Topological Order and Quantum Computation
Anyons

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Anyons in the toric code

1. **Species of anyons (topological charges)** in the toric code:
   - $\mathbb{I}$: vacuum or identity (it is always present).
   - e: it is created by Z strings.
   - m: it is created by X strings.
   - ψ: it is the simultaneous presence of e and m.
2. We can write down the **fusion rules**:
   - e and m are their own conjugate particles: if I change twice an A or B stabilizer I go back to the vacuum state:

$$e \times e = m \times m = \mathbb{I}$$

   - Definition of ψ:

$$e \times m = \psi$$

   - It follows:

$$e \times \psi = m; \quad m \times \psi = e, \quad \psi \times \psi = \mathbb{I}$$

- e and m singularly behave as bosons.

Braidings in the toric code

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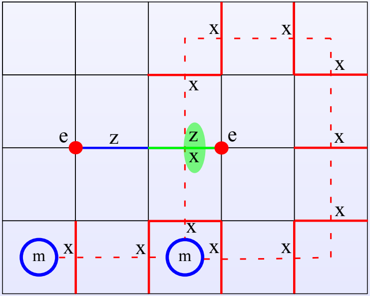
- The mutual statistics of e and m is given by $R_{em} = e^{i\frac{\pi}{2}}$.



Braidings in the toric code

- e and m singularly behave as bosons.
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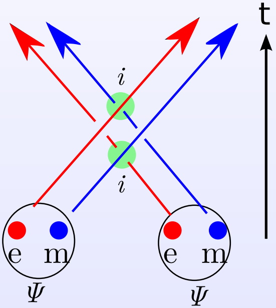
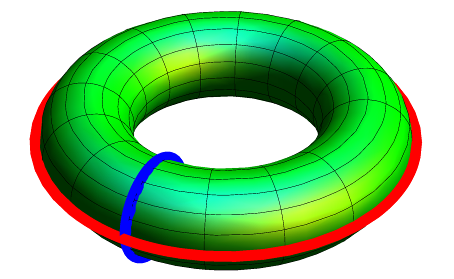
- ψ is a fermion.



Braiding and degeneracy of the ground states

Consider a generic topologically ordered system on a torus.
For each kind of anyon a in the system, we can define two string **symmetries**, T_1 and T_2, that correspond to:

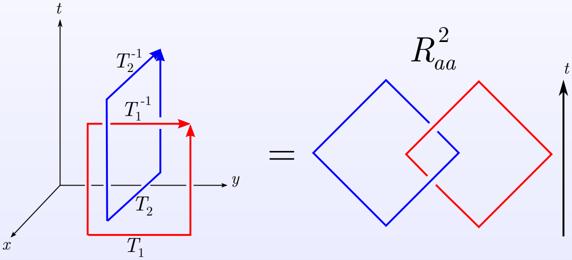
1. Create a pair of anyons.
2. Wind them around one non-trivial loop.
3. We reannihilate them.



Braiding and degeneracy of the ground states

The commutation relation between T_1 and T_2 is related to the braiding statistics R_{aa} of the anyon a:

$$T_1 T_2 T_1^{-1} T_2^{-1} :$$



- If $R_{aa}^2 = 1$, then $[T_1, T_2] = 0$, so there is no degeneracy (bosons or fermions).
- If $R_{aa}^2 \neq 1$, then $[T_1, T_2] \neq 0$, thus there are two non-commuting symmetries and the ground state of the system is degenerate.

Anyons

Localized and gapped indistinguishable objects whose exchange statistics is described by a generic unitary operator.

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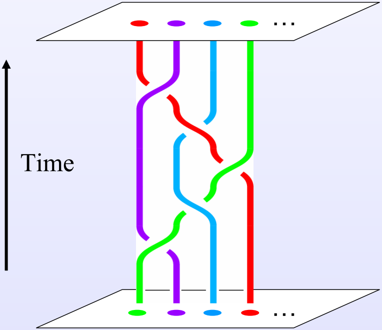
- These unitary operators describe the adiabatic evolution of the system and may be represented in terms of world lines.
- The result of the exchanges does not depend on the detail of the path the anyons undergo.



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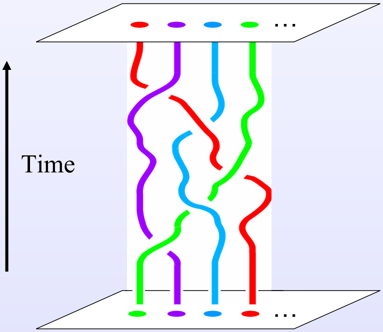
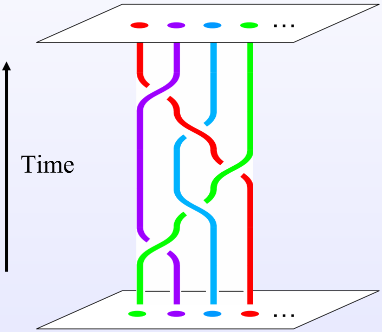


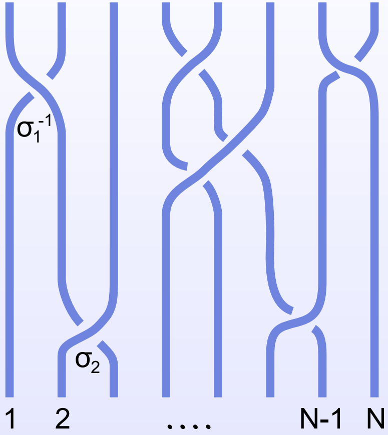
Braid Group

- For fermions or bosons, the wavefunction of a set of indistinguishable particles at fixed position depends only on their permutation.
- For anyons, instead, we must keep track of their time evolution, since $R \neq R^{-1}$.
- The anyon world lines in $2+1$D are self-avoiding strands.
- Their exchange statistics is defined by the **braid group**.



Braid Group (Oktoberfest definition)

Braid Group



- The braid group is generated by the counterclockwise and clockwise exchanges of neighboring anyons $\sigma_i, \sigma_i^\dagger$.
- Disjoint operators commute:

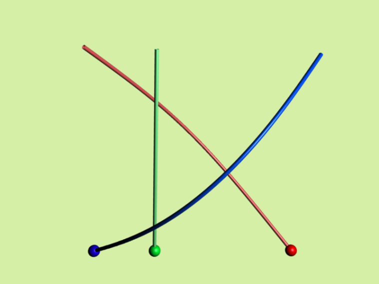
$$\sigma_i \sigma_j = \sigma_j \sigma_i \quad \text{for} \quad |i - j| > 1$$

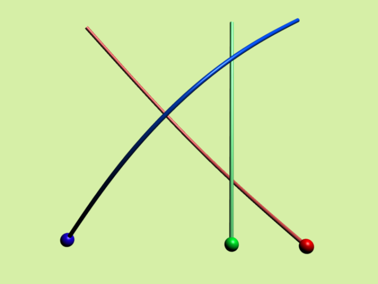
- Neighboring operators obey the Yang-Baxter relation:

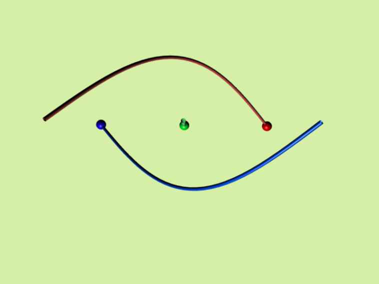
$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$$

- For $\sigma_i^2 = 1$ we recover the permutations.

Yang Baxter Braiding







Algebra relations

For non-adjacent operators:

$$[\sigma_i, \sigma_k] = 0 \quad \text{if} \quad |i-k| \geq 2$$

Yang Baxter Relations:

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$$

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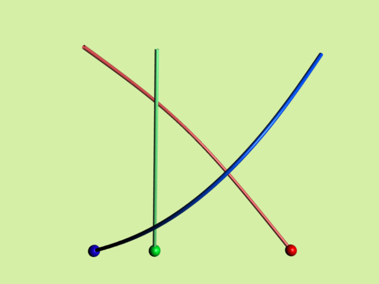

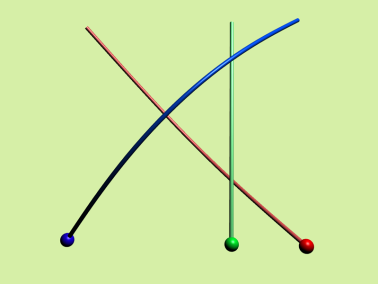

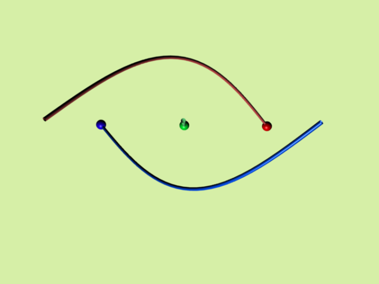

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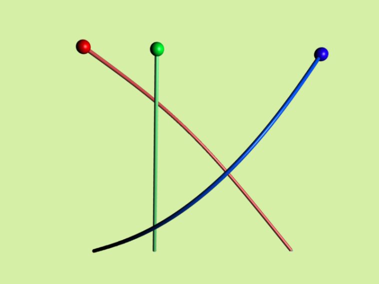

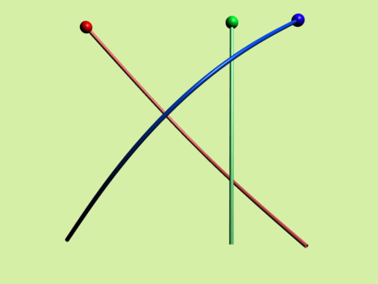

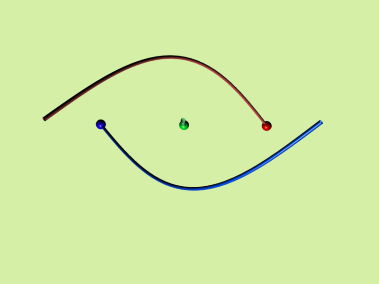

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Abelian anyons

- The easiest non-trivial representation of the braid group is provided by Abelian anyons:

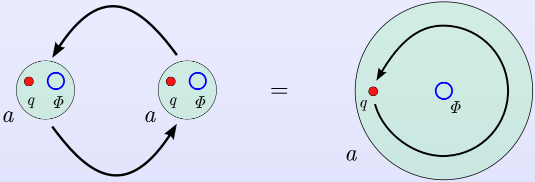
$$\sigma \to R_{aa} = e^{i\theta_a}.$$

- Abelian anyons can be described in terms of charge-flux composite objects where:

$$\theta_a = q_a \Phi_a / 2$$

- **Spin-Statistics**: The exchange of two Abelian anyons a gives the same phase as a 2π rotation of q_a around Φ_a:

$$\frac{1}{2} q_a \Phi_a \quad + \quad \frac{1}{2} q_a \Phi_a \quad = \quad q_a \Phi_a$$



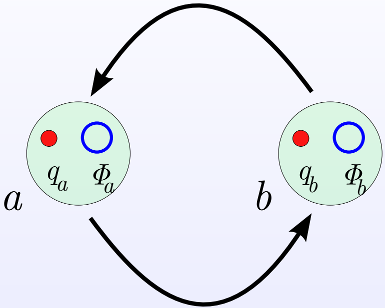
Topological spin

For two Abelian anyons:

$$a \times b = c$$

Then:

$$R^c_{ab} = \exp[i\left(\theta_c - \theta_a - \theta_b\right)/2]$$



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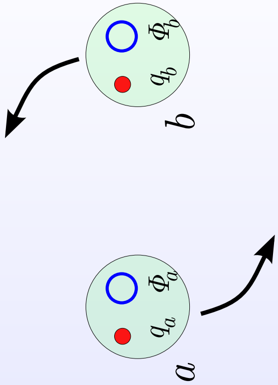

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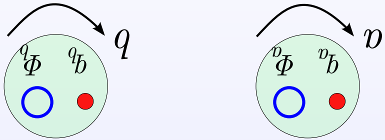

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We could also write:

$$\phi_a\left(z_1\right)\phi_b\left(z_2\right) = \frac{1}{\left(z_1 - z_2\right)^{\Delta_a + \Delta_b - \Delta_c}}\phi_c\left(z_1\right)$$

Non-Abelian Anyons

- **Non-Abelian Anyons** correspond to higher dimensional representations of the Braid group.

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- **Non-Abelian Anyons** correspond to higher dimensional representations of the Braid group.
- To obtain these higher dimensions we need to introduce a new degeneracy.
- A pair of non-Abelian anyons may assume different states, characterized by different topological charges:

$$a \times b = c \oplus d \oplus e \oplus \dots$$

- Each pair define a Hilbert space, and the braidings are unitary operators on these spaces.
- Braidings of neighboring pairs of non-Abelian anyons, in general, do not commute.

Fusion Rules

Let's consider the simple case of spin $\frac{1}{2}$ (Qubit):

$$\frac{1}{2} \times \frac{1}{2} = 0 + 1 \quad \longrightarrow \quad 2 \otimes 2 = 1 \oplus 3$$

- A particle with spin $1/2$ is described by a two-dimensional Hilbert space
- When two of them fuse, they give rise to a singlet or to a triplet. This is a **Fusion Rule**

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Non-Abelian anyons are characterized by non trivial fusion rules.

Ising Anyons / Majorana modes:

$$\sigma \times \sigma = \mathbb{I} + \varepsilon$$

$$\gamma \times \gamma = \mathbb{I} + \psi$$

Fibonacci Anyons:

$$\tau \times \tau = \mathbb{I} + \tau$$

In general one writes:

$$a \times b = \sum_c N_{ab}^c c$$

Non Abelian Anyons: main ‘ingredients’

To describe a non-Abelian anyon model we need a theory characterized by the following elements:

- **Fusion Rules**: N_{ab}^{c}
- **Associativity Rules**: $\left(F_{d}^{abc}\right)_{xy}$
- **Braiding Rules**: $\sigma \rightarrow R_{ab}^{c}$

These rules must have a coherent structure and must obey several constraints.

Fusion Rules

A non-Abelian anyonic model is defined starting from a **finite set of particles** (*Topological charges*).
These particles are linked by the fusion rules:

$$a \times b = \sum_c N_{ab}^c c \quad \longrightarrow \quad V_a \otimes V_b = \bigoplus_c N_{ab}^c V_{ab}^c \quad \longrightarrow \quad d_a d_b = \sum_c N_{ab}^c d_c$$

where $N_{ab}^c = 0, 1$; $V_{ab}^c = V_c$ are Hilbert spaces and d_i are their quantum dimension.

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where $N_{ab}^c = 0, 1$; $V_{ab}^c = V_c$ are Hilbert spaces and d_i are their quantum dimension.

a is a non-Abelian anyon if $\sum_c N_{aa}^c \geq 2$.

This means that a pair of a anyons may be found in at least two degenerate states.

Fusion Rules

- N_{ab}^{c} can be understood as a (transfer) matrix: $(N_a)_{b_i}^{b_{i+1}}$.
- Starting from the anyon b_i, N_a defines the possible states b_{i+1} that can be obtained adding a.

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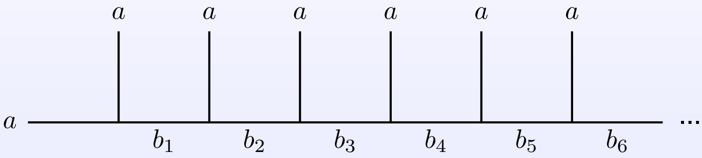
- Consider a chain of a anyons:



- A state in this chain is defined by the string $\{b_i\}$ and lives in the space:

$$V_{a_1 \dots a_n}^{b_n} = \bigoplus_{b_1, \dots, b_{n-1}} V_{a_1 a_2}^{b_1} \otimes V_{b_1 a_3}^{b_2} \otimes V_{b_2 a_4}^{b_3} \otimes \dots \otimes V_{b_{n-2} a_n}^{c}.$$

Anyon chains

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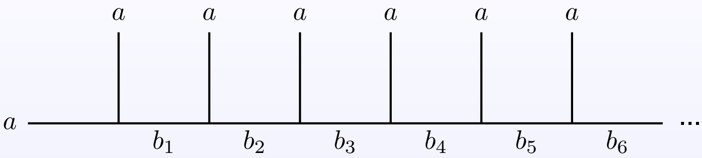


- A state in this chain is defined by the string $\{b_i\}$ and lives in the space:

$$V^{b_n}_{a_1 \ldots a_n} = \bigoplus_{b_1, \ldots, b_{n-1}} V^{b_1}_{a_1 a_2} \otimes V^{b_2}_{b_1 a_3} \otimes V^{b_3}_{b_2 a_4} \otimes \ldots \otimes V^{c}_{b_{n-2} a_n}.$$

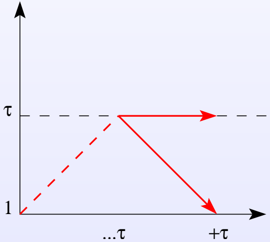
- The number of total orthogonal states (strings) is:

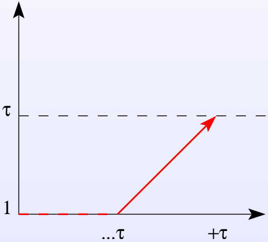
$$\dim \left(V^{b_n}_{a_1 \ldots a_n} \right) = \left(N_{a_2} N_{a_3} \ldots N_{a_n} \right)^{b_n}_{a_1} = \left[\left(N_a \right)^{n-1} \right]^{b_n}_{a} \approx d_a^{n-1}$$

- d_a is the highest eigenvalue of N_a, it is called quantum dimension of a.

- The model is characterized by two sectors: the **Vacuum** $\mathbb{I}$ and the **Fibonacci anyon** τ.
- **Fusion Rules**:

$$\tau \times \tau = \mathbb{I} + \tau \qquad\qquad \mathbb{I} \times \tau = \tau$$



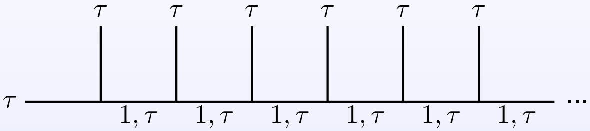


These fusion rules correspond to:

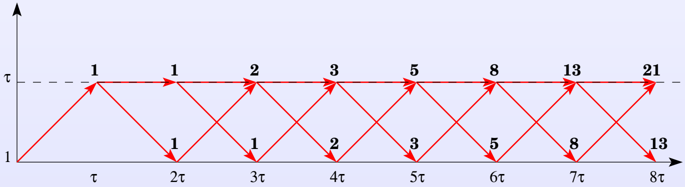
$$N_\tau = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \quad \Longrightarrow \quad d_\tau^2 - d_\tau - 1 = 0 \quad \Longrightarrow \quad d_\tau = \frac{1+\sqrt{5}}{2} \equiv \phi$$

Brattelli diagram
Fibonacci chain

$$\tau \times \tau = \mathbb{I} + \tau\,, \qquad N_\tau = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$



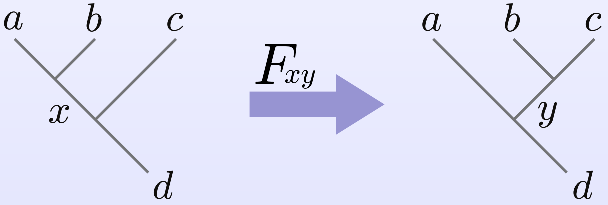
Constraint: there cannot be two consecutive vacua 1.



The number of states grows like the Fibonacci numbers.
$d_\tau = \frac{1+\sqrt{5}}{2}$ is the golden ratio!

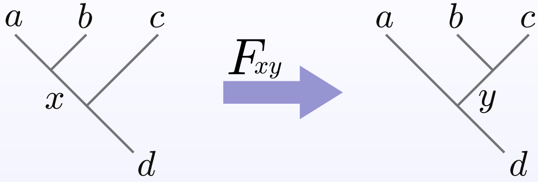
Associativity Rules
F-Matrices

- For an anyonic theory to be consistent the fusion rules N must be associative:
$$\sum_x N^x_{ab} N^d_{xc} = \sum_y N^d_{ay} N^y_{bc}$$
- These relations characterize the fusion process $abc \to d$ in the fusion space $V^d_{abc} = V^d_{(ab)c} = V^d_{a(bc)}$.
- The two descriptions of the space V^d_{abc} correspond to different orthogonal bases
- There must be a unitary operator that relates these bases:

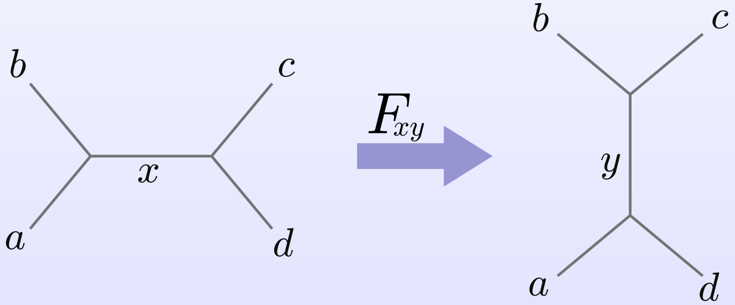


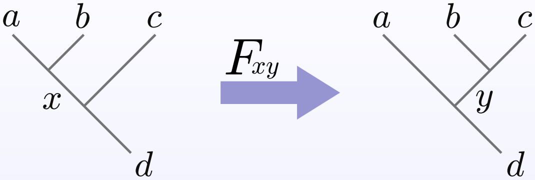
- $\left(F^{abc}_d\right)_{xy}$ is this transformation.

F Matrices



Topologically equivalent to:





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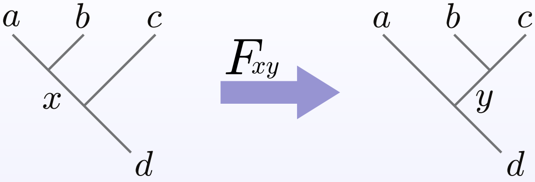

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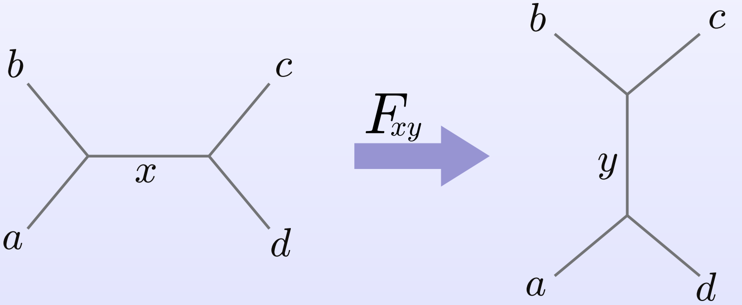
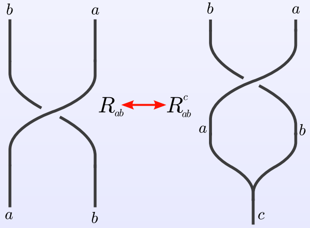


Braidings R_{ab}

A couple of anyons $a \times b$ can be in a superposition of states V_{ab}^{k} defined by the fusion rules:

$$\phi_a(z_1)\phi_b(z_2) = \frac{\phi_c(z_2)}{(z_1 - z_2)^{\Delta_a + \Delta_b - \Delta_c}} + \frac{\phi_d(z_2)}{(z_1 - z_2)^{\Delta_a + \Delta_b - \Delta_d}} + \dots$$



The clockwise exchange R_{ab} does not affect their total charge:

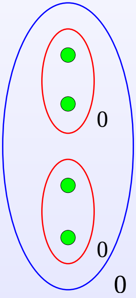
$$R_{ab} = \begin{pmatrix} R_{ab}^{a} & 0 & 0 & 0 \\ 0 & R_{ab}^{b} & 0 & 0 \\ 0 & 0 & R_{ab}^{c} & 0 \\ 0 & 0 & 0 & \ddots \end{pmatrix}$$

where: $\qquad \left(R_{ab}^{c}\right)^2 = e^{-2\pi i(\Delta_a + \Delta_b - \Delta_c)}$

The representations of the braid generators σ_i are given by combinations of F and R.

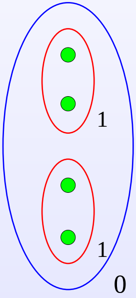
Fibonacci anyons and qubits

- Differently from Ising anyons and Majorana modes, Fibonacci anyons allow for universal quantum computation with braidings only.
- To encode a qubit we use a system of 4 anyons whose total charge is trivial:



$|0\rangle$

Each pair annihilates.

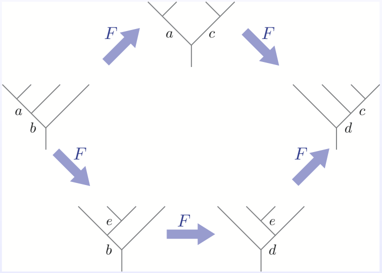


$|1\rangle$

Each pair gives a single τ

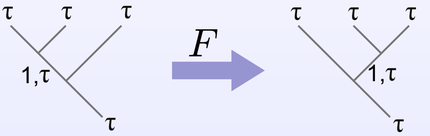
Fibonacci F - Matrix

The unitary matrix $F^{\tau\tau\tau}_{\tau}$ can be calculate from a particular constraint called pentagon equation:



$$F_{11} = F_{1\tau}F_{\tau 1}$$

$$F_{11} + F^2_{\tau\tau} = 1$$



The resulting matrix is:

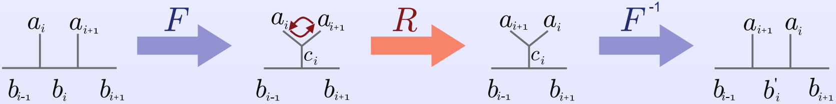
$$F = \begin{pmatrix} \varphi & \sqrt{\varphi} \\ \sqrt{\varphi} & -\varphi \end{pmatrix} \quad \text{with} \quad \varphi = d_\tau^{-1} = \frac{1-\sqrt{5}}{2}$$

Fibonacci Braidings

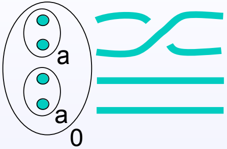
- To process a single qubit we must find the operators σ that defines the braidings.
- From the Yang-Baxter eq. (or the hexagon equation) one finds out the R matrix:

$$R = \begin{pmatrix} e^{\frac{4}{5}\pi i} & 0 \\ 0 & -e^{\frac{2}{5}\pi i} \end{pmatrix}$$

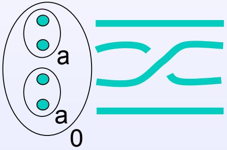
- In a Fibonacci chain, to find the representations of σ's, we need to make a basis tranformation in order to apply the R - matrix:



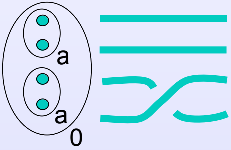
Fibonacci Braidings



$$\sigma_3 = \sigma_1 = R^{-1} = \begin{pmatrix} e^{-\frac{4}{5}\pi i} & 0 \\ 0 & -e^{-\frac{2}{5}\pi i} \end{pmatrix}$$



$$\sigma_2 = F\sigma_1 F = \begin{pmatrix} -\varphi\, e^{-i\frac{\pi}{5}} & -\sqrt{\varphi}\, e^{i\frac{2\pi}{5}} \\ -\sqrt{\varphi}\, e^{i\frac{2\pi}{5}} & -\varphi \end{pmatrix}$$

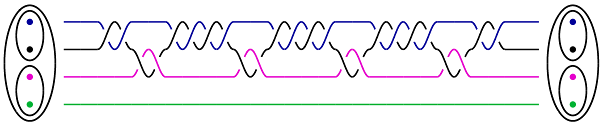


$$\sigma_1 = \sigma_3 = R^{-1} = \begin{pmatrix} e^{-\frac{4}{5}\pi i} & 0 \\ 0 & -e^{-\frac{2}{5}\pi i} \end{pmatrix}$$

Single-Qubit Gate Compiling

Bonesteel et al.

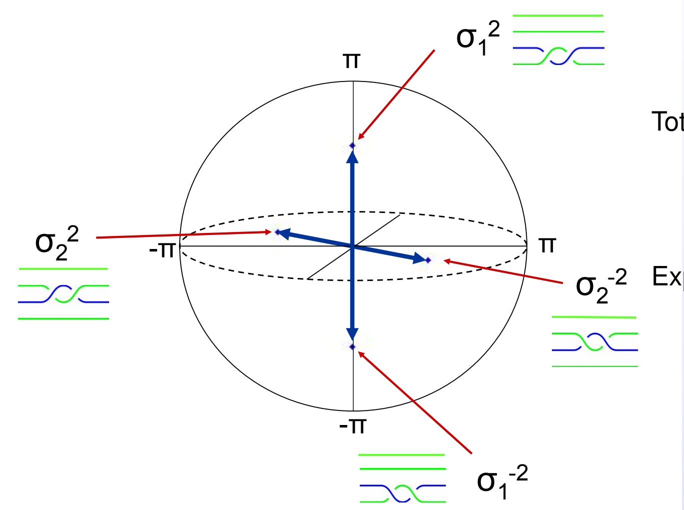
- To the purpose of Universal Quantum Computation we want to approximate, at any give accuracy, any single-qubit gate using as generators the braidings σ_1 and σ_2
- For Fibonacci anyons the elementary braidings generate an **infinite group**, dense in $SU(2)$



$$\sigma_3^{-2}\sigma_2^2\sigma_3^{-4}\sigma_2^2\sigma_3^{-4}\sigma_2^2\sigma_3^{-4}\sigma_2^2\sigma_3^{-2}$$

$$\cong -iX \pm 0.0031$$

Brute Force search
Bonesteel et al.

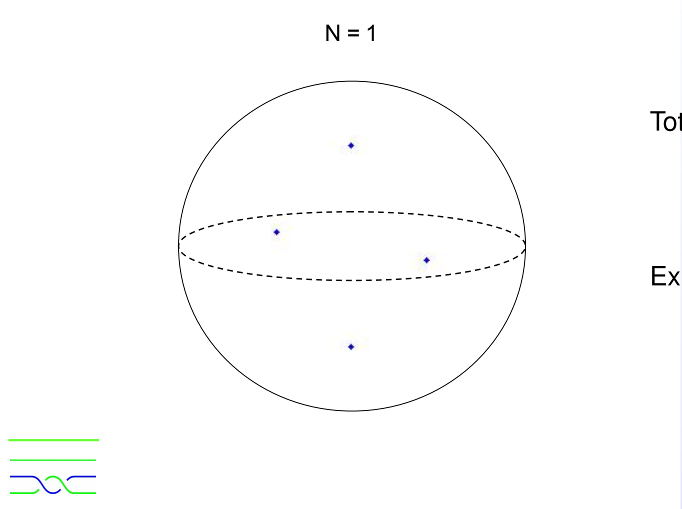


Total weaves:

$$B_N \cong 3^N$$

Expected error:

$$\varepsilon_N \cong \frac{1}{3^{N/3}}$$



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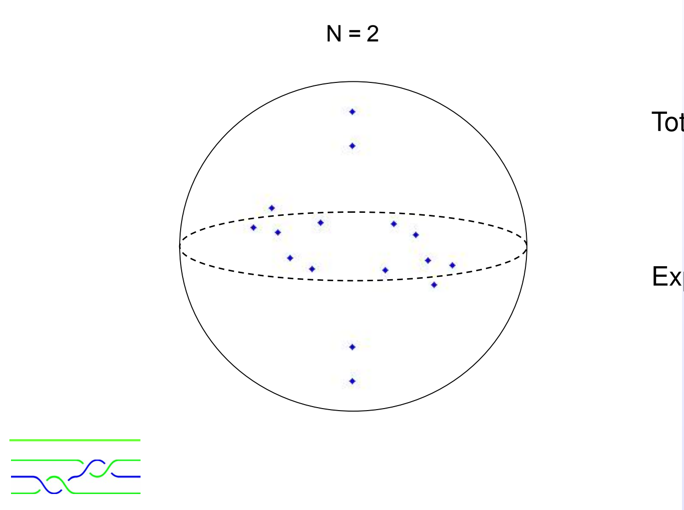

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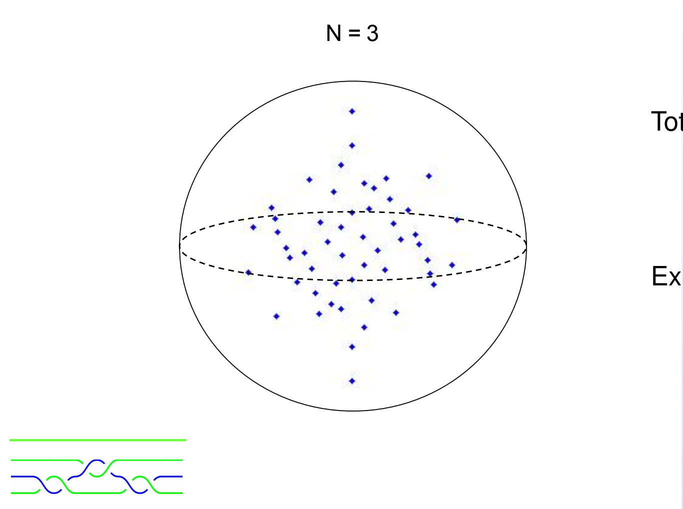

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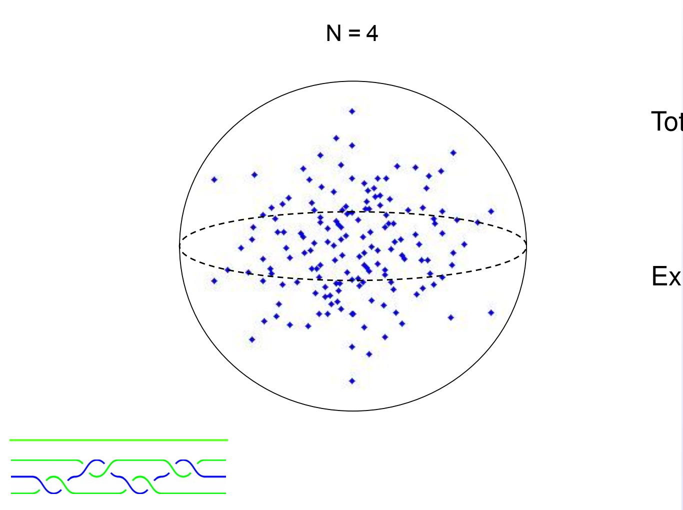

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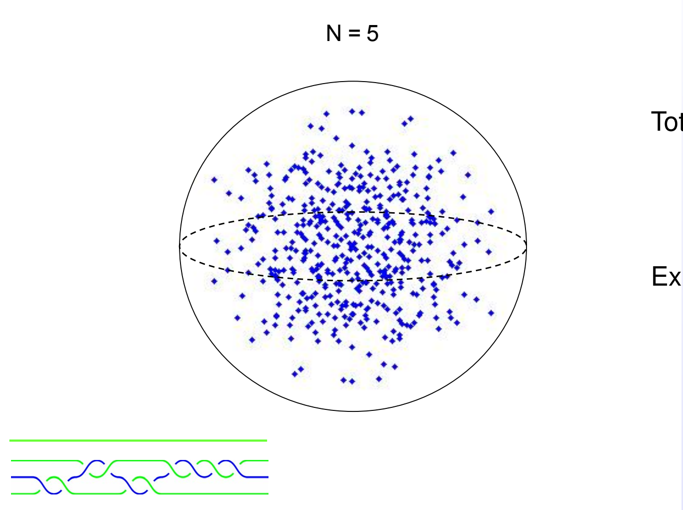

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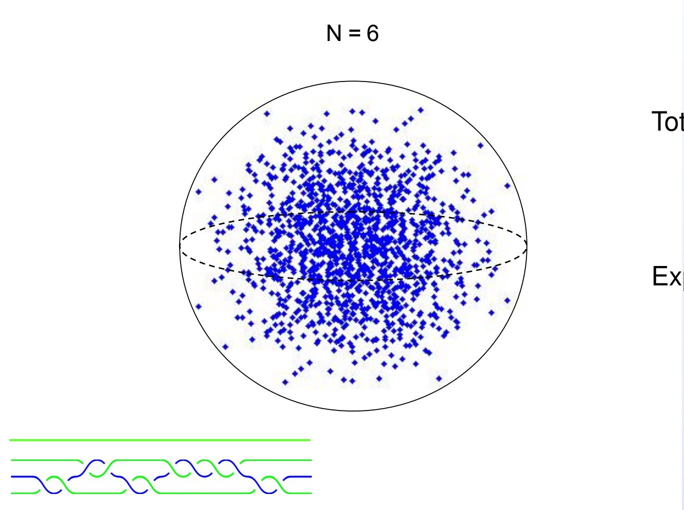
Brute Force search

Bonesteel et al.



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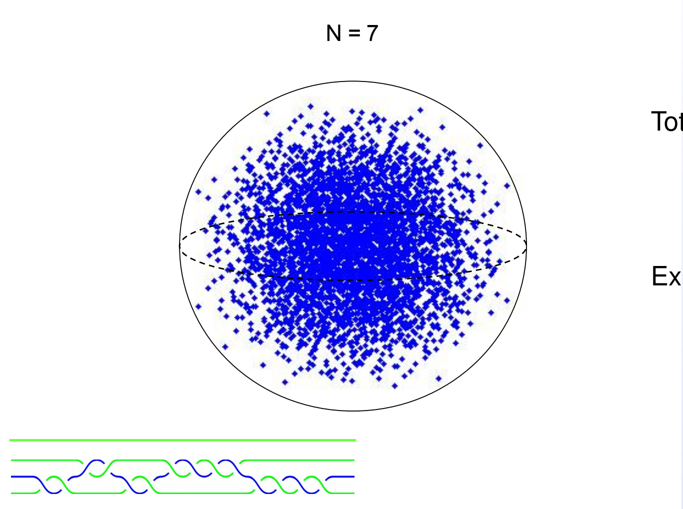

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Brute Force search
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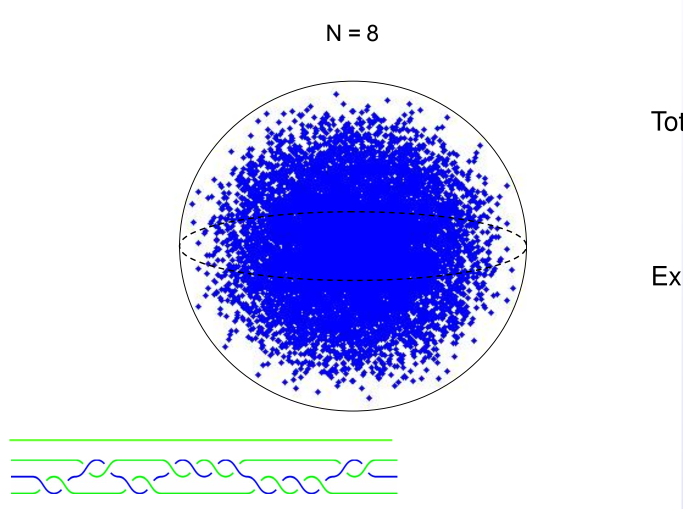

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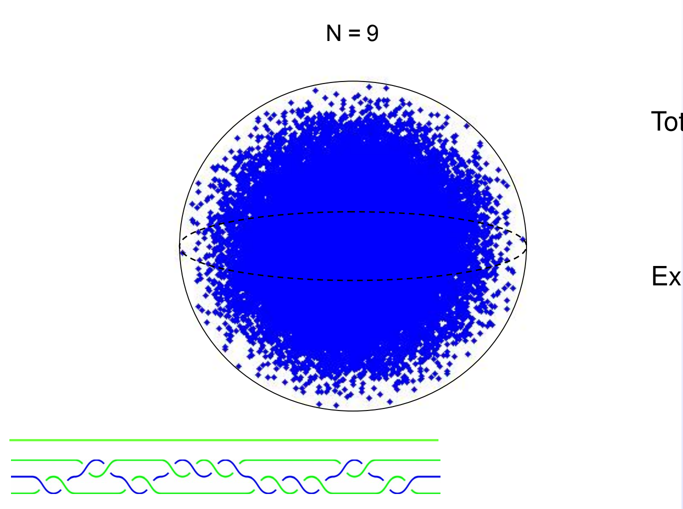
Bonesteel et al.



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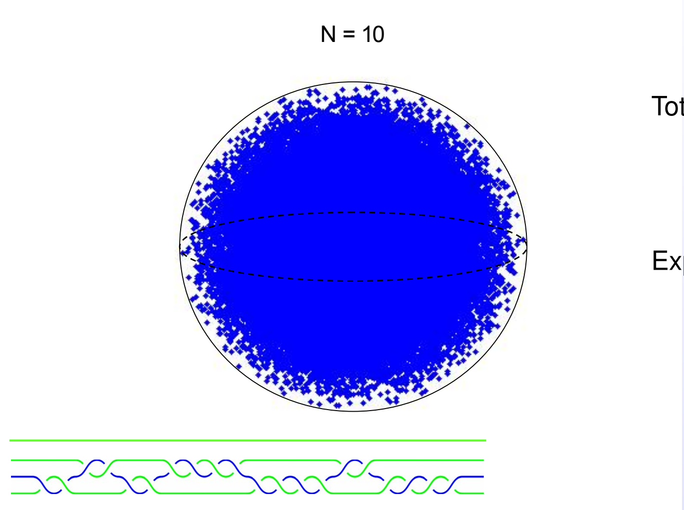

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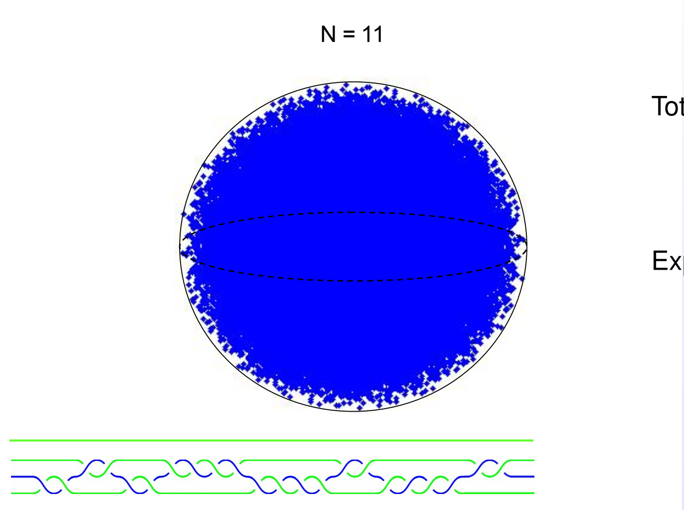
Brute Force search

Bonesteel et al.



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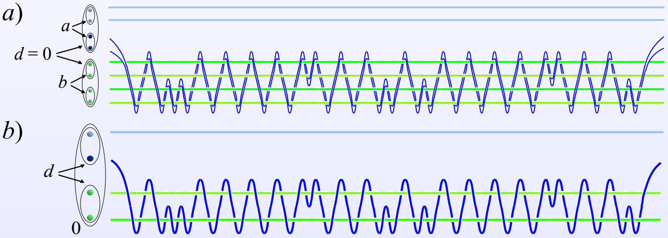

$$\leftrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$\leftrightarrow \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

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