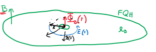
ANYONS

- emergent q.p. in 2D
- excitations over a GS [e, m in TORIC CODE]
- localized
- "Through exchange of 2 anyons, the system wavefunction may pick "ANY" phase"
  ↳ Abelian Anyons

FQH quasi-holes $\quad j_x = \sigma_{xy} E_y \quad \boxed{\sigma_{xy} = \nu \frac{e^2}{h} \quad \nu = \frac{1}{3}}$

$\nu = 1/3 \to$ Laughlin

Laughlin - Halperin



$\Phi_B(t): \quad \Phi_B(t=0) = \Phi_{BG}$

$\Phi_B(t=T) = \Phi_{BG} + \phi_0$

$\phi_0 = \frac{hc}{e}$

Adiabatic process

$$\boxed{\oint \vec{E}(t)\, d\vec{\ell} = -\int dA \frac{\partial_t B}{c} \qquad 2\pi E(t) R = -\frac{\partial_t \Phi_B}{c}}$$

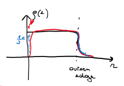
How much charge do we push out in T?

$$Q = \int_0^T \left[\oint_C d\ell\, j_r\right] dt = \int_0^T dt\, 2\pi R \cdot j(t) = \int_0^T dt\, 2\pi R\, \sigma_{xy} E(t) =$$

$$= \cancel{2\pi R} \cdot \int_0^T dt \boxed{\frac{1}{3}\frac{e^2}{h}} \left(-\frac{\partial_t \Phi_B}{\cancel{2\pi R} c}\right) =$$

$$= -\frac{1}{3}\frac{e^2}{hc}\int_0^T dt\, \partial_t \Phi_B(t) = -\frac{1}{3}\nu e \cdot \frac{e}{hc} \cdot \overbrace{\left[\Phi_{FIN} - \Phi_{IN}\right]}^{\phi_0} =$$

$$= \boxed{\boxed{-\frac{1}{3} e}} \quad \cancel{\frac{\phi_0}{\phi_0}}$$



quasi-hole : $\underbrace{\phi_a = \phi_0}$ $\quad \underbrace{q_a = +\frac{|e|}{3}}$ $\quad$ [costs energy]

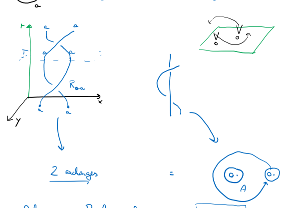
↓

excitation in the Laughlin State

"vortices"

Abelian Anyons as composite objects

$(\phi_a\ q_a)_a \qquad \nu = 1/3 : \quad \phi_a = \phi_0 \quad q_a = e/3$



Aharonov - Bohm phase $= \boxed{2\pi \frac{q_a}{e} \cdot \frac{\phi_a}{\phi_0}}$

$q_a = \nu e \qquad 2\pi \left(\frac{\nu e}{hc}\right) \underbrace{\int A\, ds}_{\phi_a}$

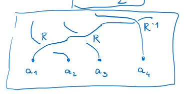
$$R_{aa}^2 = e^{i\Theta_a} = e^{i 2\pi \frac{q_a}{e}\frac{\phi_a}{\phi_0}}$$

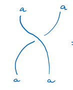
Θ_a : AB phase for a charge q_a around the flux ϕ_a

$$R_{aa}^2 = e^{i 2\pi q_a \phi_a} \Rightarrow R_{aa} = e^{i\pi q_a \phi_a}$$

Laughlin $\nu = 1/3$: $\quad R_{aa}^2 = e^{i\frac{2\pi}{3}} \Rightarrow R_{aa} = e^{i\frac{\pi}{3}}$

Spin $\quad \boxed{S_a = \frac{q_a \phi_a}{2}} \qquad$ electron $\quad q_a = e \quad \phi_a = \phi_0 \quad S_a = 1/2$





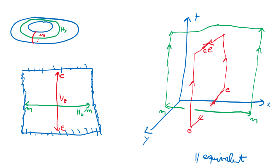
$= R_{aa}$: unitary operator [phase]
counterclockwise exchange

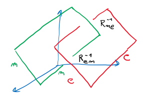
‖ In T.O. systems the degeneracy of the system is depending on its topology

TORIC CODE : symmetries $\to H_x, V_x, H_z, V_z$

$\{H_x, V_z\} = 0$

$$\boxed{H_x V_z H_x^{-1} V_z^{-1} |\Psi_{GS}\rangle = -|\Psi_{GS}\rangle}$$



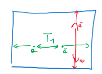


$$H_x V_z H_x^{-1} V_z^{-1} \underset{\text{Topol. equivalent}}{\simeq} R_{em}^{-1} R_{me}^{-1} = (-1)$$

Laughlin q.h. a : $q_a = \frac{e}{3} \quad \phi_a = \phi_0$

$\boxed{\text{on a torus}}$

Torus

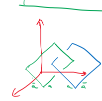


T_1 : is a process in which I nucleate a q.h and a q.P, I wind them and I re-annihilate them

$[T_1, H] = 0$

$$\boxed{T_1^{-1} T_2^{-1} T_1 T_2 |GS\rangle = R_{aa}^2 |GS\rangle}$$

$= e^{i\frac{2\pi}{3}} |GS\rangle$



$[T_1, T_2] \neq 0$

$\Rightarrow$ g.s. degeneracy on the torus

If there are anyons $\Rightarrow$ The GS is degenerate on the torus

Anyons $\Rightarrow$ T.O.

If $R_{aa}^2 = 1 \iff [T_1, T_2] = 0$

$R_{aa} = \pm 1 \Rightarrow$ bosons or fermions

More on composite Anyons

$a = (\phi_a, q_a)$

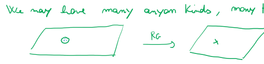
$\bar{a} = (-\phi_a, -q_a)$

$a \text{ "}\times\text{" } \bar{a} = 1\!\!1 \qquad \otimes$: it is a kind of "trivial" tensor product

↑ Topological sectors

$a \times \bar{a} = 1\!\!1 \to$ Fusion rule

We may have many anyon kinds, many topological sectors



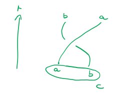
"b" $= (\phi_b, q_b)$

	$1\!\!1$	: $(e, 0)$	
Laughlin	$e/3$	: $(e/3, \phi_0)$	$-e/3$
	$2e/3$	: $(2e/3, 2\phi_0)$	$-2e/3$
composite fermion ←	"e"	: $(e, 3\phi_0)$	

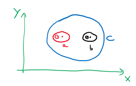
$a \times b = c \qquad c : \quad q_c = q_a + q_b$

$\phi_c = \phi_a + \phi_c$

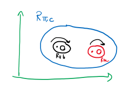
Spin $\quad \boxed{S_c = \frac{e_c \phi_c}{2}} \quad$ [in units of e, ϕ_0]



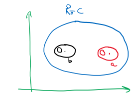
R_{ab}^c



π rotation of everything



I apply a π rotation of each body



TOTAL PROCESS:
Rigid exchange of a and b

$$\boxed{R_{ab}^c = e^{i\pi S_c} e^{-i\pi S_a} e^{-i\pi S_b}}$$

$a = b \Rightarrow q_a = q_b, \quad \phi_a = \phi_c$

$q_c = 2q_a \quad \phi_c = 2\phi_a$

$S_c = \frac{\phi_c q_c}{2} = \frac{4\phi_a q_a}{2}$

$$R_{ab}^c = e^{i 2\pi \phi_a q_a - i\frac{\pi}{2} q_a \phi_a - i\frac{\pi}{2} q_a \phi_a} = e^{i\pi q_a \phi_a}$$

Operators that create (annihilate) anyons in positions $\vec{r} = (\vec{x}, \vec{y})$

$$\phi_a(x, y) \xrightarrow[z = x+iy]{} \phi_a(z)$$

$$\text{"}\phi_a(z_1)\, \phi_b(z_2) \simeq \phi_c\left(\frac{z_1 + z_2}{2}\right) \cdot \frac{1}{(z_1 - z_2)^{\Delta_a + \Delta_b - \Delta_c}}\text{"}$$

$\Delta_a = S_a \qquad e^{i\pi(\Delta_a + \Delta_b - \Delta_c)}$

↓ CFT

$\phi_a = e^{i\alpha(\varphi - \Theta)}$

$$\langle V_a V_a^\dagger \rangle \sim \frac{1}{|z_1 - z_2|^{2\Delta_a}}$$

Statistics of Anyons $\iff$ spins $\iff$ Scaling dimensions of suitable operators

$$\Psi(x_1, x_2, x_3, \ldots) \xrightarrow{T} \Psi(x_{\sigma_1}, x_{\sigma_2} \ldots)$$

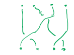
σ is a permutation

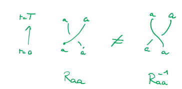
in 3D

fermions $\qquad$ property of the permutation [not of the topology]

$$\Psi(x_1, x_2, x_3 \ldots) = e^{i\pi P_\sigma} \Psi(x_{\sigma_1}, x_{\sigma_2}, x_{\sigma_3} \ldots)$$

bosons $\qquad e^{i 2\pi P_\sigma} = 1$





$\boxed{\text{The statistics of anyons in 2D is no longer described by the permutation group}}$

BRAID GROUP

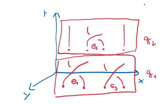
Each anyon is an "obstruction" on the plane

→ anyons do not overlap

[we keep them sufficiently far apart]

World lines ⟷ strands for braids in 2+1 D

4 Anyons

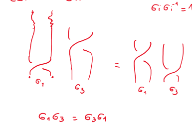


Generators : exchanges of neighboring braids

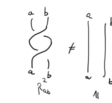
$B = \sigma_1^{a_1}\sigma_2^{a_2}\sigma_3^{a_3}\sigma_1^{a_4}\sigma_2^{a_2}\ldots$

$\sigma_1 \iff \sigma_1^{-1} \qquad \sigma_1 \sigma_1^{-1} = 1\!\!1$

ccw $\qquad$ cw $\qquad \sigma_i \sigma_i^{-1} = 1\!\!1$

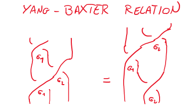


$\sigma_1 \sigma_3 = \sigma_3 \sigma_1$



If $i \neq j \pm 1 \qquad [\sigma_i, \sigma_j] = 0$

YANG - BAXTER RELATION



$$\boxed{\sigma_{i+1}\sigma_i\sigma_{i+1} = \sigma_i\sigma_{i+1}\sigma_i}$$

$\boxed{\text{What are the irreps of the Braid group?}}$

BRAID GROUP

- Abelian representations ⟷ Abelian anyons

$\sigma_i \sim R_{ab}^c = e^{i\pi[S_c - S_a - S_b]} \in U(1)$

- Non - Abelian representations

↳ σ are mapped into matrices

→ $\boxed{\text{what is the Hilbert space these matrices are acting on?}}$

Non - Abelian Anyons

"$a \times a = 1\!\!1 + b + c$"

Spin 1/2 objects

$$\boxed{\underset{2}{\frac{1}{2}} \otimes \underset{2}{\frac{1}{2}} = \underset{1}{0} \oplus \underset{3}{1} \qquad SU(2)}$$

$(\gamma_1\ \gamma_2)_{0,1} \qquad$ "$\underset{?}{\gamma} \times \gamma = \underbrace{\underset{P=+1}{1\!\!1} \oplus \underset{P=-1}{\psi}}_{\text{orthogonal states}}$"

"Tensor category theory"

Model of Anyons

- List of topological sectors ⟷ "Finite list"

$\{1\!\!1, e, m, \psi\} \to$ TORIC CODE

$\{1\!\!1, \gamma, \psi\} \to$ MAJORANAS

$\{1\!\!1, \sigma, \psi\} \to$ ISING $\quad \nu = 5/2$ Moore Read FQH

$\{1\!\!1, \tau\} \to$ FIBONACCI MODEL

- Fusion Rules $\qquad \sim$ "Clebsh Gordan coeff."

$a \times b = \sum_c N_{ab}^c \cdot c \qquad a, b, c \in$ Topol. sectors charges

$= c_1 + c_2 + c_3 \qquad \boxed{N_{ab}^c \in \{0, 1\}}$

$\boxed{\gamma \times \gamma = 1\!\!1 + \psi} \qquad \boxed{N_{\gamma\gamma}^{1\!\!1} = 1 \quad N_{\gamma\gamma}^{\psi} = 1 \quad N_{\gamma\gamma}^{\gamma} = 0}$

$$\boxed{\begin{aligned} 1\!\!1 \times \psi &= \psi \\ 1\!\!1 \times \gamma &= \gamma \\ \psi \times \psi &= 1\!\!1 \\ \psi \times \gamma &= \gamma \end{aligned}}$$

$1\!\!1$: vacuum, Cooper pairs

"$(\gamma \times \gamma) \times \gamma = \gamma$"

a is non - Abelian if

$N_{aa}^c \neq 0$ for at least 2 different c

$(a \times a) \to$ 2 states or more

$|GS, +\rangle \Rightarrow 1\!\!1$

$|GS, -\rangle \Rightarrow \psi$

- Associativity Matrices $\quad F$
- Braiding Matrices $\quad R \quad$ [represent σ's]

FIBONACCI ANYON $\quad \sim \quad$ FQH $\quad \nu = \frac{12}{5}$

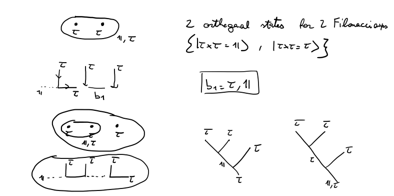
↓

Strongest kind of non - Abelian anyons ⟷ UNIVERSAL QUANTUM COMPUTATION

2 topological sectors

$\{1\!\!1, \tau\}$

$$\begin{cases} 1\!\!1 \times \tau = \tau \\ \tau \times \tau = 1\!\!1 + \tau \end{cases}$$



2 orthogonal states for 2 Fibonacci anyons

$\{|\tau \times \tau = 1\!\!1\rangle, |\tau \times \tau = \tau\rangle\}$

$\boxed{b_1 = \tau, 1\!\!1}$

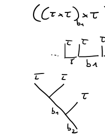
3 Fibonacci : 3 orthogonal states

$((\tau \times \tau)_{b_1} \times \tau)_{b_2}$

$|b_1, b_2\rangle = |1\!\!1, \tau\rangle$

$|\tau, 1\!\!1\rangle$

$|\tau, \tau\rangle$



$$\underbrace{V_{\tau\tau\tau\tau\tau\tau}^{b_{n-1}}}_{n} = \bigoplus_{\{b_i\}} V_{\tau_1\tau_2}^{b_1} \otimes V_{b_1\tau_3}^{b_2} \otimes V_{b_2\tau_4}^{b_3} \otimes \ldots \otimes V_{b_{n-2}\tau_n}^{b_{n-1}}$$

Hilbert space of n τ's

$$\sum_{\{b_i\}} N_{\tau\tau}^{b_1} \cdot N_{b_1\tau}^{b_2} \cdot N_{b_2\tau}^{b_3} \cdots N_{b_{n-2}\tau}^{b_{n-1}} = \dim\left(V_{\tau\tau\tau\ldots}^{b_{n-1}}\right)$$

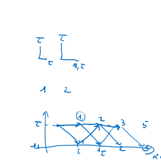
$$(N_\tau)_{b_1}^{b_2} = N_{\tau b_1}^{b_2},$$

$$\dim V = (N_\tau)^{n-1}_{(1\!\!1, b_{n-1})} \simeq \lambda^{n-1} \qquad \boxed{\lambda : \text{quantum dim of } \tau}$$

$$(\tau\tau\tau\tau)_{b_{n-2}} \times \tau = (\tau\tau\tau\tau\tau)_{b_{n-1}}$$

$\dim \sim \lambda^n$

$\lambda = \frac{1 + \sqrt{5}}{2}$



$$N_\tau \begin{pmatrix} 1\!\!1 \\ \tau \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$