BERRY PHASE

→ ① classical Berry phase definition [Berry 1984]
② Topological invariant in classical topological models
e.g. Chern number

$|\psi(\lambda+\delta\lambda)\rangle = |\psi(\lambda)\rangle + \delta\lambda\,|\partial_\lambda\psi\rangle$

$= e^{i\theta(\lambda)}|\psi(\lambda)\rangle + \delta\lambda\,|\partial_\lambda\psi\rangle$

$\langle\psi(\lambda)|\psi(\lambda+\delta\lambda)\rangle = e^{-i\theta(\lambda)}\langle\psi(\lambda)| + \delta\lambda\,\langle\partial_\lambda\psi|$

$\cdot\ |\psi(\lambda+\delta\lambda)\rangle : \quad e^{i\theta(\lambda+\delta\lambda)}|\psi(\lambda+\delta\lambda)\rangle = e^{i\theta(\lambda)}|\psi(\lambda)\rangle + \delta\lambda\,|\partial_\lambda\psi\rangle$

$e^{i\theta(\lambda)}e^{i\partial_\lambda\theta\,\delta\lambda}|\psi(\lambda)\rangle = e^{i\theta(\lambda)}|\psi(\lambda)\rangle + \delta\lambda\,|\partial_\lambda\psi\rangle$

$e^{i\partial_\lambda\theta\,\delta\lambda}|\psi(\lambda)\rangle = |\psi(\lambda)\rangle + \delta\lambda\,|\partial_\lambda\psi\rangle$

$\Rightarrow \quad i\partial_\lambda\theta\,\delta\lambda\,|\psi(\lambda)\rangle = \delta\lambda\,|\partial_\lambda\psi\rangle$

$\Rightarrow \quad i\partial_\lambda\theta\,|\psi(\lambda)\rangle = |\partial_\lambda\psi\rangle$

$\Rightarrow \quad \theta(\lambda) = -i\int_0^\lambda \langle\psi(\lambda')|\partial_{\lambda'}\psi(\lambda')\rangle\,d\lambda'$

BERRY

- Hamiltonian $H(\lambda(t))$
- Periodicity : $H(\lambda)$ periodic
- Adiabatic theorem

H is non-degenerate : discrete spectrum, no crossings in energy with $\lambda(t)$

T : final time

$T \gg \frac{1}{\Delta_{min}}$ [$\hbar=1$]

$\forall\lambda$: $H(\lambda)$: $H(\lambda)|n(\lambda)\rangle = \varepsilon_n|n(\lambda)\rangle$

$\forall\lambda$: $\{|n(\lambda)\rangle\}_n$ are orthogonal and complete [$\langle n(\lambda)|m(\lambda)\rangle = \delta_{nm}$]

Note: Σ makes sense only

$|\psi(t)\rangle = \sum_n c_n(t)\,|n(\lambda(t))\rangle$: Adiabatic theorem [Adiabatic theorem]

[No Adiabatic Berry phase : Adiabatic : Fast limit]

We want to calculate $\vec{x}(t)$ [when λ is rearranged]

$\vec{x}(t) = \underbrace{-\int \varepsilon_n(\lambda(t))\,dt}_{\text{Dynamical}} + \underbrace{\kappa(t)}_{\text{Adiabatic evolution}}$

$-i\partial_t|\psi(t)\rangle = H(\lambda(t))\,|\psi(t)\rangle$

$|\psi(t)\rangle = e^{-i\int \varepsilon_n(\lambda(t'))\,dt' + i\kappa(t)}\,|n(\lambda(t))\rangle$ [adiabatic]

$\mathcal{E}(\lambda(t)) \underbrace{\langle n(\lambda(t))|n(\lambda(t))\rangle}_{=1} e^{-i\int\varepsilon_n(\lambda(t'))dt' + i\kappa(t)} \underbrace{|\partial_t|\psi(t)\rangle}_{|\partial_\lambda\psi(\lambda(t))\rangle\dot\lambda(t)} + \mathcal{E}\,e^{-i\int\varepsilon_n(\lambda(t'))dt' + i\kappa(t)}\,\dot\lambda(t)\,|\partial_\lambda\psi(\lambda(t))\rangle = \varepsilon_n(\lambda(t))\,|\psi(t)\rangle$

$\langle\psi(t)|\,[-\dot\kappa(t)]\,|\psi(t)\rangle + \langle\psi(t)|\,\dot\lambda(t)\,|\partial_\lambda\psi(\lambda(t))\rangle = \dot\lambda(t)\,\langle\psi(t)|\partial_\lambda\psi(\lambda(t))\rangle$

$\dot\kappa(t) = i\,\langle\psi(t)|\partial_\lambda\psi(\lambda(t))\rangle\,\dot\lambda(t)$

$\Rightarrow \quad \partial_\lambda\kappa(\lambda) = \langle n(\lambda(t))|\partial_\lambda|n(\lambda(t))\rangle$

$\kappa(M) = \int_{\lambda(0)}^{\lambda(T)} d\lambda\ \underbrace{\langle n(\lambda)|\partial_\lambda|n(\lambda)\rangle}_{A(\lambda)\,\text{: Berry connection}}\cdot 1\cdot\partial\lambda =$

$= \int_{\lambda(0)}^{\lambda(T)} \underbrace{\langle n(\lambda)|\partial_\lambda|n(\lambda)\rangle}_{\text{Gauge invariant}}\,d\lambda =$

$\int_0^{2\pi} \langle n(\lambda)|\partial_\lambda|n(\lambda)\rangle\,d\lambda$

Gauge invariant quantity

does not depend on time

$= \int_0^{2\pi} \ldots\,\frac{d\lambda}{dt}\,dt$

It depends on the path only

e.g. Geometric object, Geometric phase

Berry transformation

$|n'(\lambda)\rangle = e^{-i\theta(\lambda)}\,|n(\lambda)\rangle$: I redefine the phase

$\theta(\lambda+\delta\lambda) = \theta(\lambda) + \partial_\lambda\theta(\lambda)\,\delta\lambda$

Gauge : Abelian, group is in λ space

$|\psi(t)\rangle = e^{-i\int\varepsilon_n(\lambda(t'))dt' + i\kappa(t)} \cdot e^{-i\int_0^{\lambda(t)}\partial_\lambda\theta\,d\lambda}\,|n(\lambda(t))\rangle$

$|\psi'(t)\rangle = e^{i\theta(\lambda(t))}|n(\lambda(t))\rangle$

I redo the calculation

$\kappa'(t) = \kappa'(\lambda(t)) = \kappa'(\lambda) + \theta(\lambda(t)) - \theta(0)$

$= \int_0^\lambda d\lambda\ \langle n(\lambda)|\partial_\lambda|n(\lambda)\rangle + \partial_\lambda\theta(\lambda)$

$A'(\lambda) = A(\lambda) + \partial_\lambda\theta(\lambda)$

$\kappa'(\lambda) = \kappa(\lambda) + \theta(\lambda) - \theta(0)$

This phase becomes physical when $\lambda : S^1$

$\kappa(T) = \oint_C \langle n(\lambda)|\partial_\lambda|n(\lambda)\rangle$ Berry or geometric phase [Pancharatnam phase]

To $\theta(\lambda)$ [depends only on λ] $\Rightarrow$ $\underbrace{\theta(\lambda_f) - \theta(\lambda_i)}_{\text{path independent}} = 0$ $\Rightarrow$ $\kappa = \kappa'$ and κ depends only on C in Geometric phase

$\gamma(t)$ is the area defined and after the cycle

Spin $\frac{1}{2}$

$H(\vec{r}) = \vec{B}\cdot\vec{\sigma}$

$E = \pm|\vec{B}| \qquad |\pm\rangle \neq 0 \quad \forall\,t$

We vary $\vec{\lambda}$ in time

$\vec{\lambda} \equiv |\vec{B}|\left(\sin\theta\cos\varphi,\ \sin\theta\sin\varphi,\ \cos\theta\right)$

$|\psi_\pm\rangle = \cos\frac{\theta}{2}|\uparrow\rangle + e^{i\varphi}\sin\frac{\theta}{2}|\downarrow\rangle$

$A_{\theta\varphi} \in \mathbb{Q} \qquad |\psi_\pm\rangle_\lambda$ does not depend on $|\vec{B}|$

$\partial_{|\vec{B}|}|\psi_\pm\rangle_\lambda = 0$

$A_{\theta\varphi} = \langle\psi_\pm|\,\partial_\theta|\psi_\pm\rangle =$

$= \cos\frac{\theta}{2}e^{-i\varphi}\,\partial_\theta\begin{pmatrix}\cos\frac{\theta}{2}\\ e^{i\varphi}\sin\frac{\theta}{2}\end{pmatrix} =$

$+\frac{1}{2}\cos\frac{\theta}{2}\sin\frac{\theta}{2} + \cancel{i\sin^2\frac{\theta}{2}} - \cancel{\frac{1}{2}\sin\frac{\theta}{2}\cos\frac{\theta}{2}} = 0$

$A_{\theta\varphi} = \cancel{\cos\tfrac{\theta}{2}e^{-i\varphi}}\ \partial_\varphi\begin{pmatrix}\cos\frac{\theta}{2}\\ \cancel{e^{i\varphi}}\sin\frac{\theta}{2}\end{pmatrix} = -i\sin^2\frac{\theta}{2} \quad \therefore \ \frac{1-\cos\theta}{2}$

$\theta \in [0,\pi]$ is fixed

Berry $\gamma_C = \int_S \Omega_{\theta\varphi}\,d\theta\,d\varphi = \int_0^{2\pi} A_{\theta\varphi}\,d\varphi = -i\pi\left(1-\cos\theta\right)$

$= -\frac{1}{2}\int\Omega_{\theta\varphi}\,d\theta\,d\varphi = \boxed{\int\vec{A}\cdot\vec{dl}}$

$\vec{\Omega} = \nabla\times\vec{A}$

Spherical coordinates

$\vec{\nabla} = \left(\partial_r,\ \frac{1}{r}\partial_\theta,\ \frac{1}{r\sin\theta}\partial_\varphi\right)$

$\vec{A} : \quad A_\varphi = \frac{r}{|\vec{B}|=r}\left(-\sin\frac{\theta}{2}\right) \qquad A_{\theta=0}$

$A_\theta = 0$

$\vec{\Omega} = \vec{\nabla}\times\vec{A}$

Berry curvature

$\Omega_r = \frac{1}{|\vec{B}|=r}\partial_\theta\left(\sin\theta\,A_\varphi\right) =$

$= \frac{r}{|\vec{B}|=r}\partial_\theta\sin^2\frac{\theta}{2} = \frac{r}{2}\sin\theta\cdot\frac{1}{|\vec{B}|=r} = \frac{\vec{B}}{2|\vec{B}|^3}$

$\Omega_\theta = \Omega_\varphi = 0$

$\gamma_C = \int_S \Omega_r\,r^2\sin\theta\,d\theta\,d\varphi = \int_S d\theta\,d\varphi\,\sin\theta\left(\frac{1}{2|\vec{B}|^2}\right) = -\frac{1}{2}\int_C d\varphi\,(1-\cos\theta)$

$\vec{B} = \vec{\nabla}\times\vec{A}$

$\#\ \cdots\ \exists\,\vec{B} \ \Rightarrow \quad H(\vec{k}) = \left[a_0(\vec{k})\,\mathbb{1} + \vec{a}(\vec{k})\cdot\vec{\sigma}\right]$

2×2 matrix $H(\vec{k})$

$H(\vec{k}) \quad \mathcal{U}(\vec{k}) : \quad |\psi_\pm(\vec{k})\rangle \ \Big/\ H(\vec{k})\,|\psi_\pm(\vec{k})\rangle = \left[a_0(\vec{k}) \pm |\vec{a}(\vec{k})|\right]|\psi_\pm\rangle$

$\frac{2\pi n}{\Delta k_x}$ $\frac{2\pi m}{\Delta k_y}$ $\int_{BZ} d^2k\ \frac{\mathcal{U}(\vec{k})}{(2\pi)^2}$ $\mathcal{U}(\vec{k}) = \mathcal{U}(\vec{k}+\vec{G})$

Tot numbers

$n_x, n_y \in \mathbb{Z}$ Periodic

$\Delta k_x = \Delta k_y = \frac{2\pi}{a}$ Crystal size $= a$

- $|\psi_\pm(\vec{k})\rangle$ does not depend on $a_0(\vec{k})$

I demand that the system is insulator $\Rightarrow$ $\boxed{|\vec{a}(\vec{k})| \neq 0}$ $\forall\,\vec{k}\in BZ$

Singular

$\hat{\vec{a}}(\vec{k}) = \frac{\vec{a}(\vec{k})}{|\vec{a}(\vec{k})|}$

$\vec{k}\in BZ \longrightarrow$ Sphere $\hat{\vec{a}}(\vec{k}) \in S_2 \longrightarrow (\theta(\vec{k}),\varphi(\vec{k}))$ mapping $(n,m)\in\mathbb{Z}^2$

$\theta(\vec{k}) = \theta(\vec{k}+\vec{G})$

$\varphi(\vec{k}) = \varphi(\vec{k}+\vec{G})$

Chern number

$C = \frac{1}{2\pi}\int d^2k\ F(\vec{k})$ [Total flux number]

$F(\vec{k}) = \vec{\nabla}_k\times\vec{A} = \partial_{k_x}A_{k_y} - \partial_{k_y}A_{k_x} = \left[\langle\partial_{k_x}\psi(k_x,k_y)| - \langle\partial_{k_y}\psi|\partial_{k_x}\psi\rangle\right]$

$A(\vec{k}) = \begin{pmatrix}\langle\psi_\pm(\theta,\varphi)|\partial_\theta|\psi_\pm(\theta,\varphi)\rangle\\ \langle\psi_\pm(\theta,\varphi)|\partial_\varphi|\psi_\pm(\theta,\varphi)\rangle\end{pmatrix}$

C is integer : Topological object invariant under continuous λ set of ψ

QUANTUM HALL EFFECTS — von Klitzing 1980