RG

- Basic ideas : relevant vs irrelevant
- RG in real space → 1D classical Ising → T

$H = \frac{J}{kT} \sum_{ij} \sigma_i \sigma_j \to \sum_i \sigma_i \sigma_{i+1} + h \sum_i \sigma_i$

Transfer Matrix

$Z = \mathrm{Tr}_\sigma T^N = \sum_{\{\sigma\}} \langle \sigma_1 | \sigma_2 \rangle \langle \sigma_2 | T | \sigma_3 \rangle \cdots \langle \sigma_N | \sigma_1 \rangle$ $\qquad T_{\sigma\sigma'} = \langle \sigma | \ (\ ) \ | \sigma' \rangle$

$\langle \psi(\cdot) \ \psi(\cdot+n) \rangle_c = \langle \psi(\cdot) \psi(\cdot+n) e^{-\int H(\sigma)} \rangle =$

$= \sum_{\sigma\sigma} \langle \sigma_1 | \sigma_2 \rangle \langle \sigma_2 | T \ T \ T \cdots \langle \psi \ T \rangle T_{n+1} T_{n+2} \cdots T_{n+m} \langle \psi_{n+1} \rangle T_{n+m} \cdots T_N \langle \sigma_N | \sigma_1 \rangle$

$\frac{1}{Z} \langle \sigma_1 | \sigma_2 \ T^{N-1} \sigma_1 | \sigma_1 \rangle$

$\left[\langle \psi(\cdot) \rangle e^{-\int \delta \lambda_1 \psi + \int \delta \lambda_2 \psi \psi + \cdots} \right]$

$= \frac{1}{Z} \sum_{\sigma\sigma} \langle \sigma_1 | \sigma_2 \ T^{N-1} \langle \psi_{n+1} \rangle T^{N-n-1} \sigma_1 | \sigma_1 \rangle$

$\frac{1}{Z} \langle \sigma_1 | \sigma_2 \ T^{N-1} \sigma_1 | \sigma_1 \rangle$ $\qquad = \langle \psi(\cdot) \rangle \langle \psi(\cdot) \rangle$

$\boxed{T} \to \{\lambda_i\}$ $\qquad \hat{\sigma}_i T = \lambda_i T$ $\qquad \lambda_1 \gg \lambda_2 \gg \cdots$

$= \sum_{\sigma\sigma} \langle \sigma_1 | \sigma_2 \ \lambda_1^{N-1} \langle \psi \rangle \langle \psi \rangle \sigma_1 | \sigma_1 \rangle$

$\boxed{\langle \sigma_1 | \sigma_1 \rangle \ \lambda_1^{N-1} \ \langle \psi \rangle \langle \psi \rangle \ \sigma_1 | \sigma_1 \rangle}$

$\langle \psi \psi \rangle_c \sim \boxed{\frac{\lambda_2^N}{\lambda_1^N}} = \left(\frac{\lambda_2}{\lambda_1} \right)^N = e^{-\frac{N}{\xi}}$ $\qquad \boxed{\xi} = \frac{-\mathrm{const}}{\log \frac{\lambda_2}{\lambda_1}}$

$T = \begin{pmatrix} e^{J+h} & e^{-J+h} \\ e^{-J-h} & e^{J-h} \end{pmatrix}$

λ_1, λ_2

ϕ_1, ϕ_2

$\langle \sigma_1 | \sigma_2 \ \lambda_1^{N-1} \ \sigma_2 | \sigma_1 \rangle$ $\qquad \langle \psi \psi \rangle$ $\qquad$ W/ TOPOLOGY T

$\boxed{\lambda_1^{N-1} \lambda_1^{N-1} \lambda_2^{N}}$

$\lambda_1 \gg \lambda_2$

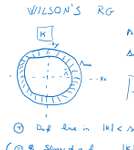
$\left(\frac{\lambda_2}{\lambda_1} \right)^N$

$Z = \sum_{\sigma\sigma} e^{N\log\lambda_1} e^{N\log\frac{\lambda_2}{\lambda_1}} \left[\langle \sigma_1 | \phi_1 \rangle \lambda_1^{N-1} \langle \phi_1 | \sigma_1 \rangle + \langle \sigma_1 | \phi_2 \rangle \lambda_2^{N-1} \langle \phi_2 | \sigma_1 \rangle \right]$

$\boxed{|\langle \phi_1 | \sigma_1 \rangle|^2}$ $\qquad \boxed{|\langle \phi_2 | \sigma_1 \rangle|^2}$

$\langle \psi \psi \rangle_c = \frac{\sum_{\sigma\sigma} e^{N\log\lambda_1} \left[\langle \sigma_1 | \phi_1 \rangle \lambda_1^{N-1} \langle \phi_1 | \psi \rangle \langle \phi_1 | \psi \rangle \phi_1 \rangle \lambda_1^{N-1} \langle \phi_1 | \sigma_1 \rangle + e^{N\log\frac{\lambda_2}{\lambda_1}} \langle \phi_1 | \sigma_1 \rangle \right]}{Z} =$

WILSON'S RG



Λ_{max} : UV cutoff in k

$\Delta x = \frac{2\pi}{\Lambda}$

$\boxed{\frac{\tilde{\Lambda}}{\Lambda_{max}}} = 1 - d\ell = \frac{1}{b}$ $\qquad b = 1 + d\ell$

$a \to ab$

① Drop fast in $|k| < \Lambda_{max}$

CG $\begin{cases} ② \text{ Slow d.o.f.} \quad |k| < \tilde{\Lambda} \\ \phi_s \text{ fast d.o.f.} \quad \tilde{\Lambda} < |k| < \Lambda_{max} \end{cases}$ ← CG: we integrate the out

③ Action for $|k| < \tilde{\Lambda}$ only

④ Rescaling $\boxed{\tilde{\Lambda}_{max} = \Lambda_{max} = b\tilde{\Lambda}}$ $\qquad x' = \frac{x}{b} = x(1 - d\ell)$

$Z = \int D\phi_s D\phi_f \ e^{-S[\phi_s, \phi_f]}$

gradients $\qquad$ no grads.

$S[\phi_s, \phi_f] = \underbrace{S_s[\phi_s] + S_s[\phi_f]}_{\text{gradients}} + S_I[\phi_s, \phi_f]$

$Z = \int D\phi_s D\phi_f \ e^{-S_s(\phi_f) - S_s(\phi_s) - S_I(\phi_f, \phi_s)} =$

$= \int D\phi_s \ e^{-S_s(\phi_s)} \int D\phi_f \ e^{-S_s(\phi_f) - S_I(\phi_f, \phi_s)} =$

$= \int D\phi_s \ e^{-S_s(\phi_s)} \underbrace{\left\langle e^{-S_I(\phi_f, \phi_s)} \right\rangle_f}_{\Xi}$

$e^{-S_I(\phi_s)}$

$\langle \phi_s \ll 1$

$S_s'(\phi_s) = S_s(\phi_s) - \log \left[\left\langle e^{-S_I(\phi_f, \phi_s)} \right\rangle_f \right] =$

$S_I \ll S_s \qquad S_s \ll 1$

$\simeq S_s(\phi_s) - \log \left[\left\langle 1 - S_I + \frac{S_I^2}{2!} + \cdots \right\rangle_f \right] =$

$= S_s(\phi_s) + \langle S_I \rangle_f - \frac{\langle S_I^2 \rangle_f}{2} + \frac{\langle S_I \rangle_f^2}{2} + \cdots$

$S_s'[\phi_s] \approx S_s(\phi_s) + \langle S_I \rangle_f + \frac{1}{2} \left[\langle S_I^2 \rangle_f - \langle S_I \rangle_f^2 \right] + \cdots$

RESCALING $\qquad S = \int \mathcal{L}[\phi] d^D x$

$\frac{\tilde{\Lambda}}{\Lambda_{max}} = 1 - d\ell$

$K \to K' = (1 + d\ell) K$

$x \to x' = (1 - d\ell) x$

$\phi \to \phi' = (1 - d\ell) \phi \to \boxed{\text{Gauss inv}}$

$\phi \to \phi' = \frac{\phi}{b} = (1 + d\ell) \phi$ $\qquad \boxed{Z = 1}$

$\varphi(\cdot) = \varphi_s(\cdot) + \varphi_f(\cdot) \to \varphi_s(\cdot) \to \varphi_s(\cdot)$

$\to \boxed{\varphi'(\cdot) = f[\varphi_s(\cdot)]}$

$S(\varphi(\cdot)) \leftrightarrow S_s(\varphi'(\cdot))$

- $\boxed{\text{Any Massless Gaussian theory is scale inv.}}$ $\qquad \xi \to \infty$

$\boxed{S = 0}$

Free massless theory

$\int d\ell \int d^D x \ \mathcal{L}(\cdot) = \int d\ell' \int d^D x' \ \mathcal{L}'(\cdot)$

$\int d\ell \int d^D x \ \mathcal{L}(\cdot) = \int d\ell' \int d^D x' \ \mathcal{L}'(\cdot)$

$\boxed{S =} \int d^D x \ \mathcal{L}(\cdot) = \int d^D x' \ \mathcal{L}'(\cdot)$

Free Gaussian field

$\boxed{\int d^D x \ (\vec{\nabla} \varphi(x))^2 = \int d^D x' \ (\vec{\nabla}' \varphi'(x'))^2}$

$Kx = Kx'$

$x \to x' = (1 - d\ell) x$

$d^D x' = (1 + D d\ell) d^D x$

$\vec{\nabla}' = \frac{\vec{\nabla}}{(1 + d\ell)} = \vec{\nabla}(1 - d\ell)$

$\int (1 + D d\ell) d^D x \left((1 - d\ell) \vec{\nabla} \varphi(x) \right)^2 =$

$\int d^D x \ \underbrace{(1 + D d\ell)(1 - d\ell)^2}_{(1 + (D-2) d\ell)} (\vec{\nabla} \varphi(x))^2 = \int d^D x' \ (\vec{\nabla}' \varphi'(x'))^2$

$(\vec{\nabla}' \varphi'(x'))^2 = (1 + (D-2) d\ell) (\vec{\nabla} \varphi(x))^2$ $\qquad \boxed{D = 2}$

$\boxed{\varphi'(x') = (1 + d\ell)^{\frac{D-2}{2}} \varphi(x)} = (1 + \frac{D-2}{2} d\ell) \varphi(x)$

1+1D Quantum system $\qquad \boxed{\varphi'(x') = \varphi(x)}$

$\boxed{\varphi'(x') = (1 + d\ell)^{\frac{D_\varphi}{2}} \varphi(x) = b^{D_\varphi} \varphi(x)}$ $\qquad \left[\varphi'(x') \right]^{\sim} = (1 + d\ell)^{-D_\varphi} \tilde{\varphi}(x)$

$S = \int d^D x \ \mathcal{L}(x) = \int d^D x \ (\nabla \varphi)^2$

$\boxed{O_\delta'(x') = b^{D_\delta} O_\delta(x)}$

$\langle O_\delta(x) \rangle = 0$

$\langle O_\delta'(x) \ O_\delta(y) \rangle_{0,c} = \langle O_\delta'(x') \ O_\delta(y') \rangle_{0,c} = b^{-2D_\delta} \langle O_\delta'(x') \ O_\delta'(y') \rangle_{0,c}$

$(x - y)^{-2} = b^{-2} (x' - y')^{-2}$ $\qquad = b^{-2D_\delta} (x' - y')^{-2}$

$\boxed{x = bx' \quad y = by'}$

$\boxed{\lambda = 2D_\delta}$

$\varphi^{\sim} \to D_\varphi$

$D_\varphi = 0$, $D_\delta = 0$ $\qquad \langle O_\delta'(x) \ O_\delta(y) \rangle < \boxed{\frac{1}{(x - y)^{2D_\delta}}} \frac{1}{(x - y)^{2D_\delta}}$ $\qquad O_\delta = \cos\{\delta \varphi(x)\}$ $\qquad \frac{D_\delta}{(x, y)}$

$S = \frac{1}{2\pi} \int d^2 x \left[\frac{\partial}{\partial x} (\partial_x \varphi)^2 + \frac{\partial}{\partial x} (\partial_x \varphi)^2 + m^2 \varphi^2 \right] =$

$= \frac{1}{2\pi} \int d\omega \, dk \ \varphi(-\omega, -k) \ \sigma^2(k, \omega) \ \varphi(\omega, k) =$

$= \frac{1}{2\pi} \int d\omega \, dk \ \varphi(-\omega, -k) \left(\frac{\omega^2}{v^2} + \frac{k^2}{v^2} + m^2 \right) \varphi(\omega, k)$

$\varphi(x) = \varphi_s(x) + \varphi_f(x)$ $\qquad q = \left(\frac{\omega}{v}, k \right)$

$|k| < \tilde{\Lambda} \quad \tilde{\Lambda} < |k| < \Lambda_{max}$ $\qquad q = |q| = \sqrt{\frac{\omega^2}{v^2} + k^2}$

$|k| < \tilde{\Lambda} \quad \tilde{\Lambda} < |k| < \Lambda_{max}$

$\int d\omega \, dk \to \int_{\tilde{\Lambda}}^{\Lambda_{max}} dq \ \varphi(q)$

$S = \int d\omega \, dk \int_0^{\tilde{\Lambda}} q \, dq \ \varphi(-q) \left(\frac{\omega^2}{v^2} + k^2 + m^2 \right) \varphi(q) + \to S_s$

$+ \int d\omega \int_{\tilde{\Lambda}}^{\Lambda_{max}} q \, dq \ \varphi(-q) \left(\frac{\omega^2}{v^2} + k^2 + m^2 \right) \varphi(q) \to S_f$

$Z = \int D\varphi_s D\varphi_f \ e^{-S_f - S_s} = \int D\varphi_s \int D\varphi_f \ e^{-S_f} \ e^{-S_s}$ $\qquad \boxed{S_I = 0}$

$S_s(\varphi_s) = \int d\omega \, dk \left[\frac{\partial}{\partial x} (\partial_x \varphi_s)^2 + \frac{\partial}{\partial x} (\partial_x \varphi_s)^2 + m^2 \varphi_s^2 \right]$

$= \int (1 + 2 d\ell) \, d\omega \, dk' \left[\frac{\partial}{\partial x} \left(\frac{\partial_{x'}}{1 + d\ell} \right)^2 \varphi_s(x')^2 + \frac{\partial}{\partial x} \left(\frac{\partial_{x'}}{1 + d\ell} \varphi_s(x') \right)^2 + m^2 \varphi_s^2(x') \right]$

$\boxed{D_\varphi = 0}$ $\quad \boxed{D = 2} \Rightarrow \varphi_s'(x') = \varphi_s(x)$

$= \int d\ell' \, dk' \ \cancel{(1 + 2 d\ell)} \left[\frac{\partial}{\partial x} (\partial_{x'} \varphi_s'(x'))^2 + \frac{\partial}{\partial x} (\partial_{x'} \varphi_s'(x'))^2 \right] +$

$+ \int d\ell' \, dk' \ \underbrace{(1 + 2 d\ell)}_{m'^2 = (1 + 2 d\ell) m^2} m^2 \ \varphi_s'^2(x')$

$\boxed{\frac{d m'^2}{d\ell} = 2 m^2}$ $\qquad \frac{d m'}{d\ell} = m$

$m'^2(\ell) = m_0^2 e^{2\ell}$ $\qquad m'(\ell) = m e^{\ell}$

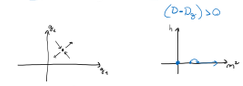
$\boxed{\text{RELEVANT}}$

$\int d^D x \ m^2 \varphi^2 \to \int d^D x' (1 + D d\ell) \ m^2 \ (1 - \underbrace{D d\ell}_{2 D = 2}) \varphi^2$

$(1 + D d\ell)(1 - D d\ell + 2 d\ell) m^2 = \underbrace{(1 + 2 d\ell) m^2}_{m'^2}$

$\boxed{m \to \tilde{m}}$

$\int d\omega \, dk \ \underbrace{m^2 \varphi^2}_{(D - D_\varphi) > 0} \to \int d\omega' \, dk' \ m'^2 (1 + 2 d\ell) \varphi^2$



$\boxed{\text{RESCALING}}$ [reflecting CG]

$\to S_s[\varphi_s]$

$\int d^D x \ (\nabla \varphi)^2 + \boxed{h \varphi + g \varphi^4} + \boxed{m_\delta O_\delta(x)}$

$\varphi(x) = \underbrace{(1 + D d\ell)^{\frac{D_\varphi}{2}}}_{(1 + D d\ell)^{\frac{D-2}{2}}} \varphi(x) = (1 + d\ell)^{\frac{D-2}{2}} \varphi(x)$

$h: \int d^D x \ h \varphi \to \int d^D x' \ (1 + D d\ell) \ h \ (1 - D_\varphi d\ell) \varphi'(x') =$

$= \int d\omega \, dk' \left(1 + (D - D_\varphi) d\ell \right) h \varphi \quad = \int d^D x \underbrace{\left(1 + (D - D_\varphi) d\ell \right) h}_{h'} \varphi'(x')$

$h' = \left(1 + \underbrace{(D - D_\varphi)}_{2 - D_\varphi} d\ell \right) h$

$2 - D_\varphi = h \frac{(2 - D_\varphi)}{2}$ $\qquad h'(\ell) = e^{\frac{(2 - D_\varphi)}{2} \ell} h(\ell)$ relevant

$g: \int d^D x \ g \varphi^4 = \int d^D x' \ (1 + D d\ell) \ g \ (1 - 4 D_\varphi) \varphi^4 =$

$= \int d^D x' \underbrace{g \left(1 + (D - 4 D_\varphi) d\ell \right)}_{g'} \varphi^4$

$D - 4 D_\varphi = D - 2(D - 2) = -D + 4$ $\qquad \begin{cases} D = 4 & : \varphi^4 \text{ is marginal} \\ D < 4 & : D \text{ dim} > 2 \Rightarrow \varphi^4 \text{ is relevant} \\ D > 4 & : D \text{ dim} < 2 \Rightarrow \varphi^4 \text{ is irrelevant} \end{cases}$

$S = \int d^D x \ (\nabla \varphi)^2 + g \ O_\delta(x)$

$O_\delta(x) \to O_\delta(x') = b^{D_\delta} O_\delta(x)$

$\int d^D x' \ (1 + D d\ell) \ g \ (1 - D_\delta d\ell) \ O_\delta'(x') =$

$\int d^D x' \ (1 + (D - D_\delta) d\ell) \ O_\delta'(x')$

$D - D_\delta > 0$: O_δ is relevant $g' > g$

$D - D_\delta < 0$: O_δ is irrelevant $g' < g$

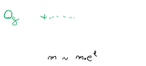
$D - D_\delta = 0$: O_δ is marginal

D_δ? $\qquad \boxed{\left\langle O_\delta^*(x) \ O_\delta(y) \right\rangle_0 = \frac{1}{|x - y|^{2D_\delta}}}$ $\qquad \xi \to \infty$ $\qquad \frac{1}{\text{dist}}$

$S = \int d^D x \ \sum_\delta g_\delta O_\delta(x)$

$\frac{d g_\delta}{d\ell} = \frac{g_\delta}{b^{(D - D_\delta)}} \quad e^{\frac{(D - D_\delta)}{b} \ell}$

$\frac{d g_\delta}{d\ell} = \underbrace{(D - D_\delta)}_{\ell} g_\delta$



$S = \int d^D x \, dt \quad (\nabla \varphi_k)^2 + (\nabla \varphi_k)^2 + \frac{g^2}{\varphi^2} \varphi^2$

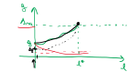
$\varphi^2 K \varphi$

$S = S_0 + \langle S_I \rangle + \frac{1}{2} [\cdot$

$\left(\frac{\varphi_k}{\varphi_k} \right) \left(\frac{\varphi}{\varphi_k} \right)$

Λ_{max} $\qquad g \sim \Lambda_{max}$

$S = \int \boxed{\varphi (\nabla \varphi)^2} + g \ O_\delta + \cdots$



$m \sim m e^{\ell}$

$\Delta_g \sim g_{phys} e^{\ell}$

$\boxed{\Delta_{g_{phys}} = \Delta_{max} e^{-\ell^*}}$

$\to \boxed{\frac{d g_\delta}{d\ell}} = (D - D_\delta) g_\delta$ $\qquad g_\delta(\ell) = e^{(D - D_\delta) \ell} g_\delta$

$\Delta_{max} = g_\delta e^{(D - D_\delta) \ell^*}$

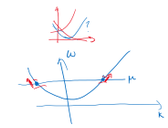
$b = e^{\ell}$

$\boxed{\frac{\Delta_{max}}{\tilde{\Lambda}} = b = 1 + d\ell}$

$\boxed{\left(\frac{\Delta_{max}}{g_\delta} \right)^{\frac{1}{D - D_\delta}} = e^{\ell^*}}$

$\boxed{\Delta_{g_{phys}}} \ \Delta_{max} e^{-\ell^*} = \boxed{\Delta_{max} \left(\frac{g}{\Lambda_{max}} \right)^{\frac{1}{D - D_\delta}}}$

$\Delta \sim \boxed{\omega_0 \left(\frac{U}{\omega_0} \right)^{\frac{1}{1 - D_\delta}}}$



$\frac{1}{L}$

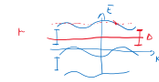
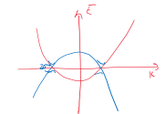
$+ \boxed{g} \psi^*(x) \psi(x) \psi^*(x + a) \psi(x + a)$

$H = \frac{k^2}{2m}$ $\qquad H = \sum_k c_k^\dagger \frac{k^2}{2m} c_k$

$\sum_k -2t \cos k \ c_k^\dagger c_k$ $\qquad$ 1D

$\sum_{k, y} -2t \left(\cos k_x + \cos k_y \right) c_k^\dagger c_k$

$\mu \in [-2t, 2t]$



$+ \boxed{\Delta} \psi_+^\dagger(k) \psi_-^\dagger(-k) + H.c.$