Group exercise

April 28, 2022

I collect here some exercises we will discuss together in class.

1 Massless bosonic 1+1D fields

Consider a 1+1D massless bosonic scalar and real field $\theta(x,\tau)$ described by the following action in Euclidean time:

$$S_0 = \frac{1}{2\pi}\int dx\, d\tau\, \frac{(\partial_\tau\theta)^2}{u} + u\,(\partial_x\theta)^2 \tag{1}$$

where u is a velocity (it is invariant under rescaling).

1. What is the scaling dimension of the θ field?
2. Based on its scaling dimension, what would you expect from correlation functions of the kind $\langle\theta(r,\tau)\theta(r',\tau)\rangle$?

These correlation function are actually quite non-trivial, and we will often use them in the next lectures. Therefore, it is worth calculating them in some detail.

1.1 Correlations of the massless bosonic field in 1+1D.

Use your favorite technique to show that the following two-point correlation function can be approximated in the following way:

$$\langle\theta(x,0)\theta(x',0)\rangle \approx -\frac{1}{2}\ln|x-x'| + \frac{1}{2}\ln\Lambda_{\text{min}}^{-1} \tag{2}$$

where it is convenient to introduce an infrared cutoff $\Lambda_{\text{min}} = 1/L$ and an ultraviolet cutoff $\Lambda_{\text{max}} = 1/a$ such that $a \ll |x-x'| \ll L$.

Hints: In full generality one can calculate $\langle\theta(x,\tau)\theta(x',\tau')\rangle$. In this case, it may be convenient to define space-time vectors $\vec{p} = (\omega/u, -k)$ and $\vec{r} =$

$(u\,[\tau-\tau'], x-x')$ such that $r=\sqrt{(x-x')^2+u^2(\tau-\tau')^2}$ and you can apply the following relations:

$$\int_0^{2\pi}\frac{d\alpha}{2\pi}e^{ipr\cos\alpha}=J_0(pr)\,, \tag{3}$$

$$\int_{\Lambda_{\min}}^{\Lambda_{\max}}dp\frac{J_0(pr)}{2p}\approx-\frac{1}{4}\ln\left(\frac{r^2+\Lambda_{\max}^{-2}}{\Lambda_{\min}^{-2}}\right)\approx-\frac{1}{2}\ln r+\frac{1}{2}\ln\Lambda_{\min}^{-1}\,, \tag{4}$$

where the approximation is valid for $a\ll|x-y|\ll L$.

2 About the dynamical scaling exponent z

In the examples we saw on rescaling, we always considered systems whose gapless fixed points display a linear dispersion $\omega=vk$. We now want to check what happens when this is not verified. Consider the Lagrangian:

$$L=\int d^{D-1}x\left[\psi^\dagger(i\partial_t)\psi+a\psi^\dagger\vec{\nabla}^2\psi\right] \tag{5}$$

1. Check that the dispersion is $\omega=ak^2$. L is indeed Gaussian and describes a gapless system, although the dispersion is no longer local.

2. Try to apply the (wrong) rescaling $x=bx'$, $t=bt'$. Show that there is no possible rescaling of the fields $\psi=b^{-D_\psi}\psi'$ that fulfills the relation:

$$\int dtd^{D-1}x\left[\psi^\dagger(i\partial_t)\psi+a\psi^\dagger\vec{\nabla}^2\psi\right]=\int dt'd^{D-1}x'\left[\psi'^\dagger(i\partial_{t'})\psi'+a\psi'^\dagger\vec{\nabla}'^2\psi'\right] \tag{6}$$

   This equation, however, must be fulfilled for any gapless and Gaussian theory.

3. Apply the rescaling $x=bx'$, $t=b^zt'$. Find z and D_ψ such that Eq. (6) is verified.