• NW

$\psi(x,y,z) \approx \boxed{\psi_{1D}(x)} \cdot \boxed{R(r,\theta)}$

$W, U, T, \underline{E} \ll \boxed{\left(\frac{2\pi}{r}\right)^2 \frac{1}{m}}$

• Ultracold atoms

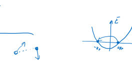
$\boxed{\Omega_{r, y, z} \gg \Gamma}$

$E \ll \Omega_{r}$

• Edges of topol. insulators.

$E \ll \Delta$

1D systems



$\rightarrow \quad \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \quad \leftarrow$ $\boxed{q = 2k_F}$

Π_p $\qquad \frac{1}{\omega - \xi_{p+q} + \xi_p}$

$\int p^2 dp \ldots$

BOSONIZATION : Mapping interacting fermions (bosons) into non-interacting bosons

chain of fermions

$\cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot$ L sites

NON - INT. $\quad H_0 = (c^\dagger_1 \ldots c^\dagger_L) \underbrace{\begin{pmatrix} & \\ & \end{pmatrix}}_{\boxed{L\times L}} \begin{pmatrix} c_1 \\ \vdots \\ c_L \end{pmatrix}$ $\quad \boxed{2L \times 2L \;: \text{for SC}}$

INT. SYSTEM : Fock space $\quad H = H_0 + H_{int} \qquad c^\dagger_{r+1} c_{r+1} c^\dagger_r c_r$

$H \rightarrow 2^L \times 2^L$

1. || NON - INTERACTING 1D Hamiltonians Dirac systems ||
2. $\boxed{\text{LUTTINGER PROBLEM} \rightarrow \text{Exactly solved } \boxed{b, b^\dagger}}$
3. Phenomenological ~~General bosonization~~ : from interacting models on the lattice to bosonic field theory
   ↳ Luttinger liquid
4. Axiomatic approach to bosonization
   → How to apply this machinery
5. JORDAN - WIGNER transf. : from fermions to hard-core bosons (spin 1/2 d.o.f)
6. $\boxed{\text{Sine - Gordon model}}$: topol. phase transition ←
   BKT phase transition

$k \in \left[0, \frac{2\pi}{a}\right] \qquad \Lambda_{max}: \frac{2\pi}{a}$

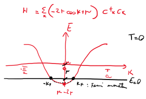
$\overset{a}{\cdot\;\cdot} \; \overset{t}{\cdot \curvearrowleft \cdot} \; \cdot \; \cdot$ $\qquad L$ sites $\qquad \frac{2\pi}{L} = \delta k = \text{"}\Lambda_{min}\text{"}$ $\qquad a = 1$

Spinless fermions

$H = -t \sum_r \left(c^\dagger_{r+1} c_r + H.c.\right) + \mu \sum_r c^\dagger_r c_r$ $\qquad$ PBC

$c_r = \sum_{k \in \left[-\frac{\pi}{a}, \frac{\pi}{a}\right)} \frac{1}{\sqrt{L}} e^{ikr} c_k \quad \rightarrow \quad \int dk\, e^{ikr} c_k$

$H = \sum_k \left(-2t \cos k + \mu\right) c^\dagger_k c_k$



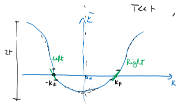
$-2t \cos k_F + \mu = 0$

$\boxed{k_F = \arccos \frac{\mu}{2t}}$

$\boxed{k_F = \pi \cdot \frac{N}{L}} = \pi \rho_0 \qquad \rho_0 = \frac{N}{L}$

$\delta k = \frac{2\pi}{L} \qquad N = \frac{2k_F}{2\pi/L}$

$T \ll t \quad , U \ll t$



$v = \frac{\partial E}{\partial k}$

$v_F = 2t a \sin k_F a$

$H = H_R + H_L = \sum_{k \approx |k_F|} + v_F (k - k_F) c^{R\dagger}_k c^R_k - v_F (k + k_F) c^{L\dagger}_k c^L_k$
($k > 0$, $k \approx +k_F$; $k < 0$, $k \approx -k_F$)

$\epsilon_R \simeq v_F (k - k_F)$

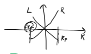
$\epsilon_L \cong -v_F (k + k_F)$

$H_R = \int dx \; -v_F \tilde{\psi}^\dagger_R \left(i\partial_x + k_F\right) \tilde{\psi}_R \qquad \tilde{\psi}_R(x) = \int \frac{dk}{\sqrt{2\pi}} e^{ikx} c^R_k$

$H_L = \int dx \; +v_F \tilde{\psi}^\dagger_L \left(i\partial_x - k_F\right) \tilde{\psi}_L$

$\boxed{\tilde{\psi}_R(x)} = \underbrace{e^{ik_F x}}_{\text{fast}} \underbrace{\psi_R(x)}_{\text{slow}}$ $\quad$ "$k - k_F$" $\qquad \frac{N}{L} \qquad \boxed{a \partial_x \psi_R \ll 1}$

$\tilde{\psi}_R(x) = e^{-ik_F x} \psi_L(x)$



$H = v_F \int dx \left[\psi^\dagger_L i\partial_x \psi_L - \psi^\dagger_R i\partial_x \psi_R\right] \qquad \psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$

$= v_F \int dx \; \psi^\dagger \sigma_z i\partial_x \psi$

$\mathcal{L} = \int dx \; \psi^\dagger \left(i\partial_t - v_F \sigma_z i\partial_x\right) \psi$

$\left(i\partial_t - iv_F \partial_x\right) \psi_L = 0 \qquad \psi_L(x + v_F t)$

$\left(i\partial_t + iv_F \partial_x\right) \psi_R = 0 \qquad \psi_R(x - v_F t)$

$H = \underbrace{-\sum_r \left(t c^\dagger_{r+1} c_r + H.c.\right) + \mu \sum_r c^\dagger_r c_r}_{H_0} + \overbrace{\boxed{\underline{U} \sum_r c^\dagger_{r+1} c_{r+1} c^\dagger_r c_r}}^{H_{int}}$
$\left[+ \sum_{r, r'} \underline{V(r')} c^\dagger_{r+r'} c_{r+r'} c^\dagger_r c_r\right]$

$H_{int} = U \sum_{k_1, k_2, k_3, k_4} \sum_r e^{-ik_1(r+1)} e^{ik_2(r+1)} e^{-ik_3 r} e^{ik_4 r} c^\dagger_{k_1} c_{k_2} c^\dagger_{k_3} c_{k_4} \frac{1}{L}$

$e^{-i(k_1 - k_2)} \boxed{\sum_r e^{ir(-k_1 + k_2 - k_3 + k_4)}} \ldots$

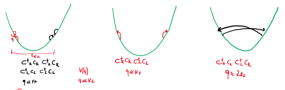
$\delta(-k_1 + k_2 - k_3 + k_4)$

$k_1 - k_2 = q = k_4 - k_3 \qquad k = k_2 \qquad k' = k_4$

$= \frac{1}{L} \sum_{q, k, k'} \underbrace{U e^{-iq}}_{V(q)} \boxed{c^\dagger_{k+q} c_k} \boxed{c^\dagger_{k'-q} c_{k'}}$ $\quad L, R$ $\qquad \boxed{U \ll 2t}$

Real space : $\quad c_r = c_{R,r} + c_{L,r} = \frac{1}{\sqrt{L}} \left[\sum_{k > -k_F} c_{R,k} e^{-i(k - k_F)r} + \sum_{k < k_F} e^{i(k + k_F)r} c_{L,k}\right]$

$c^\dagger_{R/L} c_{R/L} c^\dagger_{R/L} c_{R/L}$



$\boxed{\sum_r e^{(k_1 - k_2 + k_3 - k_4)r}} \underbrace{c^\dagger_R c_R}_{\sim 0} \underbrace{c^\dagger_R c_L}_{\sim 2k_F}$ forbidden by mom. cons.

$c^\dagger_R c_L c^\dagger_R c_L$ $\quad$ " $\quad \boxed{4k_F = \pm 2\pi}$ $\quad \boxed{k_F = \frac{\pi}{2}} \quad \boxed{\rho = \frac{1}{2}}$
$2k_F \quad 2k_F$ $\qquad$ UMKLAPP

DENSITY OPERATORS

$$\boxed{\begin{aligned} \rho_R(q) &= \sum_{k > -k_F} c^\dagger_{k,R} c_{k+q,R} \\ \rho_L(q) &= \sum_{k < k_F} c^\dagger_{k,L} c_{k+q,L} \end{aligned}}$$

$H_{int} = \frac{1}{L} \sum_q \underbrace{\left[V(q \approx 0) - V(q \approx 2k_F)\right]}_{W_{RL}(q)} \rho_R(q) \rho_L(-q) + \frac{1}{L} \sum_q V(q \approx 0) \left[\underbrace{\rho_R(q) \rho_R(-q)}_{V_{RR}(q)} + \underbrace{\rho_L(q) \rho_L(-q)}_{V_{LL}(q)}\right]$

Luttinger - Tomonaga [Haldane]

- $\underline{\underline{\rho}} \rightarrow b, b^\dagger$: bosonic
- H_{int} is quadratic in $b, b^\dagger$
- → $\boxed{H_0 \text{ is quadratic in } b, b^\dagger \; [\text{ if we begin from Dirac }]}$
- H is quadratic in b and $b^\dagger$ ⇒ exact solution [Bogoliubov]

$\boxed{\left[\rho_{R/L}(q), \rho_{R/L}(q')\right] = ?}$

$\left[\rho_R(q), \rho_R(q')\right] = \delta_{q,-q'}$

$\Rightarrow \left[\rho_R(q), \rho_R(q)\right] = \left[\sum_{k > -k_F} c^\dagger_k c_{k+q}, \sum_{k' > -k_F} c^\dagger_{k'} c_{k'-q}\right] =$

$= \sum_{k, k' > -k_F} \left(c^\dagger_k c_{k+q} c^\dagger_{k'} c_{k'-q} - c^\dagger_{k'} c_{k'-q} c^\dagger_k c_{k+q}\right) =$

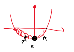
$= \sum_{k, k' > -k_F} \left[\cancel{-c^\dagger_k c^\dagger_{k'} c_{k+q} c_{k'-q}} + \delta(k+q-k') c^\dagger_k c_{k'-q} + c^\dagger_{k'} c^\dagger_k c_{k'-q} c_{k+q} - c^\dagger_{k'} c_{k+q} \delta(k'-q-k)\right] =$
$k = k'-q \qquad \pm \cancel{c^\dagger_k c^\dagger_{k'} c_{k+q} c_{k'-q}} \qquad k = k'-q$

$= \sum_{\substack{k' \\ k = k'-q > -k_F}} \underline{c^\dagger_{k'-q} c_{k'-q}} - \sum_{\substack{k' \\ k' > \max[-k_F, -k_F+q]}} c^\dagger_{k'} c_{k'} = \sum_{\max - q < k' < \max} \langle c^\dagger_{k'} c_{k'} \rangle_{T=0} = \sum_{\ldots} 1 = \frac{q}{2\pi/L} = \frac{qL}{2\pi}$

$q < 0 \quad k' > -k_F$
$q > 0 \quad k' > -k_F + q$
$k' > \max[-k_F, -k_F + q]$ $\qquad \boxed{q \ll k_F}$ $\qquad T \ll 2t \quad \boxed{T = 0}$

H_0

$\max \ldots \quad c^\dagger_{k-q} c_{k-q} \qquad c^\dagger_{k'} c_{k'}$



$$\boxed{\begin{aligned} \left[\rho_R(q), \rho_R(-q')\right] &= \delta_{qq'} \frac{qL}{2\pi} \\ \left[\rho_L(q), \rho_L(-q')\right] &= -\delta_{qq'} \frac{qL}{2\pi} \\ \left[\rho_L, \rho_R\right] &= 0 \qquad \left\{c^\dagger_R, c_L\right\} = 0 \end{aligned}}$$

$$\boxed{\begin{aligned} b_p &= \sqrt{\frac{2\pi}{|p|L}} \left[\Theta(p) \rho_R(p) + \Theta(-p) \rho_L(p)\right] \\ b^\dagger_p &= \sqrt{\frac{2\pi}{|p|L}} \left[\Theta(p) \rho_R(-p) + \Theta(-p) \rho_L(-p)\right] \end{aligned}}$$

$\left[b_p, b^\dagger_{p'}\right] = \delta_{pp'}$

$p > 0 \qquad b_p = \sqrt{\frac{2\pi}{pL}} \rho_R(p) \qquad b^\dagger_p = \sqrt{\frac{2\pi}{pL}} \rho_R(-p)$

$H_{int} = \sum_q \left(\rho_R(q)\; \rho_L(q)\right) \begin{pmatrix} V_{RR}(q) & V_{LR}(q) \\ V_{RL}(q) & V_{LL}(q) \end{pmatrix} \begin{pmatrix} \rho_R(+q) \\ \rho_L(+q) \end{pmatrix} = \sum_q \frac{|q|}{2\pi} \left(b^\dagger_q \; b_{-q}\right) \begin{pmatrix} V & V \\ V & V \end{pmatrix} \begin{pmatrix} b_q \\ b_{-q} \end{pmatrix}$

$\frac{V(q)}{L} \rho_R(-q) \rho_R(q) = \frac{V(q)}{\cancel{L}} \frac{|q|\cancel{L}}{2\pi} b^\dagger_q b_q$

$H_0 = H_R + H_L \qquad \left[\rho_R, H_L\right] = 0$
$\left[\rho_R, H_R\right]$

$\left[b_{p_0}, H_R\right]$ with $p_0 > 0$ $\qquad \boxed{\tilde{\psi}}$

$\left[\sqrt{\frac{2\pi}{Lp_0}} \rho_R(p_0), \sum_k v_F (k - k_F) c^\dagger_{Rk} c_{Rk}\right] = \sqrt{\frac{2\pi}{Lp_0}} \left[\sum_{\substack{k' > 0 \\ k' > -k_F}} c^\dagger_{k'R} c_{k'+p_0,R}, \sum_{\substack{k > 0 \\ k > -k_F}} v_F (k - k_F) c^\dagger_{Rk} c_{Rk}\right]$

$= \sqrt{\frac{2\pi}{Lp_0}} \sum_{k', k > 0} \left(c^\dagger_{k'R} c_{k'+p_0,R} c^\dagger_{kR} c_{kR} - c^\dagger_{kR} c_{kR} c^\dagger_{k'R} c_{k'+p_0,R}\right) v_F (k - k_F)$

$= \sqrt{\frac{2\pi}{Lp_0}} \left[\sum_{k', k > 0} \cancel{-c^\dagger_{k'R} c^\dagger_{kR}} c_{k'+p_0,R} c_{kR} + \boxed{\delta(k'+p_0-k) c^\dagger_{k'R} c_{kR}} + c^\dagger_{kR} c^\dagger_{k'R} \cancel{c_{kR} c_{k'+p_0,R}}\right.$
$\left. - \delta(k - k') c^\dagger_{kR} c_{k'+p_0,R}\right] v_F (k - k_F) =$

$k = k' + p_0$

$= \sqrt{\frac{2\pi}{Lp_0}} \sum_{\substack{k', k > 0 \\ k' > 0}} c^\dagger_{k',R} c_{k'+p_0,R} v_F (\cancel{k'} + p_0 - \cancel{k_F}) - \sum_{\substack{k', k > 0 \\ k = k'}} \cancel{c^\dagger_{k',R} c_{k'+p_0,R} \left(v_F (k' - k_F)\right)}$

$= \left[\sqrt{\frac{2\pi}{Lp_0}} \sum_{k'} c^\dagger_{k'R} c_{k'+p_0,R}\right] v_F p_0 = v_F p_0 b_{p_0}$

$\rightarrow \left[b_{p_0}, H_0\right] = v_F p_0 b_{p_0} \qquad p_0 > 0$ ←

$\boxed{H_0 = \sum_p v_F |p| b^\dagger_p b_p}$

$\left[b_0, b^\dagger_0 b_0\right] = b$

$H = \underbrace{\sum_p v_F |p| b^\dagger_p b_p}_{H_0} + \overbrace{\sum_{p > 0} |p| \left(b^\dagger_p \; b^\dagger_{-p}\right) \begin{pmatrix} V(p) & V(p) \\ V(p) & V(p) \end{pmatrix} \begin{pmatrix} b_p \\ b_{-p} \end{pmatrix}}^{H_{int}}$

$= \sum_p \left(b^\dagger_p \; b^\dagger_{-p}\right) \begin{pmatrix} |p|V(p) + v_F |p| & |p|V(p) \\ |p| V_p & |p|V(p) + v_F |p| \end{pmatrix} \begin{pmatrix} b_p \\ b_{-p} \end{pmatrix}$

$\tilde{b}_p = \alpha_p b_p + \beta_p b_{-p} \qquad \left[\tilde{b}_p, \tilde{b}^\dagger_{p'}\right] = \delta_{pp'}$

$= \sum_p \varepsilon_p \tilde{b}^\dagger_p \tilde{b}_p \qquad \varepsilon_p = \sqrt{p^2 (V_p + v_F)^2 - p^2 V_p^2} = |p| \sqrt{v_F^2 + 2V_p v_F} =$

$= \underline{\underline{|p|}} \boxed{v_F \sqrt{1 + \frac{2V_p}{v_F}}}$

$|p| \begin{pmatrix} v_F + V(q \approx 0) & V(q \approx 0) - V(q \approx 2k_F) \\ V(q \approx 0) - V(q \approx 2k_F) & v_F + V(q \approx 0) \end{pmatrix} \qquad \varepsilon_p = |p| \sqrt{(v_F + V_0)^2 - (V_0 - V_{2k_F})^2} =$

$= |p| \sqrt{v_F^2 + 2v_F V_0 - V_{2k_F}^2 + 2V_0 V_{2k_F}}$

φ, θ

$\varphi(x) = \int dk\; e^{ikx} b_k + e^{-ikx} b^\dagger_k$