Group exercises on Berryology

These are exercises to get a flavor of Berry phases and why we like them.

I. GENERAL DEFINITION OF THE BERRY PHASE

Berry phases are a crucial element that completes the modeling of the dynamics of quantum systems, and not only that. The two main applications of Berry phases are:

- Modeling of the adiabatic evolution of quantum systems
- Definition of topological invariants for topological materials

The first point is the most elemental and we will begin from there.

Consider a system described by a Hamiltonian which varies in time. In particular, we assume that the Hamiltonian depends on a set of parameters $\{\lambda_i(t)\}$ which, in turn are adiabatically varied in time. We consider the following definitions and hypotheses, which match the standard assumptions for the adiabatic theorem:

1. For each time t, we can define an ordered eigenbasis $\{|n(\lambda_i(t))\rangle\}$ of the Hamiltonian $H(\lambda_i(t))$, such that:
$$H(\lambda_i(t))\,|n(\lambda_i(t))\rangle = E_n(\lambda_i(t))\,|n(\lambda_i(t))\rangle\,. \tag{1}$$
This is our istantaneous eigenbasis, which clearly depends on time.
2. We assume that the eigenvalues $E_n(\lambda_i(t))$ are well separated for any time t. Typically, we are interested in the adiabatic evolution of a specific state $|n(\lambda_i(t))\rangle$. Very often this is the ground state.
3. When we consider the evolution of a given eigenstate, this eigenstate must be separated in energy by a suitable gap $\Delta(t) > 0$ with the neighboring eigenenergy states.
4. The evolution we consider is slow. Namely the total evolution time T must be sufficiently longer than the inverse of the minimum $\Delta_{\min}$ of this gap.

The **questions**, now, becomes the following. Let us consider, for simplicity, that H depends on a single parameter λ. Moreover, let us assume that at $t = 0$ we initialize the quantum system in a specific eigenstate:
$$|\psi(t=0)\rangle = |n(\lambda_i(t=0))\rangle\,. \tag{2}$$

- What is the state of the system at a generic time t?
- How do you define its phase?
- Is this phase physical? Can it be measured?
- What if λ is a vector such that $\lambda_i(t=0) = \lambda_i(t=T)$? Namely what happens when the adiabatic evolution is defined by a closed path in the parameter space?

To answer this questions **I invite you to do the following**:

- Assume that:
$$|\psi(t)\rangle = e^{i\tilde{\alpha}(t)}|n(\lambda_i(t))\rangle\,. \tag{3}$$
- Redefine:
$$\tilde{\alpha}(t) = -\int_0^t dt'\,E_n(t') + \alpha(t) \tag{4}$$
- Apply Schroedinger equation to $|\psi(t)\rangle$.
- Derive $\langle\psi(t)|\,(\partial_t\alpha(t))\,|\psi(t)\rangle$.
- Calculate $\alpha(t)$ accordingly.
- Assuming the adiabatic evolution is slow, does $\alpha(t)$ depends on how slow/fast we vary $\lambda(t)$?
- Is there a physical difference in the cases $\lambda(t=0) \neq \lambda(t=T)$ and $\lambda(t=0) = \lambda(t=T)$?

For $\lambda(t=0) = \lambda(t=T)$, α is the **Berry phase**.

II. A TOUGH BUT VERY NICE EXAMPLE: BLOCH (MOMENTUM SPACE) INTERFEROMETRY

There are many trivial textbook examples about Berry phases. Here we are a bit more ambitious, and we try to model the basic features of an experiment performed at LMU (Munich) [1] (I will upload the paper on Absalon). In this experiment, Lucia Duca and Tracy Li set up a 2D optical lattice. We assume that their lasers trap atoms in a 2D lattice with the shape of a brickwall (this is a simplification to make calculations slightly faster). They use bosonic ^{87}Rb atoms. They cool them down at very low temperature. These atoms are bosons and they condense in the ground state of the system and this is their starting point. Furthermore, they have exquisite control over the condensate spin, and in the experiment, they use as they wish two atomic states: $|\uparrow\rangle = |F = 2, m = 1\rangle$ and $|\downarrow\rangle = |F = 1, m = 1\rangle$. These states have different spin. This means that by switching on a magnetic field with a gradient, different spin species are subject to different forces. The experimentalists at LMU are very very good, and they can gently control these forces in time for each spin species as they wish.

The first experimental steps are more or less the following.

- The condensate is initialized in the bottom of the lowest energy band (or in any other poistion) in the state $|\uparrow\rangle$.
- A $\pi/2$ pulse is applied, such that now the condensate is in a linear superposition $(|\uparrow\rangle + |\downarrow\rangle)/\sqrt{2}$.
- The Stern Gerlach magnetic gradient is switched on. $|\uparrow\rangle$ and $|\downarrow\rangle$ perceive two gentle forces which vary in time in the way the experimentalist wish.

Questions:

1. The condensate in the spin $|s\rangle$ is subject to a force $\vec{F}_s(t)$ and at time $t = 0$ it occupies a state $|\vec{k}(t = 0)\rangle$ in the Brillouin zone. How does this condensate move in momentum space a a function of $\vec{F}_s(t)$?
2. Calculate the momentum space Hamiltonian $H(\vec{k})$ of the system; diagonalize it as a function of momentum.
3. Find the band touching points.
4. Consider the band touching points, how does the dispersion look like around them?

Let us move on modelling the experiment. We consider the displacement q_x, q_y in momentum space around the band touching point in $(2\pi/3, 0)$. In a neighborhood of that point the Hamiltonian can be Taylor expanded as:

$$H(q_x, q_y)/J = -\left[q_y\sigma_y - \sqrt{3}q_x\sigma_x\right] \tag{5}$$

Let us define $\tilde{q}_x = \sqrt{3}q_x$.

Let us assume now that the experimentalists move the initial condensate in momentum space in an initial position in momentum space on the ellipse $q_y^2 + \tilde{q}_x{}^2 = q_0^2$. At time $t = 0$ they apply the $\pi/2$ pulse we mentioned before, and they begin driving the spin $\uparrow$ and $\downarrow$ components of the condensate symmetrically along the two sides of the ellipse. At time $t = T$ the two components meet again in momentum space in the opposite point of the ellipse.

1. **What is the final spin state?** If they apply the opposite $\pi/2$ pulse, which spin do they measure?
2. What happens if the interferometric path of $\uparrow$ and $\downarrow$ components does not encircle a band touching point?

Hint 1: To answer the first question, I think it is convenient to write the eigenstates of (5) as a function of the parameter $\phi = \arctan(-q_y/\tilde{q}_x)$. At this point we can consider the angle ϕ as the parameter the Hamiltonian depends upon, and calculate the Berry connection associated to it.

Hint 2: To answer the second question, instead, I think it is convenient to write the Berry connection as a function of q_y and $\tilde{q}_x$. These are two parameters, and you should remember Stokes theorem. In this case, it is intuitive to understand how the circuitation of the Berry connection goes. Formally, it may be convenient (but lengthy) to calculate the Berry curvature in momentum space...

[1] L. Duca, T. Li, M. Reitter, I. Bloch, M. Schleier-Smith, and U. Schneider, Science **347**, 288 (2014).