BKT Phase transition

$\boxed{\text{Sine-Gordon model}} \to \boxed{\text{RG}}$

2D : classical : XY model
: Superfluid models : ^4He

$|\nabla\theta|^2 \qquad \partial_\tau\theta \sim \partial_x\theta$

$$H_{eff} = -J_{xy}\sum_{\langle\vec{r}\vec{r}'\rangle}\cos(\theta(\vec{r})-\theta(\vec{r}'))$$

T

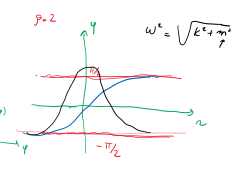
1D quantum : $S \overset{\text{Euclidean}}{=} \frac{K}{2\pi}\int d\tau\, dx\left[(\partial_\tau\varphi)^2_{U} + (\partial_x\varphi)^2 U\right] + \boxed{g\int dx\, d\tau\, \cos\beta\varphi}$

$\cos\beta\varphi \sim -1 \qquad \varphi\in[0,2\pi)$

$\varphi = \frac{\pi}{\beta} \qquad \varphi = \frac{3\pi}{\beta} \qquad \forall x$

$\sim -\frac{1}{2}g\beta^2\delta\varphi^2$

$g=2$



$\omega^2 = \sqrt{k^2+m^2}$

$\langle e^{i\theta(x)}e^{-i\theta(x')}\rangle \sim \frac{1}{|x-x'|^{\frac{1}{4}}}$

$0 \xleftarrow{L\to\infty} \langle e^{i\theta}\rangle \sim e^{-\frac{1}{2}\langle\theta^2\rangle} \sim \frac{1}{(L)^{\frac{1}{4}}}$

$\to \langle\theta^2\rangle \sim -\frac{\pi}{2}\ln\frac{\Delta_{min}}{\Delta_{max}}$

$\Delta_{min} = \frac{2\pi}{L}$

$\boxed{\text{Kosterlitz Temperature}}$

$\frac{g^2}{4K} \gtrless 2$

$K < \frac{\beta^2}{8}$: Irrelevant $\to$ T is low gapless phase : quasi-ordered [$\langle\cos\rangle$ vanishes $L\to\infty$]

$\to$ NO local order parameter

$K > \frac{\beta^2}{8}$: Relevant $\to$ T is high : gapped
vortices nucleates
Disordered phase
Magnetization is 0

$e^{i\alpha}$: annihilates a vortex
α

$e^{i\varphi}$ is ordered $\langle e^{i\varphi(x)}e^{-i\varphi(x')}\rangle \to$ const

$\langle e^{i\theta}\rangle \neq 0$ vortices condensed

$\langle e^{i\theta}e^{-i\theta}\rangle$ decay exp

$\varphi = \frac{\pi}{\beta} \quad \forall x$

$\varphi(x\to-\infty) = -\frac{\pi}{2}$

$\varphi(x\to+\infty) = +\frac{\pi}{2}$

Phase tr without a local order parameter : topol. phase tr.

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2 kinds of Sine Gordons

$H[\varphi, K]$

$H[\theta, K^{-1}]$

ξ : correl. length $\qquad \langle e^{i\theta}e^{-i\theta}\rangle \sim e^{-\frac{|x-x'|}{\xi}}$

$\xi = \frac{1}{(T-T_c)^\nu} = \frac{1}{(O(\delta_c)-O(\delta_c))^\nu}$: power law divergence for 2nd order phase tr

BKT ξ diverges in a different way $\boxed{\xi \sim e^{\frac{1}{\sqrt{T-T_c}}}}$ $\qquad \xi = \nu\Delta^{-1}$

BKT is an "infinite order" phase tran. $\qquad F \qquad \frac{\partial^2 F}{\partial K^2}$ are continuous

LUTTINGER

$$S = \underbrace{\frac{K}{2\pi}\int d\tau\, dx\left[\frac{(\partial_\tau\varphi)^2}{U} + (\partial_x\varphi)^2 U\right]}_{S_0} + \underbrace{g\int dx\cos\beta\varphi}_{S_{INT}}$$

$\boxed{\hbar = 1}$

$\varphi(x) = \varphi_s(x) + \varphi_f(x)$

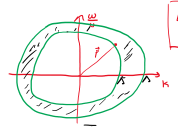
Λ : UV cutoff : $\frac{2\pi}{a}$

$\tilde{\Lambda} = (1-dl)\Lambda$

$\tilde{\Lambda} < |k| < \Lambda$

$|k| < \tilde{\Lambda}$

$|k|$



$E < \Omega_{UV} \sim 2T$ when $g(\ell) < \Omega_{UV}$

$$\underbrace{S_{eff}(\tilde{\Lambda})}_{0} \cong \underbrace{S_0(\varphi_s)}_{I} + \underbrace{\langle S_I\rangle_f}_{A} \underbrace{- \frac{1}{2}\langle S_I^2\rangle_f}_{B} + \underbrace{\frac{1}{2}\langle S_I\rangle_f^2}_{A^2} + \dots$$

$\langle\varphi_f^2\rangle_f = \int_{\tilde{\Lambda}<q<\Lambda}\frac{d^2p}{4\pi^2}\frac{\pi}{Kp^2} = \int_{\tilde{\Lambda}<p<\Lambda}2\pi\frac{dp}{2\pi}\frac{1}{Kp} = \frac{1}{2K}\ln\frac{\Lambda}{\tilde{\Lambda}}$

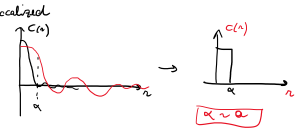
$\boxed{\frac{\Lambda}{\tilde{\Lambda}} = 1+dl}$

$\langle\varphi_f(x)\varphi_f(x')\rangle_f = \int_{\tilde{\Lambda}}^{\Lambda}dp\,\frac{J_0(px)}{2Kp} \approx \frac{C(r)}{2K}\cdot\ln\frac{\Lambda}{\tilde{\Lambda}}$

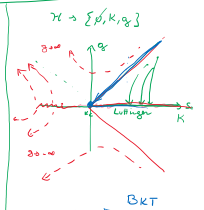
$r = \sqrt{(x-x')^2 + v^2(\tau-\tau')^2}$ $\qquad \Lambda r - \tilde{\Lambda} r = \boxed{dl\, r \ll 1}$

$C(r) \sim J_0(\Lambda r) \sim J_0(\tilde{\Lambda} r)$

$C(r)$ is localized



$\int dx\, d\tau\, C(r) = \gamma \sim \frac{\alpha^2}{U}$



$\mathcal{H} \to \{\beta, K, g\}$

$g=0$

$K > K_c$

$g \to \infty$

$\Delta = \Omega_{UV}\left(\frac{g}{\ell_{UV}}\right)^{\frac{1}{2-K}}$

$\Delta \sim g$

$\boxed{g\cos\beta\theta}$

BKT

XXZ $\boxed{\text{Heisenberg}}$

Luttinger $\qquad$ Massive

J_z/J_{xy}

- 1st order
- Gap at 1st order
- II order
- Phase diagram at II order

$$\langle S_I\rangle_f = \int dx\, d\tau\, \frac{g}{2}\left\langle e^{i(\varphi_s+\varphi_f)\beta}e^{-i(\varphi_s+\varphi_f)\beta}\right\rangle_f =$$

$$= \int dx\, d\tau\, \frac{g}{2}\left[\langle e^{i\varphi_f\beta}\rangle_f + e^{-i\varphi_s\beta}\langle e^{-i\varphi_f\beta}\rangle_f\right] =$$

$$= \int dx\, d\tau\, \frac{g}{2}\left[e^{i\varphi_s\beta}e^{-\frac{\beta^2}{2}\langle\varphi_f^2\rangle_f} + H.c.\right] =$$

$$= \int dx\, d\tau\, \frac{g}{2}\left(e^{i\varphi_s} + e^{-i\varphi_s}\right)e^{-\frac{\beta^2}{2}\underbrace{\langle\varphi_f^2\rangle_f}_{\frac{1}{2K}\ln\frac{\Lambda}{\tilde{\Lambda}}}} =$$

$$= \int dx\, d\tau\, g\cos\beta\varphi_s\cdot\left(\frac{\tilde{\Lambda}}{\Lambda}\right)^{\frac{\beta^2}{4K}} \to D(\beta,K) =$$

$$= \int dx\, d\tau\, g\cos\beta\varphi_s\cdot\left(1 - \frac{\beta^2}{4K}dl\right) =$$

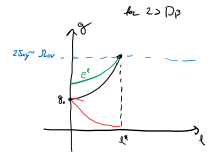
$1+1+0$ $\quad \boxed{\varphi_s(x) = \varphi_s'(x)}$

$$= \int dx'd\tau'(1+2dl)\, g\cos\beta\varphi_s'\cdot\left(1-\frac{\beta^2}{4K}dl\right) =$$

$$= \underbrace{g\left(1+\left(2-\frac{\beta^2}{4K}\right)dl\right)}_{g'}\int dx'd\tau'\cos\beta\varphi_s'$$

$\frac{dg}{dl} = \left(2-\frac{\beta^2}{4K}\right)g = (2-D_\beta)g$

for $2>D_\beta$



$K > \frac{8}{\beta^2} \to$ irrel.

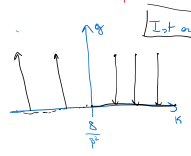
$K < \frac{8}{\beta^2} \to$ rel

$\boxed{\Delta_{gap} \sim \Omega_{UV}e^{-\ell^*}}$

$g(\ell^*) = \Omega_{UV} = g_0 e^{(2-D_\beta)\ell^*} = \Omega_{UV}$

$e^{\ell^*} = \left(\frac{\Omega_{UV}}{g_0}\right)^{\frac{1}{2-D_\beta}}$

$\boxed{\text{1st order}}$



$\frac{U}{J} = \Delta_{gap} = \Omega_{UV}\left(\frac{g_0}{\Omega_{UV}}\right)^{\frac{1}{2-D_\beta}}$

$J = \frac{U}{\Omega_{UV}}\cdot\left(\frac{\Omega_{UV}}{g_0}\right)^{\frac{1}{2-D_\beta}} \to 0$ $\qquad \frac{\Omega_{UV}}{g_0} > 1$

$D_\beta = \frac{\beta^2}{4K}$

$$\frac{A^2}{2} = \frac{\langle S_I\rangle^2}{2} =$$

$$= \int dx_1 d\tau_1\int dx_2 d\tau_2\, \frac{g^2}{2}\langle\cos\beta\varphi(x_1,\tau_1)\rangle_f\langle\cos\beta\varphi(x_2,\tau_2)\rangle_f$$

$$= \frac{g^2}{2}\left(1+\left(4-\frac{\beta^2}{2K}\right)dl\right)\int dx_1'd\tau_1'\,dx_2'd\tau_2'\,\cos\beta\varphi_s'(x_1',\tau_1')\cos\beta\varphi_s(x_2',\tau_2')$$

$$\frac{B}{2} = \frac{\langle S_I^2\rangle}{2} = \frac{g^2}{2}\int dx_1 d\tau_1\, dx_2 d\tau_2\left\langle\left(\frac{e^{i\beta\varphi(x_1)}+e^{-i\beta\varphi(x_1)}}{2}\right)\left(\frac{e^{i\beta\varphi(x_2)}+e^{-i\beta\varphi(x_2)}}{2}\right)\right\rangle_f$$

$$\frac{1}{4}\sum_{\mu\nu=\pm1}\left\langle e^{i\mu\beta\varphi(x_1)}\cdot e^{i\nu\beta\varphi(x_2)}\right\rangle_f =$$

$$\frac{1}{4}\sum_{\mu\nu}\left\langle e^{i\beta(\mu\varphi(x_1)+\nu\varphi(x_2))}\right\rangle_f =$$

$$= \frac{1}{4}\sum_{\mu\nu}e^{i\beta(\mu\varphi_s(x_1)+\nu\varphi_s(x_2))}\left\langle e^{i\beta(\mu\varphi_f(x_1)+\nu\varphi_f(x_2))}\right\rangle_f$$

$$= \frac{1}{4}\sum_{\mu\nu}e^{i\beta(\dots)}\, e^{-\frac{\beta^2}{2}\left[\langle(\varphi_f^2(x_1))\rangle_f + \langle(\varphi_f^2(x_2))\rangle_f + \mu\nu 2\langle\varphi_f(x_1)\varphi_f(x_2)\rangle_f\right]} =$$

$$= \frac{1}{4}\sum_{\mu\nu}e^{i\beta(\dots)}\, e^{-\frac{\beta^2}{2}\left[2\cdot\frac{1}{2K}\ln\frac{\Lambda}{\tilde{\Lambda}} + 2\mu\nu C(r)\frac{1}{2K}\ln\frac{\Lambda}{\tilde{\Lambda}}\right]}$$

$$= \frac{g^2}{8}\int d^2r_1 d^2r_2\sum_{\mu\nu}e^{i\beta(\mu\varphi_s(r_1)+\nu\varphi_s(r_2))}\left(\frac{\tilde{\Lambda}}{\Lambda}\right)^{\frac{\beta^2}{2K}+\frac{\beta^2}{2K}\mu\nu C(r)}$$

$$\left[1 - dl\left(\frac{\beta^2}{2K} + \frac{\beta^2}{2K}\mu\nu C(r)\right)\right]$$

$\frac{B-A^2}{2}$

$$-\frac{A^2-B}{2} = \frac{g^2}{8}dl\int d^2r_1 d^2r_2\sum_{\mu\nu=\pm1}\frac{\mu\nu}{2K}C(|r_1-r_2|)\, e^{i\beta[\mu\varphi_s'(r_1)+\nu\varphi_s'(r_2)]}$$

$\boxed{\mu\nu = \pm1}$ $\qquad C(r) = \underset{\frac{\alpha^2}{U}}{\gamma}\,\delta^2(r_1'-r_2')$

$$= \frac{g^2}{8}\frac{\gamma\, dl}{2K}\int d^2r_1'\left[e^{i\beta 2\varphi_s'(r_1')} + e^{-2i\beta\varphi_s'(r_1')}\right] =$$

$$= \boxed{\frac{g^2}{8K}\gamma\, dl}\int d^2r'\,d\tau'\,\underbrace{\cos 2\beta\varphi_s(r')}$$

$$S + \boxed{h\int dx\, d\tau\,\cos 2\beta\varphi_s(r)} \qquad \to \boxed{h(\ell=0) = 0}$$

$$\frac{\partial h}{\partial\ell} = -\frac{g^2\gamma}{8K} + (2-D_{2\beta})h = \frac{g^2\gamma}{8K} + \left(2-\frac{\beta^2}{K}\right)h$$

$\boxed{\mu = -\nu}$

$$-\frac{g^2}{8}\frac{dl}{2K}\sum_{\mu=\pm1}\int d^2r_1'\, d^2r_2'\; C(r)\, e^{i\beta\mu[\varphi_s'(r_1')-\varphi_s'(r_2')]}$$

$R = \frac{r_1+r_2}{2} \qquad r = r_1 - r_2 \qquad \hat{r}\cdot\vec{\nabla}_R\varphi_s \sim |a\cdot\nabla_R\varphi_s'|$

$$-\frac{g^2}{8K}dl\sum_\mu\int d^2R\, d^2r\;\underset{\frac{\alpha^2}{U}}{C(r)}\, e^{i\beta\mu\left[\varphi_s'\left(R+\frac{r}{2}\right)-\varphi_s'\left(R-\frac{r}{2}\right)\right]} =$$

$r < a$

$$= -\frac{g^2}{16K}dl\,\gamma\sum_{\mu=\pm1}\int d^2R\; e^{i\beta\mu|a\nabla_R\varphi_s'|} =$$

$$= -\frac{g^2}{16K}dl\,\gamma\int\frac{\beta^2a^2}{2}\cdot 2\,|\nabla_R\varphi_s'|^2\, d^2R =$$

$(\partial_x\varphi_s')^2 + U^2(\partial_\tau\varphi_s')^2$

$$= -\frac{g^2\beta^2a^2\gamma}{16KU}dl\int dx'd\tau'\left[(\partial_x\varphi_s')^2U + \frac{(\partial_\tau\varphi_s')^2}{U}\right]$$

"S_0" II

$$S_0 = \int dx\, d\tau\left[\underbrace{\left(\frac{KU}{2\pi} + \frac{g^2\beta^2a^2\gamma}{16K}dl\right)}_{\frac{K'U'}{2\pi}}(\partial_x\varphi_s)^2 + \underbrace{\left(\frac{K}{2\pi U} + \frac{g^2\beta^2a^2\gamma}{16KU^2}\right)}_{\frac{K'}{2\pi U'}}(\partial_\tau\varphi_s)^2\right]$$

$\to \boxed{U' = U} \Rightarrow$ Lorentz is satisfied

$\boxed{\begin{matrix}\alpha = a\\ \gamma = \frac{\alpha^2}{U}\end{matrix}}$

$$\frac{K'}{2\pi} = \sqrt{\left(\frac{KU}{2\pi} + \frac{g^2\beta^2a^2\gamma}{16K}dl\right)\left(\frac{K}{2\pi U} + g\dots dl\right)} = \frac{K}{2\pi} + \frac{1}{2\pi}dl\;\frac{\pi g^2\beta^4\alpha^4\gamma}{8UK}$$

$\boxed{\frac{dK_4}{dl} = \frac{\beta^4g^2\pi\alpha^4}{8U^2K_4}}$ $\qquad \boxed{\frac{dU}{dl} = 0}$ $\qquad \boxed{\frac{dg}{dl} = \left(2-\frac{\beta^2}{4K}\right)g}$

$\to \boxed{\frac{dK_0}{dl} = -\frac{\beta^4g^2\pi\alpha^4}{8U^2}K_0^3}$ $\quad K_0 = \frac{8}{\beta^2}$ $\qquad K_c = \left(\frac{4\pi}{\beta^2} - \frac{8}{\beta^2}\right)^3$ $\qquad \boxed{\frac{dg}{dl} = \left(2-\frac{\beta^2K}{4}\right)g}$ $\qquad$ XXZ $\beta = 4$

LINEARIZING close to $K = \frac{8}{\beta^2}$ $\cos\beta\theta$

$\boxed{2-\frac{\beta^2K}{4} = -t}$ $\qquad \boxed{y = \frac{4\sqrt{\pi}\alpha^2g}{U}}$

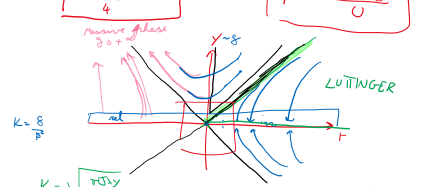
$\boxed{\begin{matrix}\frac{dt}{dl} = -y^2\\ \frac{dy}{dl} = -ty\end{matrix}}$ $\qquad \frac{\beta^2}{4}\frac{dK}{dl} = -16\pi\frac{\alpha^4}{U^2}g^2$

$\frac{dt}{dl} = \frac{dK}{dl}\cdot\frac{\beta^2}{4} = -\frac{dK}{dl} = -\frac{64\pi\alpha^4g^2}{U^2\beta^2}$

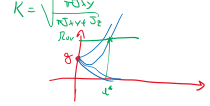
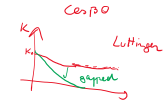
KT relations

$y^2 - t^2 = \boxed{\mu}$

$\frac{\partial\mu}{\partial\ell} = 2y\frac{dy}{dl} - 2t\frac{dt}{dl} = (-ty^2 + ty^2) = 0$



$K = \sqrt{\frac{\pi J_{xy}}{\pi J_{xy} + J_z}}$



$\cos\beta\theta$