{2x}\gamma_{2x+1}|GS\rangle$ [$\mu = 0$]

$[\gamma_1, H_+] = 0 = [\gamma_{2L}, H_+]$ ($\mu = 0$) ⇒ γ_1 and γ_{2L} are symmetries of $H_{\mu=0}$

$H_+|\psi_n\rangle = E_n|\psi_n\rangle$

$H_+\gamma_1|\psi_n\rangle = E_n(\gamma_1|\psi_n\rangle)$

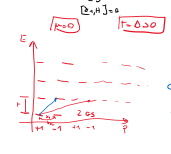
$\gamma_1\gamma_{2L} = -\gamma_{2L}\gamma_1$ 2 symmetries which do NOT commute. ⇒ All the eigenstates of H_+ are degenerate

$|\psi_0\rangle$ is the GS s.t. $i\gamma_1\gamma_{2L}|\psi_0\rangle = |\psi_0\rangle$ [even] GS is degenerate

$i\gamma_1\gamma_{2L}|\psi_0\rangle = -|\psi_0\rangle$

$[i\gamma_1\gamma_{2L}][\gamma_1|\psi_0\rangle] = -\gamma_1[i\gamma_1\gamma_{2L}|\psi_0\rangle] = -\gamma_1|\psi_0\rangle$

GS [γ_1, H] = 0



- Any local operator in the bulk which changes the fermionic parity excites the system

$\mu = 0$ $\quad P = i\gamma_1\gamma_{2L}$ $\quad d = \frac{\gamma_1 - i\gamma_{2L}}{2}$ $\quad d^\dagger = \frac{\gamma_1 + i\gamma_{2L}}{2}$

$d^\dagger d = 0, 1$ in the 2 GSs