LUTTINGER MODEL

$H = -2t \sum_k (c^\dagger_{k+1} c_k + \text{H.c.}) + U \sum c^\dagger_k c_k c^\dagger_{k+q} c_{k+q}$ $\quad c^\dagger c c^\dagger c \qquad c^\dagger c^\dagger c c$

$\to \boxed{V(q) \, \rho(q) \rho(q)}$

$\rho_q = \sum_k c^\dagger_k c_{k+q}$

$b_q \sim \sqrt{\frac{2\pi}{|q|}} \rho(q)$

$[b_q, b^\dagger_{q'}] = \delta_{qq'}$ $\qquad [H_0, b_q] = -b_q v_F |q|$

$H_0 = \sum v_F |q| b^\dagger_q b_q$

$H = H_0 + H_{int} = \sum_q |q| \, b^\dagger_q \, b_{-q} \begin{pmatrix} & \end{pmatrix} \begin{pmatrix} b_q \\ b_{-q} \end{pmatrix}$

LUTTINGER LIQUID $\qquad \omega = |q| v_F \left(\frac{1+V}{\dots}\right)$

$\to \boxed{\varphi(x)} = \int \frac{dq}{\sqrt{2\omega}} \left[b_q e^{iqx - i\omega t} + b^\dagger_q e^{-iqx + i\omega t} \right]$ $\qquad \varphi^\dagger = \varphi$

PHENOMENOLOGICAL BOSONIZATION 1D
- Derive / define rules to map inter. problems into bosonic fields

AXIOMATIC BOSONIZATION $\qquad \psi_B \sim |\psi_B| e^{i\varphi}$

N particles, L total length $\qquad \boxed{\rho_0 = \frac{N}{L}}$ $\qquad k_F = \frac{\pi N}{L}$ [spinless fermions]

Particles sitting at positions $\{x_g\}$ $\to$ Many body wave function $\psi(x_1, x_2, x_3 \dots)$ $\qquad x_1 < x_2 < x_3 \dots$

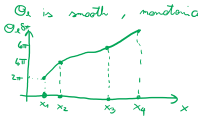
$\rho(x) = \sum_g \delta(x - x_g)$

$\bar{d} : \boxed{\frac{L}{N} = \rho_0^{-1}}$

$\boxed{x_g = g\bar{d} + u_g}$ (average) (fluctuation)

labelling or counting field $\theta_\ell(x)$: $\theta_\ell(x = x_g) = 2\pi g$

θ_ℓ is smooth, monotonically increasing



$\theta_\ell(x) \sim 2\pi \frac{x}{\bar{d}} + \text{fluctuations}$

$f(x) = i(e^{-i\theta_\ell(x)} - 1) \quad \leftarrow \quad \boxed{f(x_g) = 0} \qquad f(\theta_\ell(x))$

$\delta(f(x)) \leftrightarrow \delta$ of the zeros of f $\leftrightarrow$ δ of $x - x_g$

$\delta(f(x)) \neq 0$ for $x = x_g \quad \forall g$ $\qquad \partial_x \theta_\ell \, e^{i\theta}$

$\delta(f(x)) = \sum_{\text{zeros of } f} \delta(x - x_0) \cdot \frac{1}{|f'(x)|}$ $\qquad f'(x_g) = \partial_x \theta|_{x_g}$

$= \sum_{\text{zeros}} \frac{1}{|\nabla\theta|} \delta(x - x_g) = \frac{1}{|\nabla\theta_\ell(x)|} \sum_g \delta(x - x_g)$ $\quad \rho(x)$

$\boxed{\rho(x) = \nabla\theta_\ell(x) \cdot \sum_g \delta(\theta_\ell(x) - 2\pi g)}$ $\qquad \theta_\ell$ monotonically incr.

Poisson Summation $\qquad t \leftrightarrow \theta_\ell(x) \qquad$ sas S $\qquad \hat{S} = 1$

$\boxed{\sum_m s(t + 2\pi m) = \frac{1}{2\pi} \sum_k \hat{S}\left(\frac{k}{2\pi}\right) e^{ikt}}$

$\to \boxed{\rho(x) = \frac{\nabla\theta_\ell(x)}{2\pi} \sum_p e^{ip\theta_\ell(x)}}$

$\theta_\ell(x) = 2\pi \frac{x}{\bar{d}} - 2\theta(x) = 2\pi\rho_0 x - 2\theta(x) \doteq 2k_F x - 2\theta(x)$

$\boxed{\theta(x) = -\frac{\theta_\ell(x)}{2} + \pi\rho_0 x}$

$\rho(x) = \left(\rho_0 - \frac{\nabla\theta(x)}{\pi}\right) \sum_p e^{i2p(\pi\rho_0 x - \theta(x))}$ $\qquad$ f.o. $\quad$ s.e.

θ: varies slowly $\qquad a\partial\theta \ll 1$, $\quad \bar{d}\, \partial\theta \ll 1$ $\qquad a \sim \bar{d}$

$e^{i2p\theta}$

$p = 0 \qquad |p| = 1$ is more relevant than $|p| = 2$

$[\rho(x), \rho(x')] = 0 \quad \Rightarrow \quad [\theta(x), \theta(x')] = 0$

BOSONS	FERMIONS
$\psi_B(x)$, $\psi_B^\dagger(x)$	$\psi(x)$, $\psi^\dagger(x)$
$\rho(x) = \psi_B^\dagger(x)\psi_B(x)$	$\rho(x) = \psi^\dagger(x)\psi(x)$
$\psi_B(x) = \sqrt{\rho(x)}\, e^{i\varphi(x)}$	$\psi(x) = ?$
$[\psi_B(x), \psi_B^\dagger(x')] = \delta(x - x')$	$\{\psi(x), \psi^\dagger(x')\} = 0$

BOSONS $\qquad \boxed{\psi_B(x) = \sqrt{\rho(x)}\, e^{i\varphi(x)}}$

$[\psi_B(x), \psi_B^\dagger(x')] = \delta(x - x')$

$\sqrt{\rho(x)} e^{i\varphi(x)} e^{-i\varphi(x')} \sqrt{\rho(x')} - e^{-i\varphi(x')} \sqrt{\rho(x')} \sqrt{\rho(x)} e^{i\varphi(x)} = \delta(x - x')$

if $x \neq x'$: $[\varphi(x), \varphi(x')] = 0$, $\quad [\varphi(x), \sqrt{\rho(x')}] = 0$

$\langle \psi^\dagger \rangle = |\psi(x)| e^{i\varphi(x)}$; $\langle \psi \psi^\dagger \rangle$ = ordered; 3D = const; 1D = ? $\frac{1}{|x - x'|^\alpha}$; Mermin Wagner; 1D: quasi-order

if $x = x'$

$\sqrt{\rho(x)} \cdot \sqrt{\rho(x)} - e^{-i\varphi(x)} \rho(x) e^{i\varphi(x)} = \delta(x - x')$

$[\psi_B(x), \psi_B^\dagger(x')] = \delta(x - x')$ $\qquad x \neq x'$

$\rho(x)$

$[\rho(x), e^{-i\varphi(x)}] = + e^{-i\varphi(x)}$

$\to [\rho(x'), e^{-i\varphi(x)}] = + e^{-i\varphi(x)} \delta(x - x')$

"$[b^\dagger b, b^\dagger] = + b^\dagger$"

$\rho(x) = \left(\rho_0 - \frac{\partial_x\theta}{\pi}\right) \sum_p e^{i2p(\pi\rho_0 x - \theta(x))}$ $\qquad \boxed{p = 0}$

$\left[\rho_0 - \frac{\partial_x\theta(x')}{\pi}, e^{-i\varphi(x)}\right] = e^{-i\varphi(x)} \delta(x - x')$

$\left[\partial_x\theta(x'), e^{-i\varphi(x)}\right] = -\pi e^{-i\varphi(x)} \delta(x - x')$ $\qquad \varphi = x'$

$\partial_x\theta \sim -i\partial_\varphi$ $\qquad \Rightarrow \left[-i\partial_\varphi e^{-i\varphi} - e^{-i\varphi}(-i\partial_\varphi) = -e^{-i\varphi} + e^{-i\varphi}\cancel{(-i\partial_\varphi)} - e^{-i\varphi}\cancel{(-i\partial_\varphi)} = -e^{-i\varphi}\right]$

(1) $\boxed{[\partial_x\theta(x'), \varphi(x)] = -i\pi\,\delta(x - x')}$ $\qquad [\partial_t\varphi(x'), \varphi(x)] = -i\delta(x - x')$

$\boxed{[\theta(x'), \varphi(x)] = -i\pi\,\Theta(x' - x)}$ $\qquad \frac{1}{2}$ Sign

$\Theta(x > 0) = 1$
$\Theta(x < 0) = 0$
$\Theta(0) = 0$ bosons ; $\frac{1}{2}$ fermions

$[e^{i2p\theta(x)}, e^{-i\varphi(x)}] = 0$

$e^{i2p\theta(x')} e^{-i\varphi(x)} \overset{\text{CBH}}{=} e^{i2p\theta(x') - i\varphi(x) + \frac{1}{2} 2p[\theta(x'), \varphi(x)] + \cancel{\dots}} =$
$= e^{-i\varphi(x)} e^{i2p(\theta(x'))} e^{2p[\theta(x'), \varphi(x)]} =$
$= e^{-i\varphi(x)} e^{i2p\theta(x')} e^{-2pi\pi\Theta(x' - x)}$ $\quad \to 1 \;\; x' > x$; $\;\to 0 \;\; x' \leq x$

$[\rho(x'), e^{-i\varphi(x)}] = e^{-i\varphi(x)} \delta(x - x')$ $\qquad e^{-2pi\pi}$ or $e^0 = 1$

(2) CBH $\quad e^A e^B = e^{A + B + \frac{1}{2}[A,B] + \cancel{\dots}} = e^B e^A e^{[A,B]}$ $\qquad [[A,B], A] = 0$

$:e^{i(\alpha\varphi + \beta\theta)}:$: Vertex operators : CFT

$\psi_B(x) = \sqrt{\rho(x)}\, e^{i\varphi(x)}$

$\rho(x) = \left(\rho_0 - \frac{\partial_x\theta}{\pi}\right) \sum_p e^{i2p[\pi\rho_0 x - \theta(x)]}$ $\qquad \delta$

$\boxed{\delta = \sqrt{\delta}}$

$\psi_B(x) = \sqrt{\rho_0 - \frac{\partial_x\theta}{\pi}} \left(\sum_p e^{i2p[\pi\rho_0 x - \theta(x)]}\right) e^{i\varphi(x)}$ $\qquad p = 0$

$\simeq \sqrt{\rho_0 - \frac{\partial_x\theta}{\pi}}\, e^{i\varphi(x)}$

$\boxed{\psi_B(x) \simeq \sqrt{\rho_0}\, e^{i\varphi(x)}}$ $\qquad \boxed{\rho(x) \approx \rho_0 - \frac{\partial_x\theta}{\pi}}$

FERMIONS

$\psi(x) = \sqrt{\rho_0 - \frac{\partial_x\theta}{\pi}} \sum_p e^{i(2p+1)(\pi\rho_0 x - \theta(x))} e^{i\varphi(x)}$

$\{\psi^\dagger(x'), \psi(x)\}$

$\boxed{x' \neq x}$

$= e^{-i\varphi(x')} \sum_{p'} e^{-i(2p'+1)(\pi\rho_0 x' - \theta(x'))} \sqrt{\rho_0 - \frac{\partial_x\theta(x')}{\pi}} \sqrt{\rho_0 - \frac{\partial_x\theta(x)}{\pi}} \sum_p e^{i(2p+1)(\pi\rho_0 x - \theta(x))} e^{i\varphi(x)}$

$+ \sqrt{\rho_0 - \frac{\partial_x\theta(x)}{\pi}} \sum_p e^{i(2p+1)(\pi\rho_0 x - \theta(x))} e^{i\varphi(x)} e^{-i\varphi(x')} \sum_{p'} e^{-i(2p'+1)(\pi\rho_0 x' - \theta(x'))} \sqrt{\rho_0 - \frac{\partial_x\theta(x')}{\pi}}$

$= \dots$

$\sqrt{\rho_0 - \frac{\partial_x\theta(x)}{\pi}}\, e^{-i\varphi(x')} \sum_p e^{i(2p+1)(k_F x - \theta(x))} \underbrace{e^{(2p+1)[\theta(x), \varphi(x')]}}_{e^{-(2p+1)(-i\pi)\Theta(x - x')}} \sum_{p'} e^{-i(2p'+1)(k_F x' - \theta(x')) - (2p'+1)[\varphi(x), \theta(x')]} \underbrace{e}_{e^{+(2p'+1)[\theta(x'), \varphi(x)]}} e^{+i\varphi(x)} \sqrt{\rho_0 - \frac{\partial_x\theta(x')}{\pi}}$

$[\partial_x\theta(x), \varphi(x')] = -i\pi\delta$ $\qquad -i\pi\Theta(x' - x)$

if $x > x'$ $\quad \underbrace{e^{+i(2p+1)\pi}}_{-1} \quad \cdot \quad 1$

if $x < x'$ $\quad 1 \quad \cdot \quad \underbrace{e^{-i\pi(2p'+1)}}_{-1}$

$x = x'$

FERMIONS

$\psi = \sqrt{\rho_0 - \frac{\partial_x\theta}{\pi}} \sum_p e^{i(2p+1)(k_F x - \theta(x))} e^{i\varphi(x)} \simeq$

"RG" $\quad p = 0$ or $p = -1$

$\simeq \sqrt{\rho_0 - \frac{\partial_x\theta}{\pi}}\, e^{ik_F x - i\theta(x)} e^{i\varphi(x)} + \sqrt{\rho_0 - \frac{\partial_x\theta}{\pi}}\, e^{-ik_F x + i\theta(x)} e^{i\varphi(x)}$ + higher $|p|$

$\simeq \sqrt{\rho_0}\, e^{ik_F x - i\theta(x) + i\varphi(x) + \frac{1}{2}[\theta(x), \varphi(x)]} + \sqrt{\rho_0}\, e^{-ik_F x + i\theta(x) + i\varphi(x) - \frac{1}{2}[\theta(x), \varphi(x)]}$

$E \ll \Lambda$, T $\quad \Lambda_{UV}$ $\quad \to \psi = \underbrace{N e^{ik_F x - i\theta(x) + i\varphi(x)}}_{\psi_R,\; +k_F} + \underbrace{N e^{-ik_F x + i\theta(x) + i\varphi(x)}}_{\psi_L,\; -k_F}$ $\qquad R \qquad L$

$\rho(x) = \lim_{x' \to x} \psi^\dagger(x) \psi(x')$ $\qquad [x > x']$

$\psi_R^\dagger\psi_R \qquad \psi_R^\dagger\psi_L \qquad \psi_L^\dagger\psi_L \qquad \psi_L^\dagger\psi_R$

$\rho(x) = N^2 \Big[e^{-ik_F x} e^{i\theta(x) - i\varphi(x)} e^{ik_F x'} e^{-i\theta(x') + i\varphi(x')} +$
$e^{-ik_F x} e^{i\theta(x) - i\varphi(x)} e^{-ik_F x'} e^{i\theta(x') + i\varphi(x')} +$
$e^{ik_F x} e^{-i\theta(x) - i\varphi(x)} e^{-ik_F x'} e^{i\theta(x') + i\varphi(x')} +$
$e^{ik_F x} e^{-i\theta(x) - i\varphi(x)} e^{ik_F x'} e^{-i\theta(x') + i\varphi(x')} \Big] =$

$\lim_{x' \to x}$

$= N^2 \Big[e^{-ik_F(x - x')} e^{i[\theta(x) - \varphi(x) - \theta(x') + \varphi(x')] - \frac{1}{2}[\theta(x), \varphi(x')]} +$ $\qquad (-i\pi)$
$e^{ik_F(x - x')} e^{i[\theta(x) - \varphi(x) + \theta(x') + \varphi(x')] + \frac{1}{2}[\theta(x), \varphi(x')]} +$
$e^{ik_F(x + x')} \dots$ $\qquad e^{i2k_F x}$
$e^{-ik_F(x + x')} \dots \Big]$ $\qquad \Big\}$ F.O.

$\psi_R^\dagger\psi_R$, $\psi_L^\dagger\psi_L$ $\to$ s.e. $\to \rho_R$, ρ_L

$\psi_L^\dagger\psi_R$, $\psi_R^\dagger\psi_L$

Friedel oscilla.

$\int dx \, \langle \rho(x) \rangle$ $\qquad \langle \rho(x)\rho(x') \rangle$ $\qquad$ LL

$\psi_L^\dagger\psi_R \; \psi_R^\dagger\psi_L \to$ slow oscillating

$\lim_{x' \to x}$ $\qquad x' = x - a$ $\quad (\bar{d})$ $\qquad$ 1st Taylor

$\theta(x') \simeq \theta(x) - a\partial_x\theta(x) + \frac{a^2}{2}\cancel{\partial_x^2\theta(x)} + \cancel{\dots}$

$\theta(x) - \theta(x') \simeq +a\partial_x\theta(x) + \dots$

$k_F a$

$= N^2 \left[e^{-ik_F a} e^{i[a\partial_x\theta - a\partial_x\varphi] + i\frac{\pi}{2}} + e^{ik_F a} e^{i[-a\partial_x\theta - a\partial_x\varphi] - i\frac{\pi}{2}} \right] + \text{F.O.}$

$\downarrow$ 1st Taylor for $a \to 0$

$a\partial_x\theta, a\partial_x\varphi \ll 1$

$= N^2 \left[[1 - ik_F a + ia\partial_x\theta - ia\partial_x\varphi](+i) - i[1 + ik_F a - ia\partial_x\theta - ia\partial_x\varphi] \right] + \text{F.O.}$

$= -N^2 \left[-k_F a + a\partial_x\theta - \cancel{a\partial_x\varphi} - k_F a + a\partial_x\theta + \cancel{a\partial_x\varphi} \right] + \text{F.O.} =$

$\psi_R^\dagger\psi_R$: $\partial_x\theta - \partial_x\varphi$ $\qquad \psi_L^\dagger\psi_L$: $\partial_x\theta + \partial_x\varphi$

$= \boxed{N^2} a \left[2k_F - 2\partial_x\theta \right] + \text{F.O.}$ $\qquad N^2$ normalization

$\boxed{N^2 = \frac{1}{2\pi a}} \leftarrow \psi, \psi$

$\rho \simeq \frac{k_F}{\pi} - \frac{\partial_x\theta}{\pi} =$

$= \rho_0 - \frac{\partial_x\theta}{\pi}$

$N = \frac{1}{\sqrt{2\pi a}}$ or $\frac{1}{\sqrt{2\pi\alpha}}$ or $\sqrt{\Lambda}$

$k_F a$ NOT small $\qquad k_F = \frac{\pi}{2a}$

$e^{ik_F a}(1 + i\partial_x\theta \dots)$