4 Computational power of anyons

Topological quantum computation encodes and manipulates information by exclusively employing anyons. To study the computational power of anyons we plan to look into their fusion and braiding properties in a systematic way. This will allow us to identify a Hilbert space, where quantum information can be encoded fault-tolerantly. We also identify unitary evolutions that serve as logical gates. It is an amazing fact that fundamental properties, such as particle statistics, can be employed to perform quantum computation. As we shall see below, the resilience of these intrinsic particle properties against environmental perturbations is responsible for the fault-tolerance of topological quantum computation.

Anyons are physically realised as quasiparticles in topological systems. Most of the quasiparticle details are not relevant for the description of anyons. This provides an additional resilience of topological quantum computation against errors in the control of the quasiparticles. In particular, the principles of topological quantum computation are independent of the underlying physical system. We therefore do not discuss its properties in this chapter. The abstraction might create a conceptual vacuum as many intrinsic properties of the system might appear to be absent. For example, we shall not be concerned with the trapping and transport of anyons or with geometrical characteristics of their evolutions. In this chapter we treat anyons as classical fundamental particles, with internal quantum degrees of freedom, much like the spin. Moreover, we assume that we have complete control over the topological system, in terms of initial-state preparation and final-state identification. Details of how this can be achieved on concrete physical implementations can be found in later chapters, where explicit topological systems are considered.

This chapter presents the inception of anyonic models. It introduces the necessary steps to consistently define an anyonic theory from basic principles. The first step is to define a finite set of anyonic particles, or species. We identify the vacuum of this set as the trivial particle. Moreover, every particle should have its own antiparticle. These particles are characterised by internal degrees of freedom, which are associated with quantum numbers. Relations between these quantum numbers are obtained by the fusion rules, which dictate what types of species are obtained when combining two particles together. The next step is to verify if the defined model satisfies a set of consistency conditions. These are known as the pentagon and hexagon equations, named after their characteristic geometric configuration. A discrete set of solutions is obtained from these equations that determine the braiding properties of the particles. The fusion and braiding properties are sufficient to obtain concrete models that can be used for topological quantum computation.

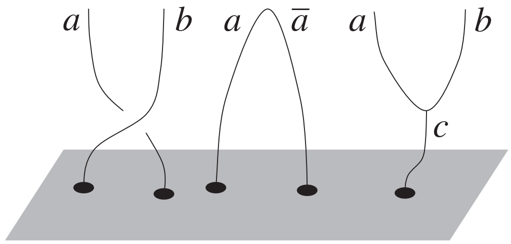

Fig. 4.1 Worldlines of particles that are positioned on a plane with time flowing downwards. An exchange of two particles, a and b, is depicted in terms of braided worldlines. A pair-creation from the vacuum of a particle a and its antiparticle $\bar{a}$ corresponds to two lines initiated at the same position. Fusion of two particles a and b is given by two lines coming together and producing a third particle c.

4.1 Anyons and their properties

We now present the fundamental properties of anyons in a systematic way. It is convenient to keep track of the anyon history by employing their worldlines. In this way we can easily visualise statistical processes and predict the time evolution of anyonic states. Examples of such processes are depicted in Figure 4.1. There, we assume that we can trap and move anyons around the plane leading to worldlines in $2 + 1$ dimensions. Exchanges of two anyons can be described by just braiding their worldlines. We can also depict the pair-creation of anyons from the vacuum as well as the fusion process that occurs when they are brought together, thereby resulting in a new anyon. Since the worldlines represent topological evolutions, no attention needs to be paid to their exact shape. We only need to focus on their global characteristics.

4.1.1 Particle types

Our starting point is to recognise that there can be a variety of different anyonic models. Each model is determined by the statistical properties of its particles. Let us consider such a particular model. To describe it we introduce finitely many different species of particles. When they are realised in a topological system they correspond to quasiparticle excitations that can be distinguished according to their properties with respect to certain physical observables. In the following we shall use the terminologies ‘particle’, ‘anyon’ and ‘quasiparticle’ in an interchangeable way.

Consider a set of particle types

$$1,\ a,\ b,\ c, \ldots, \tag{4.1}$$

where 1 corresponds to the unique vacuum, while $a, b, c, \ldots$ correspond to a finite series of different particle types. The simplest non-trivial model contains one more particle than the vacuum. Every particle, a, needs to have its own antiparticle, $\bar{a}$, which could be itself, so that they can be pair-created from the vacuum. Each particle can be locally distinguished by its topological or anyonic charge, which is a conserved quantum number. For example, the anyonic charge indicates whether a particle corresponds to the vacuum, to a boson or to a fermion. Particles with richer anyonic properties can be similarly identified. The anyonic charge is better described in combination with the rest of the anyonic particles in the system, as we shall see in the following.

4.1.2 Fusion rules of anyons

We now consider the fusion properties of finitely many anyons that belong to a given model. The fusion corresponds to bringing two anyons together and determines how they behave collectively. No interactions need to take place between the anyons. Fusion can be viewed as putting two anyons in a box and identifying the statistical behaviour of the box. For example, fusing two fermions together produces a boson. In general, the fusion rules are written as

$$a \times b = N_{ab}^c c + N_{ab}^d d + \cdots \tag{4.2}$$

These rules indicate the possible outcomes $c, d, \ldots$, listed with the $+$ symbol, that result when anyons a and b are brought together, denoted by the $\times$ symbol. The ordering of a and b is not important, so that

$$a \times b = b \times a. \tag{4.3}$$

When a and b are fused there might be several distinct mechanisms that produce particle c, enumerated by the integers N_{ab}^c.

It is possible to prepare two anyons in a certain way so that they have a unique fusion outcome. For example, two anyons produced from a vacuum pair-creation have the vacuum as their unique fusion channel. So the several possible outcomes on the right-hand side of (4.2) could be understood as different possible preparations of a and b that would result in a certain fusion outcome. Finally, the fusion process can be time-reversed. Consider the case where the fusion of a and b gives a specific fusion outcome c. When time is inverted the same process describes the splitting of anyon c into its constituent particles a and b.

Anyons are systematically characterised by their fusion behaviour. For example, Abelian anyons have only a single fusion channel

$$a \times b = c. \tag{4.4}$$

Their fusion space is one-dimensional. In contrast, non-Abelian anyons always have multiple fusion channels that give rise to higher-dimensional fusion spaces

$$\sum_c N_{ab}^c > 1. \tag{4.5}$$

This simple property is closely related to their statistical behaviour. As we shall see below, non-Abelian statistics is manifested as a non-trivial evolution between the different

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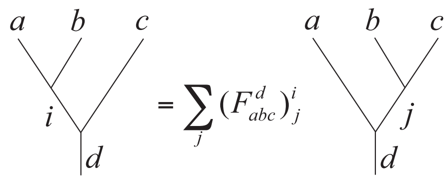

Fig. 4.2 When the order of fusion between three anyons, a, b and c, with outcome d is changed then a rotation in the fusion space is performed given by the matrix F^d_{abc}. In this diagrammatic equation the index i denotes a certain anyon, while the summation in j ranges through all possible fusion outcomes of b and c.

possible fusion outcomes given in (4.2). Abelian statistics corresponds to the evolution of a unique state by a phase factor.

When we fuse several anyons, we are free to choose the ordering in which the basic fusion processes take place. For example, three anyons, a, b and c, with total fusion channel d can be fused in two different ways. Fusing a and b might have an outcome i that is different from the outcome j of fusing b and c. These are the only two distinctive possible orders in which one can fuse three anyons. Having i and j different is consistent with having a fixed total fusion outcome, d. Explicitly, fusing i with c gives d and fusing j with a gives d as well, as shown in Figure 4.2. This is much like the different ways one can combine several spin-1/2 particles to obtain a given value for the spin of their composite. The matrix F^d_{abc} with i, j elements $(F^d_{abc})^i_j$ that relates these two different processes is called the fusion or F matrix and its action is illustrated in Figure 4.2. The dimensionality of this matrix depends on the number of possible in-between outcomes of the fusions.

The choice of fusion order is a degree of freedom in the description of several anyons. Indeed, a sequence of anyons fused in a particular order provides a set of possible in-between fusion outcomes. Another person who has exactly the same set of anyons and decides to fuse them in a different order could obtain a different sequence of in-between fusion outcomes. The F matrix can be employed to systematically translate between these two different sets. Actually, any fusion ordering can be mapped to any other with a sufficient number of F move applications, like the ones depicted in Figure 4.2. Choosing the order in which anyons are fused can be viewed as a choice of basis and the F matrix as a transformation between different bases.

4.1.3 Anyonic Hilbert space

The Hilbert space of anyons is rather unusual. It is the space of states that corresponds to the fusion process. We assign a distinct state to the time evolution of two anyons that fuse to a certain outcome. In this way, states that correspond to different fusion outcomes are automatically orthogonal to each other as we can always distinguish between different anyons. Let us denote the fusion Hilbert space of n anyons by $\mathcal{M}_{(n)}$. Since Abelian anyons have only a single fusion outcome, their fusion Hilbert space is trivial

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$$\dim(\mathcal{M}_{\text{Abelian}}) = 1. \tag{4.6}$$

Suppose we consider two non-Abelian anyons a and b with the fusion rule $a \times b = \sum_c N_{ab}^c c$, as in (4.2). In this case we assign the state

$$|\, a, b \to c; \mu\rangle \tag{4.7}$$

to each possible fusion outcome. The index $\mu = 1, \ldots, N_{ab}^c$ parameterises the possible multiplicity of a certain fusion channel. To simplify notation we restrict ourselves in the following to the case where $N_{ab}^c \leq 1$, so we can drop the index μ.

Let us have a closer look at a variety of fusion processes and their corresponding dimensionality. The hypothetical evolution of a single non-trivial anyon going through fusion with the vacuum and coming out as the vacuum is not permitted. This allows us to assign the zero-dimensional Hilbert space to this evolution

$$\dim(\mathcal{M}_{(1)}) = 0. \tag{4.8}$$

When there is an initial and a final anyon, that due to anyonic charge conservation are necessarily equal to each other, then the Hilbert space is one-dimensional

$$\dim(\mathcal{M}_{(2)}) = 1. \tag{4.9}$$

The Hilbert space of two initial non-Abelian anyons a and b with a fusion outcome c with multiplicity N_{ab}^c gives rise to equally many states. The Hilbert space of three anyons related by the fusion process is therefore given by

$$\dim(\mathcal{M}_{(3)}) = N_{ab}^c. \tag{4.10}$$

Consider now three initial anyons a, b and c that fuse to d. To evaluate the dimension of their Hilbert space we need to count all possible in-between outcomes from pairwise fusions. To be explicit we can initially fuse a and b and then fuse the outcome i of this fusion with c in order to obtain d, as illustrated in Figure 4.3(a). For each i we might write the state of this fusion process as

$$|\, i\rangle = |\, a, b \to i\rangle \,|\, i, c \to d\rangle\,, \tag{4.11}$$

where the tensor product symbol between the states of the two different fusion processes has been omitted. This state can be written as $|\, i\rangle$ when the fixed anyons a, b, c and d are implicitly assumed. If there is more than one possible outcome then the corresponding states, $|\, i\rangle$, can comprise a basis of a higher-dimensional Hilbert space denoted $\mathcal{M}_{(4)}$. Alternatively, one could consider fusing b and c and their outcome j with a to obtain d with

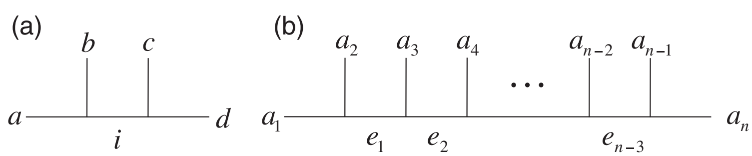

Fig. 4.3 (a) Basis states $|\, i\rangle$ for the fusion space of four ordered anyons a, b, c and d. (b) Basis states $|\, e_1 e_2, \ldots, e_{n-3}\rangle$ of n ordered anyons a_i with $i = 1, \ldots, n$.

corresponding basis states $|b,c\to j\rangle\,|j,a\to d\rangle$. Changing between these two different fusion states corresponds to the F move we described in Figure 4.2. The change of basis in the Hilbert space $\mathcal{M}_{(4)}$ of the four anyons is given by

$$|a,b\to i\rangle\,|i,c\to d\rangle=\sum_j (F^d_{abc})^i_j\,|b,c\to j\rangle\,|a,j\to d\rangle \tag{4.12}$$

or simply $|i\rangle=\sum_j (F^d_{abc})^i_j\,|j\rangle$. If we consider more initial anyons we have to specify how we order their fusions if we want to uniquely determine the basis states of their Hilbert space $\mathcal{M}_{(n)}$. For the ordering of n anyons a_i with $i=1,\dots,n$, depicted in Figure 4.3(b), we have the states

$$|\mathbf{e}\rangle=|e_1,e_2,\dots,e_{n-3}\rangle=|a_1,a_2\to e_1\rangle\,|e_1,a_3\to e_2\rangle\dots|e_{n-3},a_{n-1}\to a_n\rangle\,. \tag{4.13}$$

By a simple counting argument we can see that the number of different fusion possibilities is given by

$$\dim(\mathcal{M}_{(n)})=\sum_{e_1\dots e_{n-3}} N^{e_1}_{a_1a_2}\dots N^{a_n}_{e_{n-3}a_{n-1}}. \tag{4.14}$$

A more intuitive expression for $\dim(\mathcal{M}_{(n)})$ can be given in terms of the quantum dimension d_i of anyon i. Quantum dimension is a fancy name that refers to the dimension of the Hilbert space associated with an anyon. Starting from the fusion rules $a\times b=\sum_c N^c_{ab}c$ one can define the quantum dimension to satisfy the following relation:

$$d_a d_b=\sum_c N^c_{ab}d_c. \tag{4.15}$$

Abelian anyons, such as the vacuum, always have $d_i=1$, while non-Abelian anyons necessarily have $d_i>1$. It is worth noting that the quantum dimension does not need to be an integer. Consider now the set of n anyons, shown in Figure 4.3(b), where all a_i are identical to a. The quantum dimension characterises how fast the dimension of the Hilbert space grows when one additional a particle is inserted, i.e.,

$$\dim(\mathcal{M}_{(n)})\propto d_a^n, \tag{4.16}$$

where we assume that n is large (see Exercise 4.2). The dimension of $\mathcal{M}_{(n)}$ is always an integer as it enumerates different fusion outcomes, while d_a^n does not need to be an integer. Relation (4.16) therefore gives the proper behaviour only for large n. The important fact is that the fusion Hilbert space increases exponentially fast with the number of anyons n. Nevertheless, $\dim(\mathcal{M}_{(n)})$ is not necessarily the product of the dimensions of particular subsystems. Finally, we define the total quantum dimension of a topological model by

$$\mathcal{D}=\sqrt{\sum_i d_i^2}, \tag{4.17}$$

where the summation runs through all the anyonic species of the model. The quantity $\mathcal{D}$ can be defined for any topological model.

Before moving further let us interpret these rather obscure fusion states in terms of more conventional means. After all, when the topological model is physically realised, the fusion states have to correspond to certain quantum states of the constituent particles. We expect that the states of the microscopic system which correspond to different fusion outcomes are pairwise orthogonal. On the other hand, microscopic states that produce the same fusion outcome are considered as equivalent. This is manifested as an indistinguishability of the microscopic states in terms of their topological properties. The information on the fusion outcome is not a local property as it is encoded in the system in a non-local way. For example, consider two quasiparticles a and b prepared in a given fusion channel c. Their fusion state $|a, b \to c\rangle$ corresponds to a concrete state of the underlying microscopic system. When this state evolves adiabatically in order to fuse anyonic quasiparticles, the state that corresponds to quasiparticle c results. All the states of the constituent particles along this time evolution that describe different positions of the a and b quasiparticles are equivalent since they correspond to the same fusion state. As a conclusion the fusion states correspond, in general, to a whole family of states of the microscopic system.

4.1.4 Exchange properties of anyons

Statistics is manifested in the evolution of the wave function of two particles when they are exchanged. In two spatial dimensions particles are allowed to exhibit any arbitrary statistical evolution. To systematically assign statistical evolutions consider the effect of exchanging two anyons, a and b, when their fusion channel is fixed, i.e., $a \times b \to c$, as shown in Figure 4.4. This exchange can be viewed as a half twist of the c particle. Hence, the exchange evolution R^c_{ab} of the fusion state $|a, b \to c\rangle$ should simply be a phase factor as it corresponds to the rotation of a single particle. We can build a matrix R_{ab} by ordering the phases for all possible fusion outcomes c of a and b on its diagonal. This exchange matrix will be referred to in the following as the R matrix.

The superposition of multiple fusion outcomes in the braiding process can result in an exchange operator B, which is a non-diagonal unitary matrix. To demonstrate this, we consider the effect of exchanging a and b when these two anyons do not have a direct fusion channel. Then the F moves can be employed to change their fusion order until the exchange is acting on anyons with a certain fusion channel. In Figure 4.5 we derive diagrammatically the relation

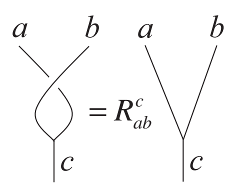

Fig. 4.4 The clockwise exchange of anyons a and b with fusion outcome c gives the phase R^c_{ab}.

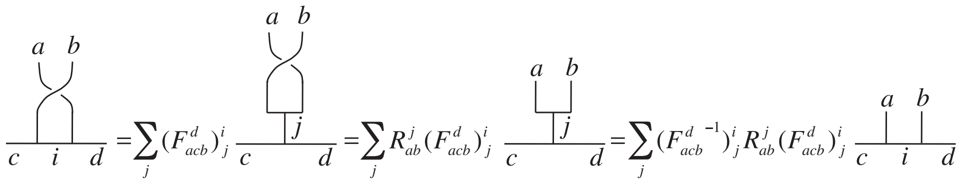

Fig. 4.5 If anyons a and b do not have a direct fusion channel then their exchange can be defined in terms of the F moves that rearrange the order of fusion. Here we depict such a series of operations that gives rise to the braiding unitary $B = F^{-1}RF$.

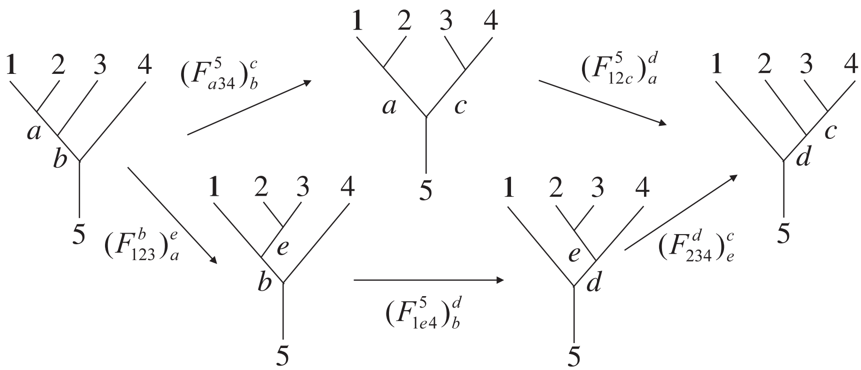

Fig. 4.6 Geometric interpretation of the pentagon identity. We start with the canonical fusion of four anyons with a fixed total outcome given by the leftmost diagram. Two different sequences of F moves applied on the initial fusion diagram arrive at the same final diagram. It is taken as an axiom that these two sequences are identical.

$$B_{ab} = F^{d\ \ -1}_{acb} R_{ab} F^d_{acb}. \tag{4.18}$$

The braiding matrix B_{ab} depends implicitly on anyons c and d. Notably, this unitary matrix can be non-diagonal due to the F transformation. The B unitary corresponds to irreducible representations of the braid group. The general properties of the braid group are analysed in Chapter 8.

4.1.5 Pentagon and hexagon identities

Arbitrary as they might seem, the F and R matrices that accompany a given set of fusion rules have to satisfy simple consistency equations. These conditions dramatically restrict the multiplicity of possible models, which satisfy the same fusion rules, to finitely many. They are called pentagon and hexagon identities (Turaev, 1994) due to their geometric interpretation and they are the subject of study of topological quantum field theory (Witten, 1989).

Let us consider Figure 4.6, where the fusion process of four anyons, 1, 2, 3 and 4, is depicted. Consider the leftmost diagram with a certain fusion ordering. We assign specific in-between fusion outcomes, a and b, that have a fixed total fusion channel, 5. By employing the two F moves depicted in the upper path it is possible to completely reverse the fusion ordering and transform the fusion diagram to the rightmost one. However, it is also possible to connect these two diagrams by following a completely different path that includes three F moves. This is depicted in Figure 4.6 as the lower path. It is an axiom that these two processes should be equivalent. Stated differently, if there is a unique interpretation of fusion states by the fusion diagrams then distinct transformations with F moves that connect the leftmost and rightmost diagrams ought to be identical. Imposing this axiom gives rise to the pentagon identity

$$(F^5_{12c})^d_a(F^5_{a34})^c_b = \sum_e (F^d_{234})^c_e(F^5_{1e4})^d_b(F^b_{123})^e_a. \tag{4.19}$$

This equation provides a relation between the matrix elements of all possible F matrices of the model. The e summation is over all possible particle types that we can have in the fusion diagrams shown in Figure 4.6.

An independent set of identities can be obtained by employing in addition the braiding processes. Consider three anyons, 1, 2 and 3, that fuse to 4 through the fusion channel a, as shown in the leftmost diagram of Figure 4.7. By alternating applications of F and R moves it is possible to interchange the fusion order of the initial anyons in two distinct ways. Demanding again that these two distinct processes correspond to the same overall procedure gives rise to the hexagon identity

$$\sum_b (F^4_{231})^c_b R^4_{1b}(F^4_{123})^b_a = R^c_{13}(F^4_{213})^c_a R^a_{12}. \tag{4.20}$$

When instead counterclockwise exchange operations R^{-1} are employed, an equivalent set of equations is obtained (Bonderson, 2007).

Finally note that the pentagon and hexagon identities become trivial for Abelian models, whose statistical phase can be arbitrary. Consistent non-Abelian anyonic models are completely determined by the pentagon and hexagon identities without the need for further conditions (MacLane, 1998). For a given number of anyon types with fixed fusion rules,

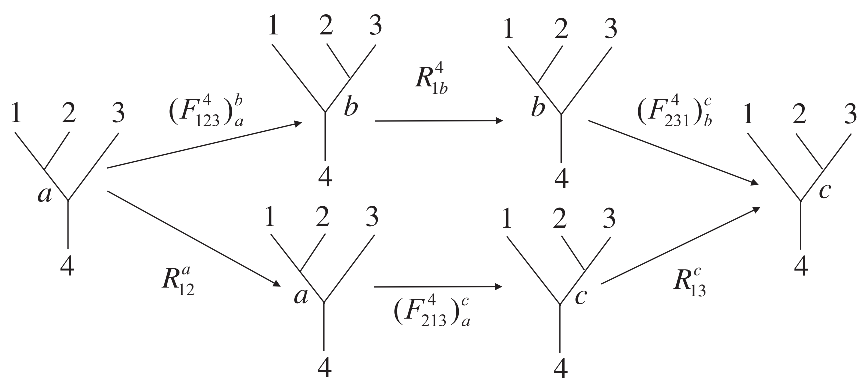

Fig. 4.7 The hexagon identity relates two distinct fusion processes of three anyons with a fixed total fusion outcome by a sequence of fusion rearrangements and braiding operations.

the solutions of these two polynomial equations give a discrete possibly empty set of F and R matrices. This resembles the discrete character of the solutions of quadratic equations. The discreteness in the F and R solutions, known as the Ocneanu rigidity (Kitaev, 2006) is in agreement with the discrete nature of topological models. Hence, topological models are not continuously connected with each other, which provides much of their resilience against erroneous perturbations.

4.1.6 Spin and statistics

It is well known that bosons have integer spin (e.g., 0) and fermions half-integer spin (e.g., 1/2). When a spin-0 particle is rotated around itself its wave function is not changed, while when a spin-1/2 particle is rotated by 2π then the fermionic wave function acquires a minus sign (Rauch *et al.*, 1975). This is in agreement with the exchange statistics of these particles. The tight relation between spin and statistics (Finkelstein and Rubinstein, 1968) also governs the behaviour of anyons. As the statistics of anyons is neither bosonic nor fermionic, the spin of anyons can take any value different from 0 or 1/2. Up to now the worldlines of anyons allowed us to keep track of their braiding history. To keep track of their self-rotations we now extend the worldlines to 'worldribbons'. This allows us to establish the connection between spin and statistics.

Consider two anyons a and b with a given fusion channel c that are exchanged k times in a clockwise fashion. Particle exchanges cannot change the fusion outcome of a and b, but they can generate phase factors, R^c_{ab}, as we have seen previously. Suppose that the quantum mechanical evolution, associated with the exchange process, remains invariant under continuous deformations of the worldribbons, due to its topological character. Then it is possible to continuously transform the k clockwise exchanges of a and b to k twists of the ribbons by an angle π, clockwise for ribbon c and counterclockwise for ribbons a and b. This is depicted in Figure 4.8. The spin-statistics theorem dictates that the amplitudes of these two processes have to be equal (Finkelstein and Rubinstein, 1968). Let us assign spins, s_a, s_b and s_c to anyons a, b and c, respectively. Clockwise rotating a spin s particle

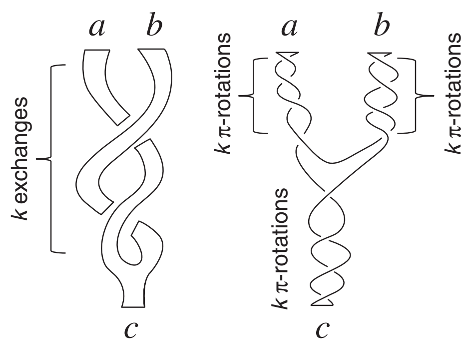

Fig. 4.8 Interpreting anyons with ribbons facilitates accounting for twists and exchanges. Clockwise exchanging k times anyons a and b can be continuously deformed to k clockwise π rotations for anyon c and k counterclockwise π rotations for both anyons a and b. These two configurations are topologically equivalent.

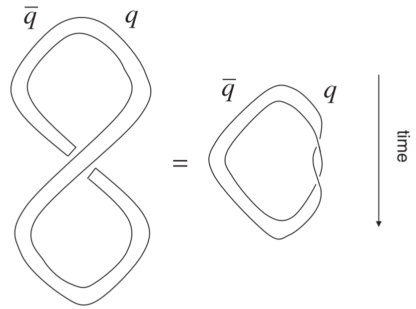

Fig. 4.9 An anyon q and an anti-anyon $\bar{q}$ denoted with their worldribbons are pair-created, exchanged and then fused to the vacuum. The exchange process can be continuously deformed to rotating one of the anyons around itself by 2π. This equivalence can be nicely verified with a belt.

by angle ϕ generates the phase factor $e^{-i\phi s}$ in front of its wave function. As the amplitude of the exchange and the twisting processes have to be the same, the twists of the particles have to generate the appropriate spin phase factors to compensate for the statistical ones. Applying this to the process of Figure 4.8, we obtain the spin-statistics theorem given in the form (Bais *et al.*, 1992)

$$(R^c_{ab})^k = e^{i\pi k s_a} e^{i\pi k s_b} e^{-i\pi k s_c}. \tag{4.21}$$

As an example, we consider the $k = 1$ case. We restrict ourselves to anyons $a = q$ and $b = \bar{q}$ that are particles and antiparticles of each other with the vacuum being their fusion channel, $c = 1$. The corresponding spins are given by $s = s_q = s_{\bar{q}}$ and $s_1 = 0$. As these anyons can be generated from the vacuum and fused to it, their evolution corresponds to a worldribbon that forms a closed loop. In Figure 4.9 we show the schematic equivalence between the process of exchanging q and $\bar{q}$ and rotating only one of them by 2π. The first evolution gives rise to a statistical phase $R^1_{q\bar{q}}$ and the second to a spin phase $e^{i2\pi s}$, where s is the spin of the q and $\bar{q}$ anyons. Hence,

$$R^1_{q\bar{q}} = e^{i2\pi s}, \tag{4.22}$$

which is exactly the spin-statistics relation. As Abelian anyons can have arbitrary values of statistic phases $e^{i\varphi}$, their corresponding spin s can take arbitrary values as well.

4.2 Anyonic quantum computation

In the previous sections we identified the Hilbert space of non-Abelian anyons and analysed the manipulations that lead to unitary evolutions of this space. We are now ready to see how to employ anyons to perform quantum computation. For that we need to implement several operations on anyons to eventually achieve the desired quantum state manipulations. Our steps follow the circuit quantum computation model. This model requires initialisation of the physical system in a well-determined quantum state, application of quantum gates

Table 4.1 Anyonic quantum computation

Quantum computation		Anyonic manipulation
State initialisation	→	Create and arrange anyons
Quantum gates	→	Braid anyons
State measurement	→	Detect anyonic charge

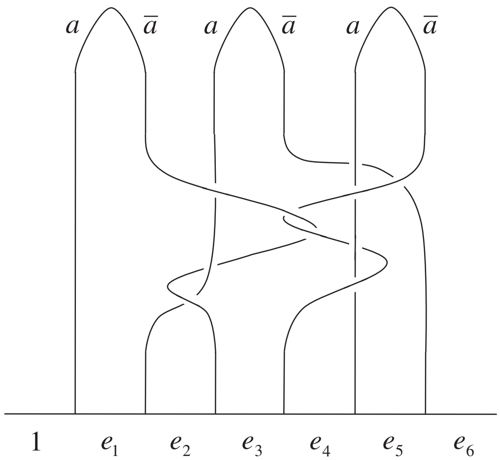

Fig. 4.10 A possible configuration of topological quantum computation. Initially, pairs of anyons, a, $\bar{a}$ are created from the vacuum. Braiding operations between them unitarily evolve their fusion state. Finally, fusing the anyons together gives a set of outcomes e_i, $i = 1, \dots,$ which encodes the result of the computation.

and measurement of the final state. To implement these steps we seek to initially create and arrange anyons, braid them together and eventually determine their anyonic charge, as summarised in Table 4.1.

4.2.1 Anyonic setting

A possible configuration and manipulation of anyons that can result in quantum computation is shown in Figure 4.10. We start with a set of anyons that are prepared in a well-defined fusion state. For example, this is possible by creating pairs of non-Abelian anyons a and $\bar{a}$ from the vacuum. The fusion state of these anyons is well known. It belongs to a Hilbert space that increases exponentially with the number, n, of anyonic pairs, $\dim(\mathcal{M}_{(n)}) \propto d_a^n$. As d_a is not always an integer, the fusion space $\mathcal{M}_{(n)}$ does not necessarily admit a tensor product structure. Nevertheless, this Hilbert space admits a subspace with qubit tensor product structure in which quantum information can be encoded in the usual way. Its dimension increases exponentially as a function of n. Hence, non-Abelian anyons are an efficient medium for storing quantum information.

Having identified the logical encoding space we now consider the gates that evolve it. Logical gates can be performed by braiding the anyons, thus evolving their fusion state by the R matrix, as shown in Figure 4.10. This operation does not affect either the type of anyons or their local degrees of freedom, but can have a non-trivial effect on the states of the fusion space $\mathcal{M}$, as we have shown in Subsection 4.1.4. In combination with the F matrices one can evolve the encoded information in a non-trivial way. Ideally, we want to be able to perform any arbitrary algorithm out of braiding anyons. Assume that the F and the R matrices span a dense set of unitaries acting on the qubits, in the sense described in Subsection 3.2.1. Then the corresponding anyonic model supports universal quantum computation implemented just by braiding anyons (Freedman *et al.*, 2002a, b). For these models it has been shown (Burrello *et al.*, 2010; Simon *et al.*, 2006) that by weaving a single anyon among many static ones it is possible to perform a universal set of gates between arbitrary qubits. Then one can employ these gates to implement quantum computation following standard quantum algorithms.

At the end of the computation we want to measure the processed information, which is encoded in the final fusion state of the anyons. This can be achieved by fusing the anyons in a series and retrieving the fusion outcomes e_i. The example given in Figure 4.10 illustrates this process at the end of the anyonic evolution. As the fusion state of the anyons can be a superposition of many different basis states $|e_1, e_2, \ldots\rangle$, the measurement of the final fusion state provides, in general, a probability distribution. This step constitutes the final read out of the computation. The braiding algorithm can be adapted to different choices of initial states of anyons and to different fusion procedures.

4.2.2 Stability of anyonic computation

Let us now have a closer look at the stability features of topological quantum computation. Initially, note that the fusion space evolution induced by anyon braiding does not depend on the details of the paths spanned by the anyons, only on their topology. The experimental control of the system inherits this resilient characteristic. Hence, an experimentalist implementing topological quantum computation does not need to be very careful in spanning these paths as long as their global characteristics are realised.

If anyons were elementary particles then they would be robust up to high energy scales. Hence, information encoded with the anyons would be resilient and we could straightforwardly perform error-free quantum computation. In reality, anyons are realised as effective particles of topological models. Thus, we need to consider the stability of these models against environmental errors. What protects the logical information encoded in these systems is the non-local encoding and the presence of a finite energy gap. Indeed, when anyons are kept far apart the information encoded in the fusion space is not accessible by local operations. Hence, environmental errors, acting as local perturbations to the Hamiltonian cannot alter the fusion states (Bravyi *et al.*, 2010). This is the fault-tolerant characteristic of anyons that makes them a favourable medium for performing quantum computation. Nevertheless, probabilistic errors on the system (e.g., due to a finite temperature) do affect the encoded space (Bravyi *et al.*, 2009). It is an important open problem to find a method that efficiently overcomes probabilistic errors with a two-dimensional system. First important steps are taken in Chesi *et al.* (2010) and Hamma *et al.* (2009).

Finally, we should emphasise that implementing universal computation solely by topological means is not the only available option. One might envision combining topological procedures with other known computational methods to optimise their resilience and efficiency. For example, quantum information can be stored in the fusion channels of anyons and thus become protected from errors compared to other quantum memory schemes. Subsequently, one might like to avoid transporting anyons in order to perform logical gates and instead perform them in a dynamical, non-topological way. A scheme has already been proposed that employs measurements of anyons in order to evolve their state, similarly to one-way quantum computation (Bonderson *et al.*, 2008; 2009). Moreover, for some models, the braiding and recombining operations might not be enough to span a universal set of gates while they still provide an efficient anyonic quantum memory. Supplementing these operations with non-topological evolutions can lead to universal quantum computational models (Bravyi, 2006; Das Sarma *et al.*, 2005).

4.3 Example I: Ising anyons

To illustrate the properties of anyonic models we now consider the example of the Ising anyons. The importance of this non-Abelian anyonic model stems from the fact that it is the most promising model for experimental realisation. As we shall see in Chapter 6, Ising anyons describe the statistical properties of Majorana fermions. The latter are currently under intense experimental investigation in the arena of fractional quantum Hall samples (Miller *et al.*, 2007), topological insulators (Fu *et al.*, 2007) and p-wave superconductors (Read and Green, 2000).

4.3.1 The model and its properties

The particle types of the Ising anyon model are the vacuum, 1, the non-Abelian anyon, σ, and the fermion, ψ. In this model the fusion rules are given by

$$\sigma \times \sigma = 1 + \psi, \quad \sigma \times \psi = \sigma, \quad \psi \times \psi = 1, \tag{4.23}$$

with 1 fusing trivially with the rest of the particles (i.e., $\sigma \times 1 = \sigma$ and $\psi \times 1 = \psi$). The first fusion rule of (4.23) signifies that if we bring two σ anyons together they might annihilate (i.e., σ can be its own antiparticle) or they might give rise to the fermion ψ. Hence, the fusion of two σ's has two possible fusion outcomes represented by the states $|\sigma, \sigma \to 1\rangle$ and $|\sigma, \sigma \to \psi\rangle$. The second rule indicates that fusing a ψ with a σ gives back a σ. In a sense, this rule states that a ψ can be absorbed by a σ without changing its anyonic charge. The third fusion rule states that when two fermions are brought together they are fused to the vacuum. Only the parity of the total number of fermions can be detected, since the composite of two ψ's is condensed to the vacuum.

Let us now give the explicit forms of the F and R matrices for the Ising model. We postpone their derivation to the next subsection. The F matrix is given by

$$F^{\sigma}_{\sigma\sigma\sigma} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \tag{4.24}$$

in the $|\sigma,\sigma \to 1\rangle$ and $|\sigma,\sigma \to \psi\rangle$ basis. It corresponds to the rearrangement of the fusion order of three σ anyons when their total fusion channel is a σ. The F matrix dictates that the in-between fusion outcomes, being the vacuum or the fermion, can be non-trivially transformed by changing the fusion order of the anyons. In the case of two σ anyons the components of the R matrix are $R^1_{\sigma\sigma} = e^{-i\pi/8}$ and $R^{\psi}_{\sigma\sigma} = e^{i3\pi/8}$, giving the matrix

$$R_{\sigma\sigma} = e^{-i\pi/8} \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}. \tag{4.25}$$

This implies that a ψ fusion channel acquires an additional $\pi/2$ phase compared to the vacuum during a π rotation due to the spin-1/2 nature of the fermion.

Let us now consider some implications of the braiding and fusion properties of the Ising anyons. Assume that one creates two pairs of anyons (σ,σ) from the vacuum, as illustrated in Figure 4.11(a). The state of the two pairs is then given by $|\sigma,\sigma \to 1\rangle\,|\sigma,\sigma \to 1\rangle$. The braiding evolution is described by the two-dimensional matrix

$$B = F^{-1}R^2F = e^{-i\pi/4} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{4.26}$$

that rotates the fusion states of each pair from $|\sigma,\sigma \to 1\rangle$ to $|\sigma,\sigma \to \psi\rangle$ up to an overall phase factor. The braiding, hence, changes the internal state of the anyons in a non-trivial way. The resulting ψ's can be further fused to the vacuum that we had started with, without violating the conservation of the total anyonic charge. Similarly, Figure 4.11(b) shows the

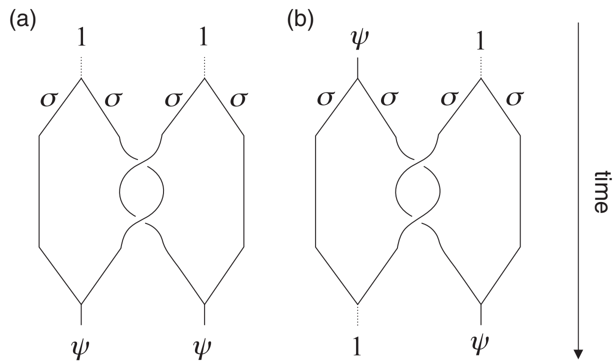

Fig. 4.11 Worldlines of Ising anyons, σ, where time is running downwards. (a) Two pairs of σ's are generated from the vacuum 1. Then an anyon from one pair is circulated around an anyon from the other pair. Finally, the anyons are pairwise fused producing fermionic outcomes. This signals a non-trivial evolution of the fusion states due to braiding. (b) A similar evolution, where two pairs of σ's are created from a fermion ψ and the vacuum 1, respectively. The braiding causes the teleportation of the fermion from one pair to the other.

generation of one pair of Ising anyons from a fermion and another one from the vacuum. The braiding process causes the fermion to be teleported from one pair to the other, even though the anyons have not been in contact with each other at any time.

Let us now describe how one could employ Ising anyons for topological quantum computation. First, we encode a qubit in a set of four σ anyons. Logical states are encoded in the different in-between fusion outcomes of four anyons, i.e. $|0_L\rangle = |\sigma,\sigma \to 1\rangle$ and $|1_L\rangle = |\sigma,\sigma \to \psi\rangle$. To encode n qubits we can employ $4n$ anyons. Logical operations between the qubits can be performed by braiding Ising anyons and changing their fusion order. As we have seen, this results in the F and R matrices given in (4.24) and (4.25), respectively. It is known that these two unitary evolutions cannot support universal quantum computation as F and R do not span the whole SU(2) group. They are restricted to the Clifford subgroup of SU(2). This model can be made universal by the addition of a phase gate which can be implemented by dynamical operations (Bravyi, 2006; Das Sarma *et al.*, 2005).

4.3.2 *F* and *R* matrices

We now explicitly calculate the F and R matrices for the Ising model. The starting point is the set of particles 1, σ and ψ and their fusion rules (4.23). Having a closer look at these rules we find that the only non-zero coefficients are $N^1_{\sigma\sigma} = 1$, $N^\psi_{\sigma\sigma} = 1$, $N^\sigma_{1\sigma} = 1$ and $N^\sigma_{\psi\sigma} = 1$. By substituting $a_1 = a_2 = a_3 = a_4 = \sigma$ into equation (4.14) and having the summation running over $e_1 = 1, \psi$ we find that $\dim(\mathcal{M}_{(4)}) = 2$, which can encode one qubit. Moreover, the quantum dimension of the vacuum is $d_1 = 1$, and ψ has quantum dimension $d_\psi = 1$ as well. So for σ we have $d_\sigma^2 = d_1 + d_\psi$, which implies $d_\sigma = \sqrt{2}$. Thus, the total quantum dimension of the Ising model is $\mathcal{D} = 2$.

Next we solve the pentagon and hexagon identities. From Figure 4.2 we see that F^4_{123} is a one-dimensional matrix, except when the anyons 1, 2, 3 and 4 are all σ. Then i and j run over the variables 1 and ψ, making $F^\sigma_{\sigma\sigma\sigma}$ a 2×2 matrix. All the one-dimensional F elements can take arbitrary complex phase values. This corresponds to a gauge degree of freedom that we conveniently fix to be $+1$ or -1.

The pentagon identity (4.19) reads

$$(F^5_{12c})^d_a (F^f_{a34})^c_b = \sum_e (F^d_{234})^c_e (F^f_{1e4})^d_b (F^b_{123})^e_a. \tag{4.27}$$

Let us initially take the particles 1, 2, 3 and 4 to be σ anyons and 5 to be the vacuum, as shown in Figure 4.12(a). Then from Figure 4.6 we see that b and d need to be σ, while a and c of (4.27) can be either 1 or ψ. Suppose $a = 1$ and $c = 1$. Then the pentagon identity becomes

$$(F^1_{\sigma\sigma 1})^\sigma_1 (F^1_{1\sigma\sigma})^1_\sigma = \sum_{e=1,\psi} (F^\sigma_{\sigma\sigma\sigma})^1_e (F^1_{\sigma e\sigma})^\sigma_\sigma (F^\sigma_{\sigma\sigma\sigma})^e_1. \tag{4.28}$$

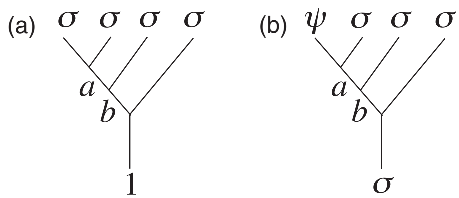

Fig. 4.12 Two initial configurations for the pentagon identity.

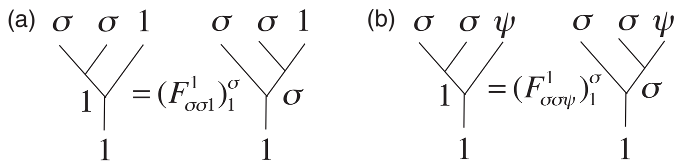

Fig. 4.13 Two trivial F moves. For the (a) configuration we choose $(F^1_{\sigma\sigma 1})^\sigma_1 = 1$. The (b) configuration corresponds to an impossible fusion process, so $(F^1_{\sigma\sigma\psi})^\sigma_1 = 0$.

The F move $(F^1_{\sigma\sigma 1})^\sigma_1$ corresponds to a trivial rearranging of anyons, as shown in Figure 4.13(a). So we set $(F^1_{\sigma\sigma 1})^\sigma_1 = 1$. Hence, we obtain the equation

$$1 = {(F^\sigma_{\sigma\sigma\sigma})^1_1}^2 + (F^\sigma_{\sigma\sigma\sigma})^1_\psi (F^\sigma_{\sigma\sigma\sigma})^\psi_1. \tag{4.29}$$

Let us now take $a = 1$ and $c = \psi$. Then the pentagon identity becomes

$$(F^1_{\sigma\sigma\psi})^\sigma_1 (F^1_{1\sigma\sigma})^\psi_\sigma = \sum_{e=1,\psi} (F^\sigma_{\sigma\sigma\sigma})^\psi_e (F^1_{\sigma e\sigma})^\sigma_\sigma (F^\sigma_{\sigma\sigma\sigma})^e_1, \tag{4.30}$$

which implies

$$(F^\sigma_{\sigma\sigma\sigma})^\psi_\psi = -(F^\sigma_{\sigma\sigma\sigma})^1_1. \tag{4.31}$$

Above we have used the condition $(F^1_{\sigma\sigma\psi})^\sigma_1 = 0$ as the corresponding process, shown in Figure 4.13(b), is forbidden. When $a = \psi$ and $c = 1$ we obtain the same condition as (4.29). Finally, when $a = c = \psi$ we have

$$1 = (F^\sigma_{\sigma\sigma\sigma})^\psi_1 (F^\sigma_{\sigma\sigma\sigma})^1_\psi + {(F^\sigma_{\sigma\sigma\sigma})^\psi_\psi}^2. \tag{4.32}$$

Let us now take particle 1 to be ψ and particles 2, 3, 4 and 5 to be σ anyons, as shown in Figure 4.12(b). The only possibility is to have $a = d = \sigma$, while b and c can be either 1 or ψ. The pentagon equation, for $b = c = 1$, now becomes

$$(F^\sigma_{\psi\sigma 1})^\sigma_\sigma (F^\sigma_{\sigma\sigma\sigma})^1_1 = \sum_{e=1,\psi} (F^\sigma_{\sigma\sigma\sigma})^1_e (F^\sigma_{\psi e\sigma})^\sigma_1 (F^1_{\psi\sigma\sigma})^e_\sigma, \tag{4.33}$$

which implies

$$(F^{\sigma}_{\sigma\sigma\sigma})^1_1 = (F^{\sigma}_{\sigma\sigma\sigma})^1_{\psi}. \tag{4.34}$$

Here we used $(F^{\sigma}_{\psi\sigma 1})^{\sigma}_{\sigma} = 1$. Finally, for $b = 1$ and $c = \psi$ we obtain

$$(F^{\sigma}_{\psi\sigma\psi})^{\sigma}_{\sigma}(F^{\sigma}_{\sigma\sigma\sigma})^{\psi}_1 = \sum_{e=1,\psi}(F^{\sigma}_{\sigma\sigma\sigma})^{\psi}_e(F^{\sigma}_{\psi e\sigma})^{\sigma}_1(F^1_{\psi\sigma\sigma})^e_{\sigma}, \tag{4.35}$$

which implies

$$(F^{\sigma}_{\sigma\sigma\sigma})^{\psi}_1 = (F^{\sigma}_{\sigma\sigma\sigma})^{\psi}_{\psi}. \tag{4.36}$$

To derive this we have set $(F^{\sigma}_{\psi\sigma\psi})^{\sigma}_{\sigma} = -1$ as setting it to $+1$ would have given a non-unitary matrix for $F^{\sigma}_{\sigma\sigma\sigma}$. Equations (4.29), (4.31), (4.32), (4.34), (4.36) can now be solved to find that the matrix $F^{\sigma}_{\sigma\sigma\sigma}$ has the following elements:

$$\begin{gathered}(F^{\sigma}_{\sigma\sigma\sigma})^{\psi}_{\psi} = -(F^{\sigma}_{\sigma\sigma\sigma})^1_1,\ (F^{\sigma}_{\sigma\sigma\sigma})^{\psi}_1 = (F^{\sigma}_{\sigma\sigma\sigma})^{\psi}_{\psi},\\ (F^{\sigma}_{\sigma\sigma\sigma})^1_1 = (F^{\sigma}_{\sigma\sigma\sigma})^1_{\psi},\ (F^{\sigma}_{\sigma\sigma\sigma})^{\psi}_1 = \pm\frac{1}{\sqrt{2}}.\end{gathered} \tag{4.37}$$

Reconstructing the F matrix from its components, we obtain

$$F^{\sigma}_{\sigma\sigma\sigma} = \pm\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \tag{4.38}$$

Hence, the pentagon equation determines the F matrix. The choice of $\pm$ sign is called the Frobenius–Schur indicator (Rowell *et al.*, 2009).

Let us now turn to the hexagon identity (4.20) given by

$$\sum_b (F^4_{231})^c_b R^4_{1b}(F^4_{123})^b_a = R^c_{13}(F^4_{213})^c_a R^a_{12}. \tag{4.39}$$

In particular, we take particles 1, 2, 3 and 4 to be all σ anyons and consider the four possibilities with a and c being either 1 or ψ. One can easily see that for $a = c = 1$ we have

$$\sum_{b=1,\psi}(F^{\sigma}_{\sigma\sigma\sigma})^1_b R^{\sigma}_{\sigma b}(F^{\sigma}_{\sigma\sigma\sigma})^b_1 = R^1_{\sigma\sigma}(F^{\sigma}_{\sigma\sigma\sigma})^1_1 R^1_{\sigma\sigma}, \tag{4.40}$$

which implies

$$\frac{1}{2}(R^{\sigma}_{\sigma 1} + R^{\sigma}_{\sigma\psi}) = \frac{1}{\sqrt{2}}{R^1_{\sigma\sigma}}^2. \tag{4.41}$$

Similarly, for $a = 1$ and $c = \psi$ we obtain

$$\frac{1}{2}(R^{\sigma}_{\sigma 1} - R^{\sigma}_{\sigma\psi}) = \frac{1}{\sqrt{2}}R^{\psi}_{\sigma\sigma}R^1_{\sigma\sigma}, \tag{4.42}$$

for $a = \psi$ and $c = 1$ we obtain

$$\frac{1}{2}(R^{\sigma}_{\sigma 1} - R^{\sigma}_{\sigma\psi}) = \frac{1}{\sqrt{2}} R^{1}_{\sigma\sigma} R^{\psi}_{\sigma\sigma} \tag{4.43}$$

and finally, for $a = c = \psi$ we obtain

$$\frac{1}{2}(R^{\sigma}_{\sigma 1} + R^{\sigma}_{\sigma\psi}) = -\frac{1}{\sqrt{2}} {R^{\psi}_{\sigma\sigma}}^{2}. \tag{4.44}$$

Combining (4.41) and (4.44) hence implies

$$R^{1}_{\sigma\sigma} = \pm i R^{\psi}_{\sigma\sigma}, \tag{4.45}$$

while adding (4.42) and (4.44) together gives

$$R^{\psi}_{\sigma\sigma} = \pm e^{-\frac{3\pi}{8} i}. \tag{4.46}$$

Similarly, we find that for both choices of sign we have the same solution $R^{\sigma}_{\sigma 1} = 1$ and $R^{\sigma}_{\sigma\psi} = i$. Note that there is a discrete multiplicity of solutions in equations (4.38), (4.45) and (4.46) corresponding to the combinations of different signs. Hence, the hexagon equation determines the R matrix when the F matrix is known.

4.4 Example II: Fibonacci anyons

In this final section we present probably the most celebrated non-Abelian anyonic model: the Fibonacci anyons. Its popularity is not only due to its simplicity and richness in structure, which supports universal quantum computation, but also to its connection to the Fibonacci series. In this model there are only two different types of anyons, the vacuum, 1 and the non-Abelian anyon, τ. The only non-trivial fusion rule is

$$\tau \times \tau = 1 + \tau. \tag{4.47}$$

The quantum dimension of τ can be obtained from $d_\tau^2 = d_1 + d_\tau$ giving $d_\tau = \phi$, where $\phi = (1+\sqrt{5})/2$ is the golden mean. This number has been used extensively by artists, such as the ancient Greek sculptor Phidias or Leonardo Da Vinci in geometrical representations of nature as it describes the ratio that is aesthetically most appealing.

It is interesting to look in detail at all the possible in-between outcomes when fusing n anyons of type τ, as shown in Figure 4.14(a). There we initially fuse the first two anyons, then their outcome is fused with the third τ anyon and so on. To each step i we assign an index e_i that indicates the outcome of the fusion at that step being either 1 or τ. The states $|e_1, e_2, \ldots, e_{n-3}\rangle$ belong to the fusion Hilbert space of the anyons, $\mathcal{M}_{(n)}$. In principle there are 2^{n-3} possible combinations of e_i's, but not all of them are allowed fusion outcomes. Let us analyse how many states $\mathcal{M}_{(n)}$ can have by counting the distinct ways in which one can fuse $n-1$ anyons of type τ to finally yield a τ. For $n = 1$ we deal with the impossible process where the vacuum turns into a τ anyon, so $\dim(\mathcal{M}_{(1)}) = 0$.

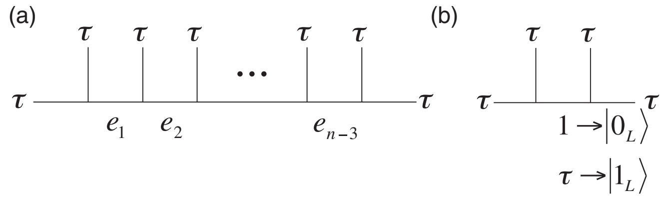

Fig. 4.14 The fusion process for Fibonacci anyons, τ. (a) A series of τ anyons are fused together ordered from left to right. The first two τ anyons are fused and then their outcome is fused with the next τ anyon and so on. (b) Four Fibonacci anyons in state τ created from the vacuum can be used to encode a single logical qubit.

The $n = 2$ case corresponds to a τ as an input and an output going through a trivial process. So $\dim(\mathcal{M}_{(2)}) = 1$. At the next fusing step, the possible outcomes are 1 or τ, giving $\dim(\mathcal{M}_{(3)}) = 1$. When we fuse the outcome with the next anyon then $1 \times \tau = \tau$ and $\tau \times \tau = 1 + \tau$, resulting in two possible τ's coming from two different processes and a single vacuum outcome. So $\dim(\mathcal{M}_{(4)}) = 2$. This signifies that four Fibonacci anyons are needed to encode a qubit. Taking all possible outcomes and fusing them with the next anyon gives a space which is three-dimensional, $\dim(\mathcal{M}_{(5)}) = 3$. Continuing this process one soon notices that the dimension of the fusion space $\dim(\mathcal{M}_{(n)})$ when n anyons of type τ are fused actually reproduces the Fibonacci series,

$$0, 1, 1, 2, 3, 5, 8, 13, \ldots \tag{4.48}$$

It is known from number theory that this dimension is approximately given by the following formula:

$$\dim(\mathcal{M}_{(n)}) \propto \phi^n,$$

in agreement with relation (4.16).

The Fibonacci anyon model can indeed realise universal quantum computation. Much like the Ising model case, the encoding of a qubit can be visualised by employing four τ anyons, as in Figure 4.14(b). There are two distinguishable ways the anyons can be fused that encode the qubit states $|0_L\rangle = |\tau, \tau \rightarrow 1\rangle$ and $|1_L\rangle = |\tau, \tau \rightarrow \tau\rangle$. To determine the possible quantum gates one needs to evaluate the F and the R matrices. From the fusion rules of Fibonacci anyons and the pentagon identity one finds the non-zero values $(F^{\tau}_{\tau\tau 1})^{\tau}_{\tau} = (F^{\tau}_{1\tau\tau})^{\tau}_{\tau} = (F^{1}_{\tau\tau\tau})^{\tau}_{\tau} = (F^{\tau}_{\tau 1\tau})^{\tau}_{\tau} = (F^{1}_{111})^{1}_{1} = 1$ and

$$F^{\tau}_{\tau\tau\tau} = \begin{pmatrix} \frac{1}{\phi} & \frac{1}{\sqrt{\phi}} \\ \frac{1}{\sqrt{\phi}} & -\frac{1}{\phi} \end{pmatrix}. \tag{4.49}$$

These solutions are unique up to a choice of gauge. Inserting these values into the hexagon identity, one obtains the following R matrix describing the exchange of two anyons:

$$R_{\tau\tau} = \begin{pmatrix} e^{4\pi i/5} & 0 \\ 0 & -e^{2\pi i/5} \end{pmatrix}. \tag{4.50}$$

It can be shown that the braiding unitaries $b_1 = R_{\tau\tau}$ and $b_2 = (F^{\tau}_{\tau\tau\tau})^{-1} R_{\tau\tau} F^{\tau}_{\tau\tau\tau}$ acting in the logical space $|0_L\rangle$ and $|1_L\rangle$ are dense in SU(2) in the sense that they can reproduce any element of SU(2) with accuracy ϵ in a number of operations that scales like $O(\text{poly}(\log(1/\epsilon)))$ (Preskill, 2004). For example, an arbitrary one-qubit gate can be performed as follows. Begin from the vacuum and create four anyons labelled τ_1, τ_2, τ_3 and τ_4. Braiding the first and second anyons implements b_1 and braiding the second and third anyons implements b_2. A measurement of the outcome upon fusing τ_1 and τ_2 projects onto $|\,0_L\rangle$ or $|\,1_L\rangle$. Similarly, by performing braiding of eight anyons and keeping in mind that $\dim(\mathcal{M}_{(8)}) = 13$ one obtains a dense subset of SU(13). Since SU(4)$\subset$SU(13), we can implement any two-qubit gate (e.g., the CNOT gate) with arbitrary accuracy. This means, the Fibonacci anyon model allows for universal computation on n logical qubits using $4n$ physical anyons (Freedman *et al.*, 2002a).

Summary

In this chapter we introduced the anyonic models in a systematic way and we derived consistency equations between their properties. For example, the spin-statistics theorem relates the spin of anyons with their braiding behaviour. Moreover, the pentagon and hexagon identities can be constructed from simple considerations of anyonic worldline diagrams. These identities establish the fusion and braiding properties of non-Abelian anyons by determining their F and R matrices.

To employ anyons for quantum computation we first identify which part of the fusion Hilbert space is ideal for encoding information. The F and R matrices are then identified as logical gate primitives that non-trivially evolved the fusion states. If the F and R unitary matrices can efficiently span the whole encoding space then the corresponding anyonic model can perform universal quantum computation.

As concrete examples we investigated the Ising and the Fibonacci models. The interest in the Ising anyons is due to their possible physical realisation with near future technology. Nevertheless, this model cannot, per se, support universal quantum computation. Supplementing it with simple dynamical phase rotations can overcome this caveat. On the other hand, the Fibonacci model is universal. Successive applications of its F and R matrices can rotate any state encoded in the fusion space to any other with well-controlled accuracy.

Exercises

4.1 For $\bar{a}$ denoting the antiparticle of the a anyon, demonstrate the following properties of N_{ab}^c: $N_{a1}^c = \delta_{ac}$, $N_{ab}^1 = \delta_{b\bar{a}}$, $N_{ab}^c = N_{ba}^c = N_{b\bar{c}}^{\bar{a}} = N_{\bar{a}\bar{b}}^{\bar{c}}$ and $\sum_e N_{ab}^e N_{ec}^d = \sum_f N_{af}^d N_{bc}^f$.

4.2 Show that starting from the definition of the quantum dimension (4.15) one can derive the asymptotic relation (4.16). [*Hint*: Consider the matrix N^c with non-negative elements $(N^c)_{ab} = N^c_{ab}$ and decompose it into eigenstates and eigenvalues (Preskill, 2004; Verlinde, 1988).]

4.3 Starting from pairs of Ising anyons created from the vacuum can we generate an entangled state? [*Hint*: See Brennen *et al.* (2009).]

4.4 Show that the F and R matrices of the Fibonacci model given in (4.49) and (4.50) satisfy the pentagon and hexagon identities.