MAJORANA KITAEV CHAIN

$H = +\frac{i}{2}\sum_n \mu c_n c_n - \frac{t}{2}\sum_n (c_n^\dagger c_{n+1} + h.c.) - \frac{\Delta}{2}\sum_n (c_n c_{n+1} + h.c.)$

$H \to \tilde H(k) = \tilde d(k)\cdot\tilde\sigma + \tilde g(k)\,\mathbb{1}$

$\gamma_{2n-1} = c_n^\dagger + c_n$ , $\gamma_{2n} = i(c_n^\dagger - c_n)$ ; $c_n^\dagger = \frac{\gamma_{2n-1} - i\gamma_{2n}}{2}$ , $c_n = \frac{\gamma_{2n-1}+i\gamma_{2n}}{2}$

$\{\gamma_m, \gamma_n\} = 2\delta_{mn}$ ; $\gamma_m^2 = \mathbb{1}$ , $\gamma_m^\dagger = \gamma_m$

$c_n^\dagger c_n = \frac{1 + i\gamma_{2n-1}\gamma_{2n}}{2}$ $\Rightarrow$ $(-1)^{c_n^\dagger c_n} = i\gamma_{2n-1}\gamma_{2n}$

1+D

$H = -\frac{\mu}{2}\sum_n i\gamma_{2n-1}\gamma_{2n} - \frac{t}{2}\sum_n i\gamma_{2n}\gamma_{2n+1}$

$\mu=0$ $\Rightarrow$ $[\gamma_{2n}, H] = 0$ $\forall n$ $\Rightarrow$ TOPOLOGICAL PHASE

$\gamma_{2n}$ and $\gamma_{2n}$ for $\mu=0$ are zero-energy localized modes $[\gamma_n, H]\neq 0$ $\Rightarrow$ GS degeneracy

Fermion: $N$ sites on a lattice $\Rightarrow 2^N$ eigenstates of $c_n^\dagger c_n = 0,1$

Majorana: $2N$ modes, $2^{2N} = (2^N)^2$ — Quotient dimension of the $\gamma$'s

GS's: $i\gamma_{2n}\gamma_{2n+1}|GS\rangle = +|GS\rangle$ $\forall n$ — NON LOCAL FERMIONIC PARITY $i\gamma_1\gamma_{2N} = \pm 1$ — 2 GS's

CAN WE UNDERSTAND THE TOPOLOGICAL PHASE FROM A DIFFERENT POINT of VIEW?

$\mathbb{Z}_2$ symmetry $P = (-1)^{\sum_n c_n^\dagger c_n}$ $\Rightarrow$ $P = \prod_n (i\gamma_{2n-1}\gamma_{2n})$

Jordan-Wigner (in terms of Majoranas) $\gamma_n \cdot P(\sigma_n) = P(\sigma_1, \sigma_2)$

$\begin{cases} \gamma_{2n-1} = \sigma_n^x \prod_{q<n}\sigma_q^z \\ \gamma_{2n} = i\sigma_n^y\sigma_n^x \prod_{q<n}\sigma_q^z = \sigma_n^y \prod_{q<n}\sigma_q^z \end{cases}$ — Must be consistent with the JW in the notes

$\sigma_n^y = -i\sigma_n^x\sigma_n^z$

$\gamma_1\ \gamma_2\ \gamma_3\ \gamma_4\ \gamma_5\ \gamma_6$

$1\ 2\ 3\ 4\ 5\ 6$

$\sigma_1^x\ \sigma_1^y\ \sigma_2^x\ \sigma_2^y\ \sigma_3^x\ \sigma_3^y$

$\mu$ term: $i\gamma_{2n}\gamma_{2n+1} = i\,\sigma_n^y\sigma_n^x \prod_{q<n}\sigma_q^z\sigma_q^z\sigma_{n+1}^x \prod_{q<n+1}\sigma_q^z = i\,\sigma_n^y\sigma_{n+1}^x = -\sigma_n^x\sigma_n^z\sigma_{n+1}^x = \sigma_n^x\sigma_{n+1}^x$

$t$ term: $i\gamma_{2n}\gamma_{2n+1} = i\,\sigma_n^y\sigma_n^x \prod_{q<n}\sigma_q^z\sigma_q^z\sigma_{n+1}^x \prod_{q<n+1}\sigma_q^z = (-i)\sigma_n^x\sigma_n^z\sigma_{n+1}^x\sigma_{n+1}^z = \sigma_n^x\sigma_{n+1}^x$

$H = -\frac{\mu}{2}\sum_n \sigma_n^z - \frac{t}{2}\sum_n \sigma_n^x\sigma_{n+1}^x$ — $t=0$

TRANSV. MAGN. FIELD — FERROMAGN. PAIRING

KITAEV MODEL [Kitaev] — NON-LOCAL UNITARY MAP JW — ISING [Lenz-Ising]

What is the topological phase in the "Ising" description?

- It becomes the ordered symmetry broken phase!

1D TOPOLOGICAL ORDER — NON LOCAL MAP — SPONTANEOUS $S\vec{B}$

$\mathbb{Z}_2$ symmetry

$P = (-1)^{\sum_n c_n^\dagger c_n} \longleftrightarrow \prod_n \sigma_n^z = P$

$|GS+\rangle = |++++\dots\rangle$ , $|GS-\rangle = |----\dots\rangle$ — $\sigma_n^x|+\rangle = +|+\rangle$ , $\sigma_n^x|-\rangle = -|-\rangle$

↓ Symmetry broken GSs

$P|GS_\pm\rangle = |GS_\mp\rangle$ : different from before

either $\begin{cases} P|GS,+\rangle = +|GS,+\rangle \\ P|GS,-\rangle = -|GS,-\rangle \end{cases}$ $\Rightarrow$ $|GS, i\gamma_1\gamma_{2N}=+1\rangle = \frac{1}{\sqrt2}[|GS+\rangle + |GS-\rangle]$ , $|GS, i\gamma_1\gamma_{2N}=-1\rangle = \frac{1}{\sqrt2}[|GS+\rangle - |GS-\rangle]$

Ising: the ferromagnetic phase extends up to $|h| > |t|$ ↕ Kitaev: two-fold degeneracy extends up to $|\mu| > |t|$ — $N\to\infty$

$-\Delta\cos 2\varphi$ $\quad \varphi = 0, \pi$

ISING MAJORANA KITAEV CHAIN

- $t=0$ and $t=\Delta$ : they are zero-energy Majorana edge modes $\Rightarrow$ two-fold degeneracy
- $\mu < t$ : the system is in the ferromagnetic phase in the ISING description $\Rightarrow$ two-fold degeneracy [$N\to\infty$]

$\Rightarrow$ We want to check that zero-modes survive for $|\mu/t| < 1$

$H = -\frac{\mu}{2}\sum_n i\gamma_{2n-1}\gamma_{2n} - \frac{t}{2}\sum_{n=1}^{N-1} i\gamma_{2n}\gamma_{2n+1}$

$\mu=0$ $\Rightarrow$ $[\gamma_1, H] = 0$

$t=0$ $\quad [\gamma_1, H] = \left[\gamma_1, -\frac{\mu}{2} i\gamma_1\gamma_2\right] = -i\frac{\mu}{2}\gamma_2$ , $i\frac{\mu}{2}\gamma_1\gamma_2\gamma_1\gamma_2 = -i\frac{\mu}{2}\gamma_2$

$= -i\mu\gamma_2$

$[\gamma_1, H] = \left[\gamma_1, -\frac{\mu}{2}\gamma_1\gamma_2 - \frac{t}{2}\gamma_2\gamma_3\right] = -i\mu\gamma_2 - i t\gamma_3$

$[\gamma_3, H] = \left[\gamma_3, -i\frac{t}{2}\gamma_2\gamma_3 - i\frac{\mu}{2}\gamma_3\gamma_4\right] = it\gamma_2 - i\mu\gamma_4$

$[\gamma_{2n-1}, H] = it\gamma_{2n-2} - i\mu\gamma_{2n}$

$[\gamma_{2n}, H] = i\mu\gamma_{2n-1} - it\gamma_{2n+1}$

We assume $\left|\frac{\mu}{t}\right| < 1$, and we want to build a zero-mode

$\Gamma^{L(1)} = \gamma_1$ $\Rightarrow$ $\left[\Gamma^{L(1)}, H\right] = -i\mu\gamma_2$

$\Gamma^{L(2)} = \gamma_2 + \frac{\mu}{t}\gamma_3$ $\Rightarrow$ $\left[\Gamma^{L(2)}, H\right] = -i\mu\gamma_2 + it\frac{\mu}{t}\gamma_2 - it\frac{\mu}{t}\gamma_4$

$\Gamma^{L(3)} = \gamma_3 + \frac{\mu}{t}\gamma_2 + \frac{\mu^2}{t^2}\gamma_5$ $\Rightarrow$ $\left[\Gamma^{L(3)}, H\right] = -it\frac{\mu}{t^2}\gamma_6$

$\Gamma^{L(n)} = \sum_{q=0}^{n}\left(\frac{\mu}{t}\right)^q \gamma_{2n-2q}$ $\Rightarrow$ only odd Majoranas

$\left[\Gamma^{L(n)}, H\right] = -it\left(\frac{\mu}{t}\right)^n \gamma_{2n}$

$\to 0$ for $n\to\infty$

FINITE SIZE $n=1\dots N$ $\quad \left[\Gamma^{L(n)}, H\right] = -it\left(\frac{\mu}{t}\right)^N \gamma_{2N}$

THERMODYNAMIC LIMIT $\quad \left[\Gamma^{L(n)}, H\right] \xrightarrow{N\to\infty} 0$ for $\left|\frac{\mu}{t}\right| < 1$

At $\mu=t$ [critical point, topological phase transition], this is false

For $\mu<t$ exponential decay with length $\sim \log\frac{t}{\mu}$

$\Gamma^L$ : "almost" commutes with $H$ : is an localized on the edge [it decays in the bulk]

$\Gamma^{R(1)} = \gamma_{2N}$

$\Gamma^{R(n)} = \sum_{q=0}^{n}\left(\frac{\mu}{t}\right)^q \gamma_{2N-2q}$

$\to$ only even!

$\left\{\Gamma^{R(n)}, \Gamma^{L(n)}\right\} = 0$ $\Rightarrow$ 2 fold degeneracy of the spectrum [one for GSs] for $N\to\infty$

(odd, odd)

For finite $N$ we have a small splitting

- Consider $\mu \neq t$ $\Rightarrow$ do perturbation theory from $H(\mu=0) = H_0$ ; $V = i\frac{\mu}{2}\sum_n \gamma_{2n-1}\gamma_{2n}$

At order $N-1$, the states are degenerate.

At order $N$ you get a splitting $\left(\frac{\mu}{t}\right)^N t$

- We consider the 2 GSs at $\mu=0$ $\quad |GS, i\gamma_1\gamma_{2N} = \pm 1\rangle$ — we get an effective Hamiltonian $H_{eff} = i\frac{\mu}{t^{N-1}}\Gamma^L\Gamma^R$
- This is true also for $t \neq \Delta$ $\Rightarrow$ there are 2 localized edge zero-modes for $\mu\neq t$
- This is true if we add disorder

$H = -\frac{i}{2}\sum_n \mu_n \gamma_{2n-1}\gamma_{2n} - \frac{i}{2}\sum_n t_n \gamma_{2n}\gamma_{2n+1}$

$\Gamma^L = \sum_{q=0}^{n}\prod_{k<q}\left(\frac{\mu_k}{t_k}\right)\cdot\gamma_{2q-1}$ — it decays as far as "$\mu_k < t_k$"

- This is true at any interface between a topol. and a trivial phase $t_n \Delta_n$

Trivial phase : Topological phase

$\mu^1 > t^1$ ; $\mu^2 < t^2$ [vacuum]

$c_1^\dagger c_1$

If $t^1 = 0$ and $\mu^2 = 0$ $\quad [\gamma_1, H] = 0$

If $t^1 < \mu^1$ and $\mu^2 < t^2$

Exponentially localized "zero-mode" at each TR / TOPO interface [almost]

$H_0 = H_{t=0} = -\frac{\mu}{2}\sum_n i\gamma_{2n-1}\gamma_{2n}$ $\Rightarrow$ GS is such that $i\gamma_{2n}\gamma_{2n+1} = +1$

$V = H_\mu = -\frac{\mu}{2}\sum_n i\gamma_{2n-1}\gamma_{2n}$ $\qquad i\gamma_{2n-1}\gamma_{2n} | i\gamma_{2n}\gamma_{2n+1} = +1) = (i\gamma_{2n}\gamma_{2n+1} = -1)$

$\Rightarrow \left[i\gamma_{2n-1}\gamma_{2n}\right]\left[\gamma_{2n}\gamma_{2n+1}\right] = -\left[i\gamma_{2n}\gamma_{2n+1}\right]\left[i\gamma_{2n-1}\gamma_{2n}\right]$

2 Excitons with total energy $2t$

$\gamma_1$ $\gamma_2$ $\gamma_3$ $\gamma_4$ $\gamma_5$ $\gamma_6$ — $\Gamma^L$ "order" — $\Gamma^R$ "order"

$P = \frac{\mu}{\Delta_{2n}}(i\gamma_{2n-1}\gamma_{2n})$ : Vodka you go back to a ground state

$\equiv i\gamma_1\gamma_{2N}$

10 TOPOLOGICAL ORDER (PROTECTED by $\mathbb{Z}_2$ symmetry) [TOPOLOGICAL PHASE] | KITAEV CHAIN [Kitaev]

① We have a gap for $\mu \neq t$

$\Delta \to t$ ; $\pm|\mu-t|$ ; $\pm|\mu+t|$ ; $\sim \pm|\mu-t|\cdot k^2$ ; $\sim \pm\Delta k$ ; $\mu=t$

② GS degeneracy for open boundary conditions

$\mathbb{Z}_2$ symmetry $\quad P = (-1)^{\sum_n c_n^\dagger c_n} = \prod_n (i\gamma_{2n-1}\gamma_{2n})$

$P|GS,\pm\rangle = |GS,\pm\rangle$ ; $P|GS-\rangle = -|GS-\rangle$

③ $\forall$ Local operator $V_R$ :

$\langle GS,i|\,V_R\,|GS,j\rangle = \bar V \delta_{ij} + c(R,i,j)$

and that $c$ decays exponentially with the distance from the boundary

$V_R$ is local : $f(c_{R,i}, c_{R,i}^\dagger, \text{with } i=0,\dots,n)$ — $\frac{m}{t}\to 0$ for $t\to 0$

example of $V$ at the edge is $c_1 \sim \frac{\gamma_1 - i\gamma_2}{2}$ and for $\frac{\mu}{t}\ll 1$ $\gamma_1 \sim \Gamma^L$

$\langle GS,+|\,\gamma_1\,|GS,-\rangle \neq 0$ $\Rightarrow$ strong $c$ at the edge

$V$ in the bulk : the overlap with $\Gamma^L$ and $\Gamma^R$ is exponentially small

$i\gamma_1\gamma_{2L}$ is left extended : $i\gamma_1\gamma_{2L} \sim i\Gamma^L\Gamma^R$, $V_R$ in the bulk

$\forall\, V_R$ in the bulk $\quad [i\Gamma^L\Gamma^R, V_R]$ is exponentially small

$\Rightarrow \langle GS,i|\,V_R\,|GS,j\rangle \simeq \bar V \delta_{ij}$

Robustness against local operators in the bulk

Img

$|GS+\rangle = |++++\rangle + |----\rangle$

$|GS-\rangle = |++++\rangle - |----\rangle$

$\sigma_{2L}^x$

$\langle GS,+|\,\sigma_{2L}^x\,|GS,-\rangle \neq 0$ even for $n=37$

③ Local Indistinguishability

$\forall$ observable $O_R$ which is local :

$\langle GS,i|\,O_R\,|GS,j\rangle = \bar O_R\,\delta_{ij} + o(L,i/\xi)$

I An observable is a Hermitian operator that commutes with $P$ $\quad [O_R, P] = 0$ — $C\alpha\beta^\dagger_R$ is not an observable, $C\alpha^\dagger_R C\beta_R$ is an observable

II $o(L) \sim e^{-L/\xi}$

for $L\to\infty$ $\quad \langle GS,i|\,O_R\,|GS,j\rangle = \bar O_R\,\delta_{ij}$

$O_R$ does not distinguish $|GS+\rangle$ and $|GS-\rangle$

To distinguish them I need $i\Gamma^L\Gamma^R$, which is non-local

The GS information is protected:

① against operators that do not change $P$

② against operators far from the edge

PHYSICAL INGREDIENTS for the KITAEV CHAIN

$H = -\mu\sum_n c_n^\dagger c_n - \frac{t}{2}\sum_n\left(c_n^\dagger c_{n+1} + h.c.\right) - \frac{\Delta}{2}\sum_n\left(c_n c_{n+1} + h.c.\right)$

$\Rightarrow$ Spinless fermions

Zero particles with the same spin

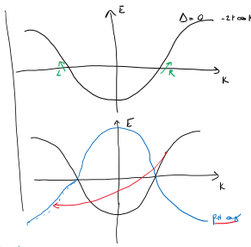
e.g. $c^\dagger_{\uparrow,n} c^\dagger_{\uparrow,n+1}$

$\Delta(k) \xrightarrow{I} -\Delta(k)$ p-wave

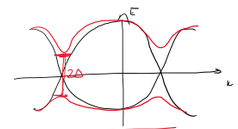
Not a standard s-wave SC $c^\dagger_{\uparrow,n} c^\dagger_{\downarrow,n}$

p wave SC are very rare in nature

$Sr_2RuO_4$ ?




$I(k) \to -E(-k)$ PH




$E = \pm\sqrt{\Delta^2\sin^2 k + (\mu - t\cos k)^2}$

$H = \int dx\ i v_F \psi_L^\dagger \partial_x \psi_L - i v_F \psi_R^\dagger \partial_x \psi_R + \Delta\left(\psi_R^\dagger \psi_L^\dagger + H.c.\right)$

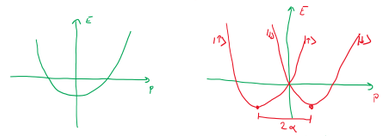
Only 1 pair of $\psi_L, \psi_R$ will interact through $\Delta$.

- In reality we always have $e^-$ with spin 1/2 and cooper pairing

2010 : Lutchyn, Das Sarma / Oreg and coworkers

Ideal case spinless p-wave SC

Physical scenario
- Spin-orbit coupling
- Zeeman term
- s-wave SC



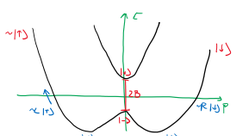

$H = \frac{p^2}{2m}$ for both spins $\longrightarrow$




spin-orbit

$H = \frac{p^2}{2m} + \frac{\alpha p_x \sigma_z}{m} = \frac{(p+k_F\sigma_z)^2}{2m} - \frac{k_F^2}{2m}$ ↓ shift

$\longrightarrow$




$H \sim \frac{p^2}{2m} + \frac{\alpha p_x \sigma_z}{m} + \vec B\vec\sigma$ — $B\alpha\beta^\dagger$

$B^x \sim \langle\psi_\uparrow^\dagger \psi_\downarrow^\dagger\rangle \neq 0$