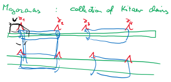
NON-ABELIANANYON MODELS

- List of the topological sectors [which kind of anyons do we have?]
  → Finite set
  Majoranas : $\{1, \gamma, \psi\}$ (1 → vacuum)
  Fibonacci : $\{1, \tau\}$
  Ising anyons : $\{1, \sigma, \psi\}$ (σ → $e/4$, ψ → e) → FQH $\frac{5}{2}$
- Fusion rules : how anyons merge → N_{ab}^c
  $a \times b = \sum_c N_{ab}^c \, c$, $N_{ab}^c \in \{0,1\}$
  → How the Hilbert space grows
- Associativity rule : Unitary mapping between bases that describe the same system
  $(a \times b) \times c \leftrightarrow a \times (b \times c)$
  F - matrix
- Braiding rules : representations of the braid group
  $R_{ab}^c \rightarrow B = F^{\dagger} R F$

Majoranas : collection of Kitaev chains



$d_1 = \frac{\gamma_1 - i\gamma_2}{2}$ $d_1^{\dagger} = \frac{\gamma_1 + i\gamma_2}{2}$ | $d_1^{\dagger}d_1, d_2^{\dagger}d_2 \ldots$

$\gamma_1 \times \gamma_2 = 1 + \psi$

$\gamma_1 \times \gamma_2 \times \gamma_3 \times \gamma_4$; $1, \psi$

$i\gamma_1\gamma_2 \cdot i\gamma_3\gamma_4 = -\gamma_1\gamma_2\gamma_3\gamma_4$

$|0\rangle_L$ = (1, 1 → 1) $\quad -\gamma_1\gamma_2\gamma_3\gamma_4 = +1$

$|1\rangle_L$ = (ψ, ψ → 1)

$i\gamma_1\gamma_2 = i\gamma_3\gamma_4$ because I am requiring $-\gamma_1\gamma_2\gamma_3\gamma_4 = 1$

$\sigma_z = i\gamma_1\gamma_2 = i\gamma_3\gamma_4 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $\quad |0\rangle_L = |00\rangle$, $|1\rangle_L = |11\rangle$ $\quad n_i = d_i^{\dagger}d_i$

$i\gamma_1\gamma_2|00\rangle = +|00\rangle$ $\quad i\gamma_1\gamma_2|11\rangle = -|11\rangle$

$i\gamma_2\gamma_3 = ?$

$i\gamma_1\gamma_2 [i\gamma_2\gamma_3|00\rangle] = -[i\gamma_2\gamma_3]\cdot(i\gamma_1\gamma_2)|00\rangle = -[i\gamma_2\gamma_3|00\rangle]$

$\Rightarrow i\gamma_2\gamma_3|00\rangle = |11\rangle = |1\rangle_L$

$i\gamma_1\gamma_4 = i\gamma_2\gamma_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sigma_x$ $\quad$ Parity constr. $-\gamma_1\gamma_2\gamma_3\gamma_4 = 1$

$(i\gamma_1\gamma_4)(i\gamma_2\gamma_3) = 1$

$\sigma_x\sigma_z = -i\sigma_y = i\gamma_2\gamma_3\, i\gamma_1\gamma_2$

$-i\sigma_y = i\gamma_3 i\gamma_1$

$\sigma_y = i\gamma_1\gamma_3$

$\sigma_z = i\gamma_1\gamma_2$

$\gamma_2 \times \gamma_3 = 1 + \psi$

$|i\gamma_2\gamma_3\rangle = \alpha|00\rangle + \beta|11\rangle$ (eigenstates of σ_x ; eigenstates of σ_z)

$|i\gamma_2\gamma_3 = +1\rangle = |+\rangle_L = \frac{|0\rangle_L + |1\rangle_L}{\sqrt{2}}$

$|i\gamma_2\gamma_3 = -1\rangle = |-\rangle_L = \frac{|0\rangle_L - |1\rangle_L}{\sqrt{2}}$

There exist a unitary mapping from the eigenstates of σ_z (Fusion of $\gamma_1 \times \gamma_2$) to the eigenstates of σ_x (Fusion of $\gamma_2 \times \gamma_3$)

$F = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\frac{1}{\sqrt{2}}$ $\quad F^{\dagger} = F^{-1} = F$ $\quad$ Hadamard

$[F^{\dagger}\sigma_z F = \sigma_x]$

$(((\gamma_1 \times \gamma_2) \times \gamma_3) \times \gamma_4) = 1$ $\quad (\gamma_1 \times (\gamma_2 \times \gamma_3)) \times \gamma_4 = 1$

$= \sum (F^{abc}_d)_{x,y}$ $\quad x, y \in \{1, \psi\}$

$d_1 = \frac{\gamma_1 - i\gamma_2}{2}$ $\quad d_2 = \frac{\gamma_3 - i\gamma_4}{2}$

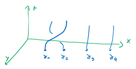
Majorana modes as Non-Abelian anyons

4 Majorana modes with fixed fermionic parity ↔ 1 qubit

Braiding : you physically move anyons one around the others → Adiabatic process

→ 2 states

$|\psi\rangle_{t=0} \xrightarrow{\text{Evolution [Braiding]}}$ $|\psi(t=T)\rangle = U(T)\,|\psi(t=0)\rangle$ — Unitary operator 2×2



R_{ab}^c — The braiding of a and b conserves c

Eigenbasis of $i\gamma_1\gamma_2$ $[\gamma_1 \times \gamma_2]$ $\quad i\gamma_1\gamma_2 = a$ ↑ phase : $R^a_{\gamma_1\gamma_2}$ $\quad a = 1, \psi$

$(a \times b) = c$

$(R_{ab}^c)^2 = e^{i(2\pi S_c - 2\pi S_a - 2\pi S_b)}$

$\gamma \times \gamma = 1 + \psi$ (even n° of fermions ; odd n° of fermions)

$(R^{1}_{\gamma\gamma})^2 = e^{i[2\pi S_1 - 4\pi S_\gamma]}$

$(R^{\psi}_{\gamma\gamma})^2 = e^{i[2\pi S_\psi - 4\pi S_\gamma]}$

Majoranas : I do not care about the common overall phase $e^{-i4\pi S_\gamma}$: merges with the dynamical phase

1 : boson : integer spin

ψ : fermion : half-integer spin

$S_1 = 0$

$S_\psi = 1/2$

$(R^{\psi}_{\gamma\gamma})^2 = e^{i\pi}\,(e^{-i4\pi S_\gamma})$

$(R^{1}_{\gamma\gamma})^2 = 1 \cdot e^{-i4\pi S_\gamma}$

$R^2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} e^{-i4\pi S_\gamma} = e^{\pm i\frac{\pi}{2}\sigma_z} e^{i\alpha}$

$R = \begin{pmatrix} 1 & 0 \\ 0 & \pm i \end{pmatrix} e^{i\alpha} = e^{\pm i\frac{\pi}{4}\sigma_z} e^{i\alpha}$ (SU(2), U(1))

$= (\cos\frac{\pi}{4} \pm i\sigma_z \sin\frac{\pi}{4})$

[Time reversal, parity cay]

$\begin{pmatrix} e^{i\pi/4} & 0 \\ 0 & e^{-i\pi/4} \end{pmatrix} = e^{i\pi/4}\begin{pmatrix} 1 & \\ & -i \end{pmatrix}$

Non-Abelian SU(2) piece

$R_{\gamma\gamma} = e^{i\frac{\pi}{4}\sigma_z} = e^{i\frac{\pi}{4}(i\gamma_1\gamma_2)} = [\cos\frac{\pi}{4} - (\sin\frac{\pi}{4})\gamma_1\gamma_2]$

What happens to the γ operators?

$R^{\dagger}_{\gamma_1\gamma_2}\,\gamma_1\,R_{\gamma_1\gamma_2} = [\cos\frac{\pi}{4} + \gamma_1\gamma_2\sin\frac{\pi}{4}]\,\gamma_1\,[\cos\frac{\pi}{4} - \gamma_1\gamma_2\sin\frac{\pi}{4}]$ ($U^{\dagger}(t)$, $U(t)$)

$= \frac{1}{2}(1 + \gamma_1\gamma_2)\gamma_1(1 - \gamma_1\gamma_2) = \frac{1}{2}(\gamma_1 - \gamma_2 + \gamma_1\gamma_2\gamma_1 - \gamma_1\gamma_2\gamma_1\gamma_1\gamma_2) =$

$= \frac{1}{2}(\gamma_1 - \gamma_2 - \gamma_2 - \gamma_1) = -\gamma_2$

$\gamma_1 \rightarrow -\gamma_2$

$\gamma_2 \rightarrow \gamma_1$

$R_{\gamma_1\gamma_2}\,\gamma_3\,R^{\dagger}_{\gamma_1\gamma_2} = \gamma_3$

Different operation

I preserve $i\gamma_2\gamma_3$

$R_{\gamma_2\gamma_3} = e^{i\frac{\pi}{4}\sigma_x} = e^{i\frac{\pi}{4}(i\gamma_2\gamma_3)} = e^{i\frac{\pi}{4}\sigma_x}$

I go in the basis of σ_x, then everything is the same as before

$R_{\gamma_1\gamma_2}$: $R_{\gamma_2\gamma_3} = F^{\dagger} R_{\gamma_1\gamma_2} F = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} e^{i\frac{\pi}{4}\sigma_z}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = e^{i\frac{\pi}{4}\sigma_x}$

$= R^{x}_{\gamma_1\gamma_2}$ (phase)

$= \sum_y (F^{\gamma\gamma\gamma}_{\gamma})_{xy}$

$= \sum_x (F^{\gamma\gamma\gamma}_{\gamma})_{xy} R^{y}_{\gamma_2\gamma_3}$

$= \sum_{xx'} (F^{\gamma\gamma\gamma}_{\gamma})_{xy} R^{y}_{\gamma\gamma} \cdot (F^{\gamma\gamma\gamma}_{\gamma})^{\dagger}_{yx'}$

The exchange of γ_2 and γ_3 in the $i\gamma_1\gamma_2$ basis reads

$FRF^{\dagger} = B = e^{i\frac{\pi}{4}\sigma_x}$

$= e^{i\frac{\pi}{4}\sigma_z} = \begin{pmatrix} e^{i\frac{\pi}{4}} & 0 \\ 0 & e^{-i\frac{\pi}{4}} \end{pmatrix}$

$= e^{i\frac{\pi}{4}\sigma_x} = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\sigma_x = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$

$R_{\gamma_1\gamma_2}$ does not commute with $R_{\gamma_2\gamma_3}$

⟹ Non-Abelian anyons

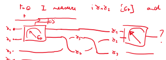
Yang-Baxter must hold

$R_{\gamma_1\gamma_2} R_{\gamma_2\gamma_3} R_{\gamma_1\gamma_2} = R_{\gamma_2\gamma_3} R_{\gamma_1\gamma_2} R_{\gamma_2\gamma_3}$

$R\; FRF^{\dagger}\; R = FRF^{\dagger} R FRF^{\dagger}$

$|[0]\rangle_L$

t=0 I measure $i\gamma_1\gamma_2$ $[\sigma_z]$ and I get $|[0]\rangle_L$



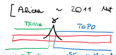
$R^2_{\gamma_2\gamma_3} = e^{i\pi/4\,\sigma_x}$ $\quad R^2 = e^{i\frac{\pi}{2}\sigma_x} = i\sigma_x = i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ — NOT

$i\gamma_1\gamma_2 = i\gamma_3\gamma_4$ because we fix the total parity

Qubit 1 ; Qubit 2 — → It creates entanglement

Majoranas in nanowires

[Alicea ~ 2011 Nat Phys]

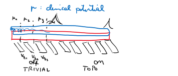


TRIVIA / TOPO

μ is outside the topol. phase ; SC with μ tuned in the Topol. phase

μ : chemical potential

off TRIVIAL ; on TOPO

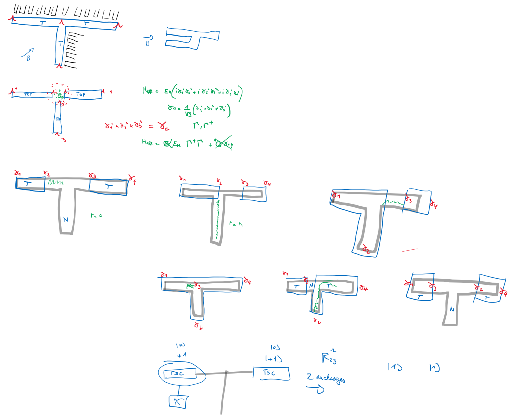


$H_{eff} = E_M(i\gamma_1'\gamma_2' + i\gamma_2'\gamma_3' + i\gamma_3'\gamma_1')$

$\gamma_0 = \frac{1}{\sqrt{3}}(\gamma_1' + \gamma_2' + \gamma_3')$

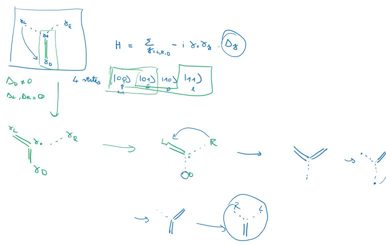
$\gamma_1' \times \gamma_2' \times \gamma_3' = \gamma_c$ $\quad \Gamma, \Gamma^{\dagger}$

$H_{eff} = E_M \Gamma^{\dagger}\Gamma$



$H = \sum_{k=1,2,3} -i\gamma_k\gamma_0$

$\Delta_0 \neq 0$; $\Delta_L, \Delta_R = 0$; 4 states $|00\rangle$ $|01\rangle$ $|10\rangle$ $|11\rangle$



Majoranas [Kitaev]

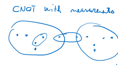
- Single qubit gates only belonging to the group of the cube generated by $e^{i\frac{\pi}{4}\sigma_z}, e^{i\frac{\pi}{4}\sigma_x}$

You are not able to have $e^{i\vec{\alpha}\vec{\sigma}}$

You cannot cover SU(2)

Majoranas are not universal

Kitaev : CNOT with measurements of 4 Majoranas



Fibonacci : Everything can be done with braidings

$e^{i\frac{\pi 7}{10}\sigma_z}$ $\quad F^{\dagger}e^{i\frac{\pi 7}{10}\sigma_z}F$