BOSONIZATION

- Real Space - Phenomenological Bosonization

$\rho \to \rho = (\rho_0 - \frac{1}{\pi}\partial_x\phi)\sum_p e^{2ip(\pi\rho_0 x - \phi)}$

$\psi_R = \sqrt{\rho}\, e^{i\theta}$

- $[\frac{1}{\pi}\partial_x\theta(x), \phi(x')] = -i\pi\delta(x-x')$ → $[\theta(x), \phi(x')] = -i\pi\Theta(x-x')$

Fermions: $\{\psi(x), \psi^\dagger(x')\} = \delta(x-x')$

$\psi(x) = \boxed{\frac{1}{\sqrt{2\pi a}}}\left[e^{ik_F x} e^{-i(\phi-\theta)} + e^{-ik_F x} e^{i(\phi+\theta)}\right]$

$H_L = -t\sum_i (c^\dagger_{i+1}c_i + H.c.) + H_{int}$

$c_n \to \psi(x)$

$-t\sum_i (c^\dagger_{i+1}c_i + H.c.) \to \int dx\, -t[\psi^\dagger(x+a)\psi(x) + H.c.]$

$\sum_i \to \int \frac{dx}{a}$

$H_L \to H(\phi,\theta) \xrightarrow{\text{Taylor, up to } (\partial_x\phi)^2} H(\phi,\theta)$; $e^{i2k_F x}$ → negligible

1. CBH: $e^A e^B = e^{A+B+\frac{1}{2}[A,B]}$
2. $[\theta(x), \phi(y)] = -i\pi\Theta(x-y)$, $\Theta(x) = \begin{cases} 1 & x>0 \\ 1/2 & x=0 \\ 0 & x<0 \end{cases}$
3. $\langle e^{i\sum_j (\alpha_j\phi(x_j) + \beta_j\theta(x_j))}\rangle = e^{-\frac{1}{2}\langle X^2\rangle}$, $\langle X\rangle = 0$ Thermodynamic limit $L\to\infty$

$\langle e^{i(\alpha\phi(x_1)+\beta\theta(x_2))}\rangle = e^{-\frac{\alpha^2}{2}\langle\phi^2\rangle - \frac{\beta^2}{2}\langle\theta^2\rangle - \alpha\beta\langle\phi(x_1)\theta(x_2)\rangle}$

$\langle\phi\theta\rangle = \text{number} = i\frac{\pi}{2}\Theta(x-x')$

4. $\langle\phi(x,t)\phi(0,0)\rangle \cong -\frac{1}{2K}\ln r$ $\quad r = \sqrt{|x^2 - v^2t^2|}$

$\langle\theta(x,t)\theta(0,0)\rangle \cong -\frac{K}{2}\ln r$ $\quad a \ll r \ll L$; $\Lambda_{max}^{-1} \ll r \ll \Lambda_{min}^{-1}$

K: LUTTINGER

ϕ and θ massless bosonic fields

$K = 1$: non-interacting spinless fermions

Spinless fermions	$K<1$: repulsive int.	
	$K>1$: attractive int	
Bosons	$K=\infty$: free bosons	[$\delta(x-x')$ interaction is absent]
	$K=1$: Hard-core bosons	[$\delta(x-x')$ " is diverging]
	$K<1$: long range repulsive int	

$H_L \to H(\phi,\theta)_{K,v}$

$H = \frac{v}{2\pi}\int dx\, K(\partial_x\phi)^2 + \frac{1}{K}(\partial_x\theta)^2$

Simpler starting point

- $L = \frac{\hbar}{2\pi}\int dx\, \frac{(\partial_t\theta)^2}{v} - v(\partial_x\theta)^2$ real time; v is a velocity scale, v is the Fermi velocity, $K=1$

$\partial_t\frac{\delta\mathcal{L}}{\delta\partial_t\theta} + \partial_x\frac{\delta\mathcal{L}}{\delta\partial_x\theta} = 0$

$\partial_t^2\theta - v^2\partial_x^2\theta = 0 \Rightarrow \boxed{E^2 = v^2k^2}$

$(\partial_t - v\partial_x)(\partial_t + v\partial_x)\theta = 0$

$(\partial_t - v\partial_x)\theta_R = 0$

$(\partial_t + v\partial_x)\theta_L = 0$

QUANTIZATION

$\Pi_\theta = \frac{\partial\mathcal{L}}{\partial(\partial_t\theta)} = \frac{\hbar}{\pi v}\partial_t\theta$

$[\Pi_\theta(x,0), \theta(y,0)] = -i\hbar\delta(x-y)$

$= [\frac{\hbar}{\pi v}\partial_t\theta(x,0), \theta(y,0)]$

ϕ is the DUAL field of θ

We define: $\boxed{\partial_t\theta = v\partial_x\phi\ ;\ \partial_x\theta = \frac{\partial_t\phi}{v}}$ duality

$[\frac{\hbar}{\pi v}v\partial_x\phi(x,0), \theta(y,0)] = -i\hbar\delta(x-y)$

$[\partial_x\phi(x,0), \theta(y,0)] = -i\pi\delta(x-y)$

$= [\phi(x,0), -\partial_x\theta(y,0)]$

Neglecting boundaries

$\int dx[\partial_x\phi(x), \theta(y)] = \int dx\,[\partial_x\phi\cdot\theta - \theta\partial_x\phi] \overset{\text{by parts}}{=} \int dx\, -\phi\partial_x\theta + \partial_x\theta\,\phi$

$= \int dx\,[\phi, -\partial_x\theta]$

$= [\partial_x\theta(y,0), \phi(x,0)] = -i\pi\delta(x-y)$

"Basis change"

- $\phi_R = \phi - \theta$ $\qquad \phi_L = \phi + \theta$
- $(\partial_t + v\partial_x)\phi_R = (\partial_t + v\partial_x)(\phi - \theta) = \partial_t\phi - \cancel{v\partial_x\theta} + \cancel{v\partial_x\phi} - \cancel{\partial_t\theta} = 0$

$\phi_R(x - vt)$

- $(\partial_t - v\partial_x)\phi_L = 0 \Rightarrow \phi_L(x + vt)$

$\psi = \frac{1}{\sqrt{2\pi a}}\left[e^{ik_F x}e^{i\phi_R} + e^{-ik_F x}e^{i\phi_L}\right]$

$H = \int dx\, \Pi_\theta\partial_t\theta - L = \frac{\hbar}{\pi}\int dx\,\frac{(\partial_t\theta)^2}{v} - \frac{\hbar}{2\pi}\int dx\left[\frac{(\partial_t\theta)^2}{v} - v(\partial_x\theta)^2\right] =$

$= \frac{\hbar}{2\pi}\int dx\,\frac{(\partial_t\theta)^2}{v} + v(\partial_x\theta)^2 \overset{\text{Duality}}{=} \frac{\hbar}{2\pi}\int dx\, v(\partial_x\phi)^2 + v(\partial_x\theta)^2$

Π_ϕ : $\partial_t\phi = \frac{\partial H}{\partial\Pi_\phi} \Rightarrow \mathcal{L}(\phi) = \frac{\hbar}{2\pi}\int dx\,\frac{(\partial_t\phi)^2}{v} - v(\partial_x\phi)^2$

$\partial_t^2\phi - \partial_x^2\phi = 0$

$v\partial_t\frac{\partial_x\theta}{\pi} - v\partial_x\frac{\partial_t\theta}{\pi} = 0$

$\partial_t(-\delta\rho) - \partial_x\frac{\partial_t\theta}{\pi} = 0$

$-\partial_t\rho - \partial_x\left(\frac{\partial_t\theta}{\pi}\right) = 0 \Rightarrow$ Continuity equation

$\boxed{j = -\frac{\partial_t\theta}{\pi} = -v\frac{\partial_x\phi}{\pi}}$

$\partial_x\theta$: related to the density

$\partial_x\phi$: proportional to the current

$H = \frac{\hbar}{2\pi}\int dx\,(\partial_x\phi)^2 + (\partial_x\theta)^2$ (current, density)

$\langle j(x)\, j(y)\rangle = \frac{v^2}{\pi^2}\langle\partial_x\phi(x)\,\partial_x\phi(y)\rangle = \frac{v^2}{\pi^2}\partial_x\partial_y\langle\phi(x)\phi(y)\rangle$

CORRELATIONS

$K=1$

$L = \frac{1}{2\pi}\int dx\,\frac{(\partial_t\theta)^2}{v} - v(\partial_x\theta)^2$ $\quad \tau = it$

$-S = -\frac{1}{2\pi}\int dx\, d\tau\,\frac{(\partial_\tau\theta)^2}{v} + v(\partial_x\theta)^2$

$Z = \int D\theta\, \text{Exp}\left[-\frac{1}{2\pi}\int dx\, d\tau\left(\frac{(\partial_\tau\theta)^2}{v} + v(\partial_x\theta)^2\right)\right]$

Momentum space

$\theta(x,\tau) = \int\frac{dk\,d\omega}{2\pi}\, e^{ikx - i\omega\tau}\theta(k,\omega)$

$(\partial_\tau\theta)^2 = \int\frac{dk\,d\omega\,dk'\,d\omega'}{4\pi^2}(-i\omega)e^{ikx-i\omega\tau}\theta(k,\omega)\,e^{ik'x-i\omega'\tau}\theta(k',\omega')(-i\omega')$

$\int d\tau\, dx\,(\partial_\tau\theta)^2 = \int\frac{dk\,d\omega\,dk'\,d\omega'}{4\pi^2}\int d\tau\, dx\; e^{i(k+k')x}e^{-i(\omega+\omega')\tau}\,\theta(k,\omega)\theta(k',\omega)$

$2\pi\delta(k+k')\; 2\pi\delta(\omega+\omega')$

$= \int dk\, d\omega\;\omega^2\,\theta(k,\omega)\theta(-k,-\omega)$

$Z = \int D\theta\,\text{Exp}\left[-\int\frac{dk\,d\omega}{2\pi}\theta(-k,-\omega)\left(\frac{\omega^2}{v} + vk^2\right)\theta(k,\omega)\right]$

$= \left[-\int\frac{dk\,d\omega}{2}\theta(-k,-\omega)\, G^{-1}(k,\omega)\,\theta(k,\omega)\right]$

$\boxed{G(k,\omega)} = \frac{\pi v}{\omega^2 + v^2k^2} = \langle\theta(k,\omega)\theta(-k,-\omega)\rangle = G(-k,-\omega)$

$Z(\eta) = \int D\theta\, D\eta\;\text{Exp}\left[-\int\frac{dk\,d\omega}{2}\theta(\ldots)\left(\frac{\omega^2}{v}+vk^2\right)\theta(\ldots) + \eta(-k)\theta(k)\right]$

$\langle\theta(k)\theta(k')\rangle = \partial_{\eta(k)}\partial_{\eta(-k')}\, Z(\eta)\Big|_{\eta\to 0}$

$\int\frac{dk\,d\omega}{2\pi}\int\frac{dk'\,d\omega'}{2\pi}\, G(k,\omega) \to G(k,k',\omega,\omega')\,\delta(k-k')\,\delta(\omega-\omega')$

$\langle\theta(x,\tau)\theta(y,0)\rangle = \int\frac{dk\,d\omega}{4\pi^2}\, G(k,\omega)\, e^{i\omega\tau - ik(x-y)} =$

$= \int\frac{dk\,d\omega}{4\pi^2}\,\frac{\pi v}{\omega^2 + v^2k^2}\, e^{i\omega\tau - ik(x-y)}$

$d\omega = dp_0\, v \qquad \vec{p} = (\frac{\omega}{v}, -k) \qquad \vec{r} = (v\tau, x-y)$

$= \int\frac{d^2p}{4\pi}\,\frac{1}{p^2}\, e^{i\vec{p}\cdot\vec{r}} = \int_0^\infty\frac{p\,dp\,d\alpha}{4\pi}\,\frac{1}{p^2}\, e^{ipr\cos\alpha} =$

$= \int_0^\infty\frac{dp}{2}\,\frac{1}{p}\, J_0(pr) \to \int_{\Lambda_{min}}^{\Lambda_{max}}\frac{dp}{2p}\, J_0(pr)$

$\Lambda_{min}, \Lambda_{max}$ $\quad \Lambda_{max}^{-1} \ll r < \Lambda_{min}^{-1}$

$\simeq -\frac{1}{4}\ln\left(\frac{r^2 + \Lambda_{max}^{-2}}{\Lambda_{min}^{-2}}\right) \simeq -\frac{1}{2}\ln r - \frac{1}{2}\ln\Lambda_{min}$

$r = \sqrt{(x-y)^2 + v^2\tau^2} = \sqrt{(x-y)^2 - v^2t^2}$ $\quad \tau = it$

$\langle\theta(x,t)\theta(y,0)\rangle \simeq -\frac{1}{2}\ln r - \frac{1}{2}\ln\Lambda_{min}$

$\langle\phi(x,t)\phi(y,0)\rangle \simeq -\frac{1}{2}\ln r - \frac{1}{2}\ln\Lambda_{min}$

$\boxed{\langle\theta^2\rangle = \int_{\Lambda_{min}}^{\Lambda_{max}}\frac{dp}{2p} = \frac{1}{2}\ln\frac{\Lambda_{max}}{\Lambda_{min}}}$

$\langle\psi_L^\dagger(x)\,\psi_L(y,0)\rangle = \langle\frac{1}{2\pi a}e^{ik_Fx}e^{-i(\phi+\theta)(x)}\, e^{-ik_Fy}e^{+i(\phi+\theta)(y,0)}\rangle$

$= \frac{e^{ik_F(x-y)}}{2\pi a}\langle e^{-i(\phi(x)+\theta(x))}e^{i(\phi(y)+\theta(y))}\rangle =$

$-i\pi\Theta(x-y) \quad +i\pi\Theta(y-x)$

$= \frac{e^{ik_F(x-y)}}{2\pi a}\langle e^{-i\phi(x)+i\phi(y)-i\theta(x)+i\theta(y)+\frac{1}{2}[\theta(x),\phi(y)]+\frac{1}{2}[\phi(x),\theta(y)]}\rangle$

$\boxed{x>y}$ $\quad \langle\phi\rangle = \langle\theta\rangle = 0$

$= -i\frac{e^{ik_F(x-y)}}{2\pi a}\langle e^{-i\phi(x)+i\phi(y)-i\theta(x)+i\theta(y)}\rangle =$

$= \ldots e^{-\frac{1}{2}\langle\phi^2(x)\rangle - \frac{1}{2}\langle\phi^2(y)\rangle - \frac{1}{2}\langle\theta^2(x)\rangle - \frac{1}{2}\langle\theta^2(y)\rangle + \frac{2}{2}\langle\phi(x)\phi(y)\rangle + \langle\theta(x)\theta(y)\rangle} =$

$= -i\frac{e^{ik_F(x-y)}}{2\pi a}\, e^{-\cancel{\frac{1}{2}}\cdot\cancel{2}\ln|x-y| - \cancel{\ln\Lambda_{min}} - \ln\frac{\Lambda_{max}}{\cancel{\Lambda_{min}}}} = -i\frac{e^{ik_F(x-y)}}{2\pi a}\cdot\frac{1}{|x-y|}\, e^{-\ln\Lambda_{max}} =$

$\Lambda_{max} \sim \frac{1}{a}$

$= -i\frac{e^{ik_F(x-y)}}{2\pi\cancel{a}}\;\cancel{a}\;\frac{1}{|x-y|}$

$H_L \to H(\phi,\theta)$

$\boxed{H = \int dx\, A(\partial_x\phi)^2 + B(\partial_x\theta)^2 = \frac{\hbar}{2\pi}\int dx\, vK(\partial_x\phi)^2 + \frac{v}{K}(\partial_x\theta)^2}$

$\boxed{\phi' = \sqrt{K}\phi \qquad \theta' = \frac{\theta}{\sqrt{K}}}$

$[\theta'(y), \phi'(x)] = [\theta(y), \phi(x)]$ ⇒ canonical tr.

$H' = \frac{\hbar}{2\pi}\int dx\, v(\partial_x\phi')^2 + (\partial_x\theta')^2 v$

$[\partial_x\theta', \phi'] = -i\pi\delta$

$[\Pi_{\phi'}, \phi'] = -i\hbar\delta$

$\partial_t\phi' = \frac{\partial H'}{\partial\Pi_{\phi'}}$

$\boxed{\mathcal{L} = \frac{\hbar}{2\pi}\int dx\,\frac{(\partial_t\phi')^2}{v} - (\partial_x\phi')^2 v}$

$\Rightarrow \langle\phi'(x)\phi'(y)\rangle = -\frac{1}{2}\ln r \quad \cancel{-\frac{1}{2}\ln\Lambda_{min}}$

$= K\langle\phi(x)\phi(y)\rangle = -\frac{1}{2}\ln r \Rightarrow \boxed{\langle\phi(x)\phi(y)\rangle \cong -\frac{1}{2K}\ln r}$

$\boxed{\langle\theta(x)\theta(y)\rangle = -\frac{K}{2}\ln r}$

$\cos\beta\phi$

SPIN [Fermions]

$\psi_\uparrow, \psi_\downarrow \quad \to \psi_s$

ϕ_s, θ_s

$\psi_s = \frac{K_s}{\sqrt{2\pi a}}\left[e^{ik_Fx}e^{-i(\phi_s-\theta_s)} + e^{-ik_Fx}e^{-i(\phi_s+\theta_s)}\right]$

$[\partial\theta_s(x), \phi_{s'}(x)] = -i\pi\delta_{ss'}\delta(y-x) \Rightarrow \{\psi_s(x), \psi_{s'}^\dagger(y)\} = \delta(x-y)$

$[\theta_s, \theta_{s'}] = 0 = [\phi_s, \phi_{s'}]$

I want to impose $\{\psi_s(x), \psi^\dagger_{s'}(y)\} = \delta_{ss'}\delta(x-y)$

Without K_s ⇒ $[\psi_\uparrow(x), \psi_\downarrow(y)] = 0$ ⇒ wrong

$K_\uparrow, K_\downarrow$

⇒ fulfill the Clifford algebra

$\{K_i, K_j\} = 2\delta_{ij} \qquad (K_i)^2 = 1 \qquad \boxed{K_\uparrow K_\downarrow = -K_\downarrow K_\uparrow}$

Majorana modes $\qquad \{\psi_\uparrow, \psi_\downarrow\} = 0$, $\{\psi_\uparrow, \psi_\downarrow^\dagger\} = 0$

Symmetry between ↑ and ↓

$H = \sum_s\frac{v_s}{2\pi}\int dx\left[K(\partial_x\phi_s)^2 + \frac{(\partial_x\theta_s)^2}{K}\right] + \int dx\left[g_\phi(\partial_x\phi_\uparrow)(\partial_x\phi_\downarrow) + g_\theta(\partial_x\theta_\uparrow)(\partial_x\theta_\downarrow)\right]$

$= \int dx\,(\partial_x\phi_\uparrow\,\partial_x\phi_\downarrow\,\partial_x\theta_\uparrow\,\partial_x\theta_\downarrow)\begin{pmatrix}\frac{Kv_F}{2\pi} & g_\phi/2 & & \\ g_\phi/2 & \frac{Kv_F}{2\pi} & & \\ & & \frac{v_F}{2\pi K} & g_\theta/2 \\ & & g_\theta/2 & \frac{v_F}{2\pi K}\end{pmatrix}\begin{pmatrix}\partial_x\phi_\uparrow \\ \partial_x\phi_\downarrow \\ \partial_x\theta_\uparrow \\ \partial_x\theta_\downarrow\end{pmatrix}$

$\theta_\rho = \frac{\theta_\uparrow + \theta_\downarrow}{\sqrt{2}} \qquad \theta_\sigma = \frac{\theta_\uparrow - \theta_\downarrow}{\sqrt{2}}$

$\phi_\rho = \frac{\phi_\uparrow + \phi_\downarrow}{\sqrt{2}} \qquad \phi_\sigma = \frac{\phi_\uparrow - \phi_\downarrow}{\sqrt{2}}$

$[\partial_x\theta_a, \phi_b] = i\pi\delta_{ab}\delta(x-y) \qquad a,b = \rho,\sigma$

$H = \sum_{a=\rho,\sigma}\frac{v_a}{2\pi}\int dx\left[(\partial_x\phi_a)^2K_a + \frac{(\partial_x\theta_a)^2}{K_a}\right] = \int dx\,(\partial_x\phi_\rho\,\partial_x\phi_\sigma\,\partial_x\theta_\rho\,\partial_x\theta_\sigma)\begin{pmatrix}\frac{v_\rho K_\rho}{2\pi} & & & \\ & \frac{v_\sigma K_\sigma}{2\pi} & & \\ & & \frac{v_\rho}{K_\rho 2\pi} & \\ & & & \frac{v_\sigma}{2\pi K_\sigma}\end{pmatrix}\begin{pmatrix}\partial_x\phi_\rho \\ \partial_x\phi_\sigma \\ \partial_x\theta_\rho \\ \partial_x\theta_\sigma\end{pmatrix}$

$\frac{v_\rho K_\rho}{2\pi} = \frac{v_FK}{2\pi} + \frac{g_\phi}{2} \qquad \frac{v_\sigma K_\sigma}{2\pi} = \frac{v_FK}{2\pi} - \frac{g_\phi}{2}$

$\frac{v_\rho}{2\pi K_\rho} = \frac{v_F}{2\pi K} + \frac{g_\theta}{2} \qquad \frac{v_\sigma}{2\pi K_\sigma} = \frac{v_F}{2\pi K} - \frac{g_\theta}{2}$

$\rho_{TOT} \cong \left[\rho_{0\uparrow} + \rho_{0\downarrow} - \frac{\partial_x\theta_\uparrow}{\pi} - \frac{\partial_x\theta_\downarrow}{\pi}\right] = \rho_{TOT,0} - \sqrt{2}\frac{\partial_x\theta_\rho}{\pi}$, $\quad \rho_{TOT,0} = \frac{N_\uparrow + N_\downarrow}{L}$

"=" $c^\dagger_\uparrow c_\uparrow + c^\dagger_\downarrow c_\downarrow$

$\psi^\dagger_\uparrow\psi_\uparrow + \psi^\dagger_\downarrow\psi_\downarrow$

$j_{charge} \propto \partial_x\phi_\uparrow + \partial_x\phi_\downarrow \sim \sqrt{2}\phi_\rho$

Spin density

$\sigma = \rho_{0\uparrow} - \rho_{0\downarrow} - \frac{\partial_x\theta_\uparrow}{\pi} + \frac{\partial_x\theta_\downarrow}{\pi} = \rho_{0\uparrow} - \rho_{0\downarrow} - \sqrt{2}\frac{\partial_x\theta_\sigma}{\pi}$

$j_{spin} \sim \sqrt{2}\frac{\partial_x\phi_\sigma}{\pi} \sim j_\uparrow - j_\downarrow$

$\boxed{K = K^\dagger}$

$K^2 = 1$

$\int dx\;\psi^{\dagger L}_\uparrow\psi^L_\downarrow = \int\boxed{K_\uparrow K_\downarrow}\, e^{-i(\phi_\uparrow+\theta_\uparrow)+i(\phi_\downarrow+\theta_\downarrow)}$ $+H.c.$

$K_\uparrow K_\downarrow \qquad K_\uparrow K_\downarrow\, e^{-i(\phi_\uparrow+\theta_\uparrow-\phi_\downarrow-\theta_\downarrow)}$

$\psi^\dagger_\downarrow\psi_\uparrow \qquad K_\downarrow K_\uparrow\, e^{-i(\phi_\downarrow+\theta_\downarrow+\phi_\uparrow+\theta_\uparrow)}$

$-K_\uparrow K_\downarrow$

$iK_\uparrow K_\downarrow\;\sin(\phi_\uparrow+\theta_\uparrow-\phi_\downarrow-\theta_\downarrow)$

$-K_\uparrow K_\downarrow K_\uparrow K_\downarrow = K_\uparrow^2K_\downarrow^2 = 1$

$+\sin^2$

$(-1)^{c^\dagger_\downarrow c_\downarrow} = \tau_\downarrow$

S_L

$K_\uparrow = (-1)^{N_\downarrow}$

$K_\downarrow = 1$

$\psi_\downarrow, \psi_\downarrow^\dagger$

$\{\psi_\uparrow, \psi_\downarrow\}$