Sketches and hints for the solution of some of the exercises

A. Hints for Exercise 1.1

I consider the degree of freedom $\tau_r = s_r s_{r+1} = \pm 1$. It represents a domain wall: $\tau = +1$ if the two neighboring spins are parallel, and $\tau = -1$ vice versa. We observe that $s_r s_{r+2} = \tau_r \tau_{r+1}$, because $s_{r+1}^2 = 1$. This implies that we can rewrite the Hamiltonian as:

$$H = -J_1 \sum_r \tau_r - J_2 \sum_r \tau_r \tau_{r+1} \,. \tag{1}$$

Disregarding boundary effects, this Hamiltonian is completely equivalent to the classical 1D Ising model in a longitudinal field given by J_1 which we considered during our lectures. Following the same steps we find the related transfer matrix:

$$T = \begin{pmatrix} e^{\beta J_2 + \beta J_1} & e^{-\beta J_2} \\ e^{-\beta J_2} & e^{\beta J_2 - \beta J_1} \end{pmatrix} . \tag{2}$$

Its eigenvalues are:

$$\lambda_\pm = e^{\beta J_2} \cosh \beta J_1 \pm \sqrt{e^{2\beta J_2} \sinh^2 \beta J_1 + e^{-2\beta J_2}} \,. \tag{3}$$

Disregarding bounday effects we approximate:

$$Z \approx \operatorname{Tr} T^N = \lambda_+^N + \lambda_-^N \to \lambda_+^N \,, \tag{4}$$

where the final limit holds for the thermodynamic limit $N \to \infty$. Therefore we obtain:

$$F/N \to -J_2 - \beta^{-1} \ln \left[\cosh \beta J_1 + \sqrt{\sinh^2 \beta J_1 + e^{-4\beta J_2}} \right] . \tag{5}$$

When considering $\tau_0 = s_1$, it is easy to see that:

$$s_r = \prod_{j=0}^{r-1} \tau_j \,. \tag{6}$$

This implies that s_r is a non-local string of the τ variables from one edge to the $r-1$ position. Essentially, it matches the Jordan-Wigner string you may have seen in CMT1 for a classical system.

B. Exercise 1.2

1. I define $K = \beta J$. The transfer matrix in the basis $\{|+1\rangle, |0\rangle, |-1\rangle\}$ reads:

$$\left\langle s_{r+1} \left| e^{J\beta s_r s_{r+1}} \right| s_r \right\rangle = e^{K s_r s_{r+1}} = \begin{pmatrix} e^K & 1 & e^{-K} \\ 1 & 1 & 1 \\ e^{-K} & 1 & e^K \end{pmatrix} \tag{7}$$

2. The global symmetry $s \leftrightarrow -s$ is implemented by:

$$U = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \tag{8}$$

   We may thus rewrite the transfer matrix in a basis given by $\{|0\rangle, |+\rangle, |-\rangle\}$, where $|\pm\rangle = (|+1\rangle \pm |-1\rangle)/\sqrt{2}$. $|0\rangle$ and $|+\rangle$ are even under the exchange U, $|-\rangle$ is odd. In this basis, T reads:

$$T = \begin{pmatrix} 1 & \sqrt{2} & 0 \\ \sqrt{2} & e^K + e^{-K} & 0 \\ 0 & 0 & e^K - e^{-K} \end{pmatrix} \tag{9}$$

The eigenvalues are:

$$\lambda_1 = \frac{2\cosh K + 1}{2} + \sqrt{2 + \left(\frac{2\cosh K - 1}{2}\right)^2} \tag{10}$$

$$\lambda_2 = 2\sinh K \tag{11}$$

$$\lambda_3 = \frac{2\cosh K + 1}{2} - \sqrt{2 + \left(\frac{2\cosh K - 1}{2}\right)^2} \tag{12}$$

λ_1 is the largest eigenvalue for any K.

3. In the thermodynamic limit ($N \to \infty$), $Z = \operatorname{Tr} T^N \approx \lambda_1^N$, thanks to the Perron Frobenius theorem. Therefore:

$$F = -\beta^{-1} \ln Z \to -\beta^{-1} N \ln \lambda_1 \tag{13}$$

Therefore:

$$F/N \to -\beta^{-1} \ln \left[\frac{2\cosh K + 1}{2} + \sqrt{2 + \left(\frac{2\cosh K - 1}{2}\right)^2}\right] =$$

$$= -J - \beta^{-1} \ln \left[\frac{1 + e^{-2\beta J} + e^{-\beta J}}{2} + \sqrt{2e^{-2\beta J} + \left(\frac{1 + e^{-2\beta J} - e^{-\beta J}}{2}\right)^2}\right] \tag{14}$$

4. One can Taylor expand λ_1/λ_2 in e^{-K} for large K. In particular we have:

$$\xi^{-1} \sim \ln\left(\frac{\lambda_1}{\lambda_2}\right) \approx \ln\left(\frac{e^K + 3e^{-K} + \dots}{e^K - e^{-K} + \dots}\right) \approx \ln\left(1 + 4e^{-2K} + \dots\right) \approx 4e^{-2K}\,. \tag{15}$$

This implies that ξ diverges at zero temperature ($\mathcal{T} \to 0$) as $\xi \propto e^{\frac{2J}{k_B \mathcal{T}}}$. This divergence indicates that at zero temperature there is an effective phase transition and the system becomes ordered, namely the spin-spin correlation does not decay.

C. Hints for Exercise 2.1

1. *1+1D massless case*

The action for the massless 1D Dirac field in Euclidean time ($\tau = it$):

$$S = \int dx d\tau\ \bar{\psi}\left(\partial_\tau + i v_F \sigma_z \partial_x\right)\psi \tag{16}$$

By rescaling $x = bx'$ and $\tau = b\tau'$ we get:

$$S = \int b^2 dx' d\tau'\ \bar{\psi}\left(\frac{1}{b}\partial_{\tau'} + i v_F \sigma_z \frac{1}{b}\partial_{x'}\right)\psi \tag{17}$$

To obtain that S is scale invariant, as required in a gapless system, we need to impose $\psi = \psi' b^{-1/2}$. Therefore the scaling dimension of ψ is 1/2.

2. *1+1D massive case*

The action associated to the mass term is proportional to:

$$S = \int dx d\tau\ m\bar{\psi}\sigma_y\psi\,. \tag{18}$$

By rescaling we get:

$$S = \int dx' d\tau' \; bm\bar{\psi}'\sigma_y\psi' \, . \tag{19}$$

Therefore we can define $m' = mb > m$. m grows under rescaling, hence it is relevant. As expected, it opens a gap, as it can be verified by explicitly calculating the dispersion relation.

3. Equations of motions

Check them in any field theory book

4. Scaling of the 2D Dirac theory

You must redo the previous calculations now with the integrals in $\int dxdyd\tau$. You get that in the gapless case $\psi = \psi' b$, thus the scaling dimension is 1. The mass term has again the same relevant behavior $m' = mb$.

5. Dirac correlation function

I consider the Fourier transform of the fields in Euclidean time to be given by:

$$\psi(y,\tau) = \int \frac{dkd\omega}{2\pi} e^{i\omega\tau + iky}\psi(k,\omega) \, . \tag{20}$$

The Dirac Green's function from the action in Euclidean time reads:

$$G_0(k,\omega) = -\frac{1}{i\omega - v_F\sigma_z k} \, . \tag{21}$$

We must calculate its Fourier transform:

$$G(y,\tau,x,0) = -\int \frac{dk\, d\omega}{4\pi^2} \frac{e^{-i\omega\tau - ik(y-x)}}{i\omega - v_F\sigma_z k} \, . \tag{22}$$

We integrate in k, the pole is in $k = i\omega\sigma_z/v_F$. I assume for simplicity that $y > x$ (the other case is similar). To proceed, it is convenient to distinguish what happens for $\omega\sigma_z > 0$ and $\omega\sigma_z < 0$.

- For $\omega\sigma_z > 0$ we must choose a contour in the lower half plane of the complex k parameter. I obtain:

$$\begin{aligned} G(y,\tau,x,0) = \int \frac{dk\, d\omega\sigma_z}{4\pi^2 v_F} \frac{e^{-i\omega\tau - ik(y-x)}}{k - i\omega\sigma_z/v_F} &= -i\int \frac{d\omega\sigma_z}{2\pi v_F}\Theta(\omega\sigma_z)e^{-i\omega\tau + \omega\sigma_z(y-x)/v_F} = \\ -i\int_0^\infty \frac{d\omega'}{2\pi v_F} e^{-i\omega'\sigma_z\tau + \omega'(y-x)/v_F} &= \frac{i}{2\pi}\frac{1}{-iv_F\tau\sigma_z + (y-x)} = \frac{1}{2\pi}\frac{i}{(y-x) + v_F t\sigma_z} \, ; \end{aligned} \tag{23}$$

  where I considered $\omega' = \sigma_z\omega$ and $\tau = it$.

- For $\omega\sigma_z < 0$ we must choose a contour in the upper half plane of the complex k parameter. I obtain:

$$\begin{aligned} G(y,\tau,x,0) = i\int \frac{d\omega\sigma_z}{2\pi v_F}\Theta(-\omega\sigma_z)e^{-i\omega\tau + \omega\sigma_z(y-x)/v_F} &= \\ i\int_{-\infty}^0 \frac{d\omega'}{2\pi v_F} e^{-i\omega'\sigma_z\tau + \omega'(y-x)/v_F} &= \frac{i}{2\pi}\frac{1}{-iv_F\tau\sigma_z + (y-x)} = \frac{1}{2\pi}\frac{i}{(y-x) + v_F t\sigma_z} \, . \end{aligned} \tag{24}$$

For both cases we get the same result, and, in particular, σ_z multiplies $v_F t$ in such a way that the left mover correlation depends on $y - x + v_F t$ only, and the right mover correlation on $y - x - v_F t$, which is consistent with ther chiral linear dispersion. From this we see that the correlation function decays algebraically, as expected in a critical system.

6. The scaling dimension

The definition of the rescaling of the fields corresponds to the relation: $\psi'(x') = b^{D_\psi}\psi(x)$. Therefore:

$$\left\langle \psi_L'^{\dagger}(y',t')\psi_L'(x',0)\right\rangle = b^{2D_\psi}\left\langle \psi_L^{\dagger}(y,t)\psi_L(x,0)\right\rangle = \frac{ib^{2D_\psi}}{y-x+v_F t}\,, \tag{25}$$

where I used the two-point correlation function calculated in Eq. (23). Eq. (23) additionally implies:

$$\left\langle \psi_L^{\dagger}(y',t')\psi_L(x',0)\right\rangle = \frac{i}{y'-x'+v_F t'} = \frac{ib}{y-x+v_F t}\,. \tag{26}$$

At the critical fixed point, the Lagrangian of the system is invariant under rescaling, therefore, as written in the exercise text, we have:

$$\left\langle \psi_L'^{\dagger}(y',t')\psi_L'(x',0)\right\rangle = \left\langle \psi_L^{\dagger}(y',t')\psi_L(x',0)\right\rangle\,. \tag{27}$$

By considering the three previous equations, and comparing the right hand sides of Eqs. (25) and (26), we obtain the expected $D_\psi = 1/2$.

7. Massive case

The Lagrangian for the massive 1D Dirac field is:

$$\mathcal{L} = \int dx\,\psi^\dagger\left(i\partial_t - iv_F\sigma_z\partial_x + m\sigma_y\right)\psi\,, \tag{28}$$

where ψ is a two-component spinor. The mass term is relevant because $2-2D_\psi = 1 > 0$. The Green's function reads:

$$G(k,\omega) = \frac{1}{\omega + v_F k\sigma_z + m\sigma_y} = \frac{\omega - v_F k\sigma_z - m\sigma_y}{\omega^2 + v_F^2k^2 + m^2} \tag{29}$$

This is a 2×2 matrix, and we are asked to calculate the Fourier transform of its diagonal terms:

$$G_{\uparrow\uparrow \text{ or } \downarrow\downarrow}(y,t,x,0) = \int \frac{dk\,d\omega}{4\pi^2}e^{-i\omega t + ik(y-x)}\frac{\omega - v_F k\sigma_z}{\omega^2+v_F^2k^2+m^2} = v_F^{-1}\left(i\partial_t + v_F\sigma_z i\partial_y\right)\int\frac{d^2p}{4\pi^2}\frac{e^{i\vec{p}\vec{r}}}{p^2+m^2}\,, \tag{30}$$

where, for simplicity, I redefined $\vec{p} = (-\omega, v_F k)$ and $\vec{r} = \left(t, \frac{y-x}{v_F}\right)$. I use Eq. 12 in the exercise to get:

$$G_{\uparrow\uparrow \text{ or } \downarrow\downarrow}(y,t,x,0) = \frac{1}{2\pi v_F}\left(i\partial_t + v_F\sigma_z i\partial_y\right)K_0\left(|m||\vec{x}|\right)\,. \tag{31}$$

We know that $\partial_t r = t/r$ and $\partial_y r = (y-x)/v_F^2 r$. Now it is a matter of using the approximations and replacing $t=0$, $r = |y-x|/v_F$. For large m we get:

$$G(y,x) \approx \frac{-ime^{-mr}}{2v_F\sqrt{2\pi mr}}\text{sign}\,(y-x)\,\sigma_z\left(1+\frac{1}{2mr}\right) \tag{32}$$

In this massive situation it is crucial to observe that the correlation function decays exponentially with the distance, with a correlation length given by v_f/m since $r = |y-x|/v_F$. This is because the mass term is relevant and drives the system away from the gapless fixed point. Only at a gapless point correlation functions decay algebraically. For the massless limit $mr \ll 1$, then we recover an algebraic decay of the kind we derived before.

As mentioned before, to maintain the correlation function invariant under the rescaling of fields and coordinates, the necessary rescaling of the mass is $m' = bm$. This can be easily seen from Eq. (32): by taking $m' = bm$ and $r' = r/b$ the terms mr remain invariant. The overall mass dependence in front, instead, gives rise to the rescaling of the fields and it is consistent with $\psi'(x') = b^{1/2}\psi(x)$. In conclusion, the scaling dimension of m is 1 and we get $m(l) = m(0)e^l$.

D. Hints for Exercise 2.3

The calculation of the correlation function for bosonic fields is made in detail in Sec. VB of the 1D models notes. C decays as $r^{-\beta^2/2}$, as it can be seen by the factor of the correlation that depends on the distance $e^{\beta^2 \langle \theta(x)\theta(x')\rangle}$. Hence the scaling dimension is $D_\beta = \beta^2/4$. Therefore, the perturbation S_I is relevant for $2 - \beta^2/4 > 0$, thus $\beta < \sqrt{8}$. $\alpha = \beta/\sqrt{K}$, thus S_I' is relevant for $2 - \alpha^2 K/4 > 0$, hence $K < 8/\alpha^2$.