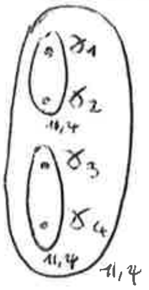
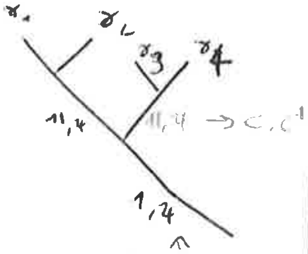


$=$



pair:

$\gamma_1 \times \gamma_2 = 1\!\!1 + \psi$

$\left.\begin{array}{l} 1\!\!1 \to |0\rangle \\ \psi \to |1\rangle \end{array}\right\}$ Fermionic number

II quantiz.

$c, c^\dagger$ / $c|0\rangle = 0$, $c^\dagger|0\rangle = |1\rangle$

$c = \dfrac{\gamma_1 - i\gamma_2}{2}$ $\qquad c^\dagger = \dfrac{\gamma_1 + i\gamma_2}{2}$

$\{c, c^\dagger\} = 1$, $\{\gamma_i, \gamma_j\} = 2\delta_{ij}$

$c^\dagger c|0\rangle = 0$

$c^\dagger c|1\rangle = |1\rangle$

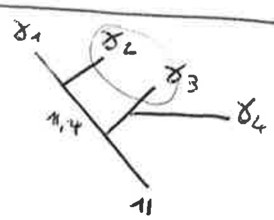
$c^\dagger c = \dfrac{1 - i\gamma_1\gamma_2}{2}$

$i\gamma_1\gamma_2 = \left\{\begin{array}{l} +1 \\ -1 \end{array}\right.$ $\Big\}$ Fermionic parity

---

2 pairs

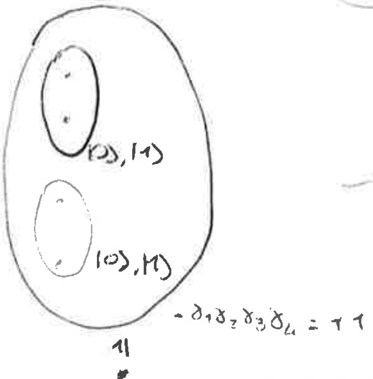
$(i\gamma_1\gamma_2)(i\gamma_3\gamma_4) = \pm 1$



States : $|00\rangle$, $|11\rangle$

$c_1 = \dfrac{\gamma_1 - i\gamma_2}{2}$

$c_2 = \dfrac{\gamma_3 - i\gamma_4}{2}$



$$i\gamma_1\gamma_2 = \begin{array}{c} |00\rangle \ \ |11\rangle \\ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{array} = \sigma_z \qquad i\gamma_2\gamma_3 = ?$$

$$i\gamma_1\gamma_2 \left[i\gamma_2\gamma_3 |00\rangle \right] = -i\gamma_2\gamma_3 \, i\gamma_1\gamma_2 |00\rangle = -i\gamma_2\gamma_3|00\rangle$$

$$\Rightarrow \; i\gamma_2\gamma_3 |00\rangle = |11\rangle \quad \Rightarrow \quad i\gamma_2\gamma_3|11\rangle = |00\rangle \quad \Rightarrow \quad i\gamma_2\gamma_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sigma_x$$

$$F = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \frac{1}{\sqrt{2}} \quad / \quad F^\dagger \, i\gamma_1\gamma_2 \, F = i\gamma_2\gamma_3 \quad \Rightarrow \quad \boxed{F^{\gamma\gamma\gamma}_{\gamma} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \frac{1}{\sqrt{2}}}$$

Braidings ?

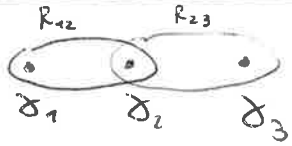
$\gamma \times \gamma = 1\!\!1 + \psi$

$$\left(R^{1\!\!1}_{\gamma\gamma}\right)^2 = e^{i(\overset{0}{\Theta_{1\!\!1}} - 2\Theta_\gamma)}$$

$$\left(R^{\psi}_{\gamma\gamma}\right)^2 = e^{i(\Theta_\psi - 2\Theta_\gamma)}$$

$\Theta_\psi = \pi$ [Fermion rotation of 2π ... spin 1/2 !]

$$\Rightarrow \; R^2 \propto \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} = e^{i\frac{\pi}{2}\sigma_z} \quad \Rightarrow \quad R \propto e^{i\frac{\pi}{4}\sigma_z}$$



$$R_{23} = F^\dagger R_{12} F = e^{i\frac{\pi}{4}\sigma_x} = e^{i\frac{\pi}{4}(i\gamma_2\gamma_3)}$$

$(R_{23})^2 = \sigma_x = \text{NOT}$ $\rightarrow$ experimental signature

R_{12} and R_{23} are rotations of $\frac{\pi}{2}$ $\Rightarrow$ they generate the cube group!

$\rightarrow$ closed group not dense in $SO(3)$ or $SO(2)$

KITAEV CHAIN

p-wave Hamiltonian:

$$H = +\sum_x \mu\, c_x^\dagger c_x - \frac{t}{2}\sum_x \left(c_x^\dagger c_{x+1} + c_{x+1}^\dagger c_x\right) - \sum_x \left(\frac{\tilde{\Delta}}{2} c_x^\dagger c_{x+1}^\dagger + \frac{\tilde{\Delta}^*}{2} c_{x+1} c_x\right)$$ H

$U(1)$ symmetry is broken $\rightarrow$ charge is conserved only mod 2

$\Downarrow$

$\mathbb{Z}_2$ symm: $c_x \rightarrow -c_x \quad \forall x$

If $\tilde{\Delta} = \Delta e^{i\varphi}$ we apply a gauge tr: $c_x^\dagger \rightarrow e^{i\varphi/2} c_x^\dagger$ Gauge

$$\Rightarrow H = +\sum_x \mu\, c_x^\dagger c_x - \frac{t}{2}\sum_x \left(c_x^\dagger c_{x+1} + c_{x+1}^\dagger c_x\right) - \frac{\Delta}{2}\sum_x \left(c_x^\dagger c_{x+1}^\dagger + c_{x+1} c_x\right)$$

We study the case $\Delta = t$

$$\Rightarrow H = +\sum_x \mu\, c_x^\dagger c_x - \frac{i\Delta}{2}\sum_x \Big[\underbrace{i\left(c_x^\dagger - c_x\right)}_{\gamma_{2x}} \underbrace{\left(c_{x+1} + c_{x+1}^\dagger\right)}_{\gamma_{2x+1}}\Big]$$

$$\gamma_{2x-1} = c_x + c_x^\dagger \qquad \gamma_{2x} = i\left(c_x - c_x^\dagger\right)$$ Majo

$$\gamma = \gamma^\dagger \quad , \quad \{\gamma_i, \gamma_j\} = 2\delta_{ij}$$

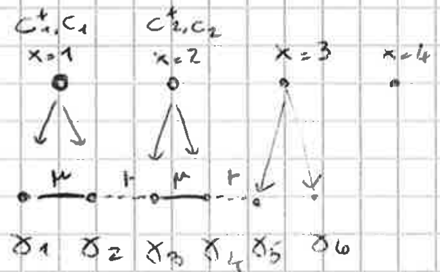
$$c_x = \frac{\gamma_{2x-1} - i\gamma_{2x}}{2} \qquad c_x^\dagger = \frac{\gamma_{2x-1} + i\gamma_{2x}}{2}$$

$$c_x^\dagger c_x = \frac{1 - i\gamma_{2x-1}\gamma_{2x}}{2} \quad \Rightarrow \quad i\gamma_{2x-1}\gamma_{2x} = (-1)^{c_x^\dagger c_x}$$

$\boxed{\text{Ferm. parity}}$

$$H = -\frac{\mu}{2}\sum_x i\gamma_{2x-1}\gamma_{2x} - \frac{t}{2}\sum_x i\gamma_{2x}\gamma_{2x+1}$$

$\longrightarrow$ Staggered configuration



This is the Ising model [via Jordan - Wigner] : $\boxed{JW}$

$$\begin{cases} \gamma_{2r-1} = \sigma_{x,r} \prod_{j<r} \sigma_{z,j} \\ \gamma_{2r} = i\sigma_{z,r}\sigma_{x,r} \prod_{j<r} \sigma_{z,j} \end{cases} \rightarrow \quad \gamma = \gamma^{\dagger} \quad \{\gamma_i, \gamma_j\} = 2\delta_{ij}$$

$$i\gamma_{2r-1}\gamma_{2r} = \sigma_{z,r}$$

$$i\gamma_{2r}\gamma_{2r+1} = \sigma_{x,r}\,\sigma_{x,r+1}$$

$$\Rightarrow H = -i\frac{\mu}{2}\sum_r \gamma_{2r-1}\gamma_{2r} - i\frac{t}{2}\sum_r \gamma_{2r}\gamma_{2r+1} \Rightarrow H = -\frac{\mu}{2}\sum_r \sigma_{z,r} - \frac{t}{2}\sum_r \sigma_{x,r}\sigma_{x,r+1}$$

Majorana $\qquad\qquad$ $\boxed{\text{Ising}}$

Fermionic Parity $\prod_r i\gamma_{2r-1}\gamma_{2r} \longleftarrow$ Global symm: $\prod_r \sigma_{z,r}$ $\boxed{\text{ISING}}$ Sym

$$\parallel$$
$$(-1)^F$$

$\langle\sigma_z\rangle = 0$, GS = $|+++\rangle \pm |---\rangle$

Phases

Ferromagnetic phase : Broken Symmetry and degeneracy for $\frac{\mu}{t} \to 0$

Paramagnetic phase : Symmetric phase ; No degeneracy $\langle\sigma_z\rangle \neq 0$

- -

Paramagnetic phase may be understood for $\frac{t}{\mu} \to 0$

$\Rightarrow \quad i\gamma_1\gamma_2 = 1 \quad \dots \quad i\gamma_{2r-1}\gamma_{2r} = 1$

$[c_r^{\dagger}c_r = 0]$

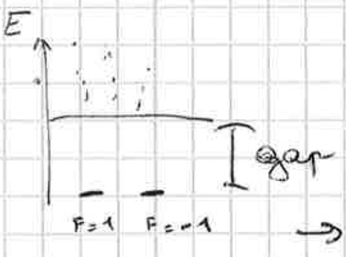
Ferromagnetic phase : $\rightarrow$ TOPOLOGICAL PHASE

Zero Modes

$\Rightarrow$ Uncoupled Majorana modes for $\mu \to 0$

$i\gamma_1\gamma_{2L}$ specify the GS $\quad$ [γ_1 and γ_8 are sym which do not commute]

Protection



$\rightarrow$ Protection given by fermionic parity

No operator which conserves the F may allow transitions

No quadratic operator in γ's ! No local measurement can distinguish F !!

TOPOLOGICAL PHASE $\mu \ll t$

$$\overset{\mu}{\cdots}\quad\overset{t}{\text{---}}\quad\overset{\mu}{\cdots}\quad\overset{t}{\text{---}}\quad\cdots$$

$\gamma_1 \quad \gamma_2 \quad \gamma_3 \quad \gamma_4 \quad \gamma_5 \quad \gamma_6$

$$-i\frac{\mu}{2}\gamma_1\gamma_2 - i\frac{t}{2}\gamma_2\gamma_3 - i\frac{\mu}{2}\gamma_3\gamma_4 - i\frac{t}{2}\gamma_4\gamma_5 \;\cdots$$

Zero Mode ?

$[\Gamma, H] = 0$?

$$[\gamma_1, H] = -i\mu\gamma_2$$

$$[\gamma_2, H] = +i\mu\gamma_1 - it\gamma_3$$

$$[\gamma_3, H] = it\gamma_2 - i\mu\gamma_4$$

$\vdots$

$$[\gamma_{2n-1}, H] = it\gamma_{2n-2} - i\mu\gamma_{2n}$$

$$[\gamma_{2n}, H] = i\mu\gamma_{2n-1} - it\gamma_{2n+1}$$

$$\left[\gamma_1 + \frac{\mu}{t}\gamma_3, H\right] = -i\frac{\mu^2}{t}\gamma_4$$

$$+\frac{\mu^2}{t^2}\gamma_5 \qquad = -i\frac{\mu^3}{t^2}\gamma_6$$

$$\Gamma_L = \sum_{n=0}\left(\frac{\mu}{t}\right)^n\gamma_{2n+1} \qquad \Gamma_L^2 = \sum_{n=0}^{N}\left(\frac{\mu}{t}\right)^{2n}$$

$$\Rightarrow [\Gamma_L, H] = \left(\frac{\mu}{t}\right)^{N-1}\mu\,\gamma_{2N} \quad \rightarrow \text{decays exp with } N$$

Γ_L is localized $= \sum_{n=0} e^{-\frac{n}{\xi}}\gamma_{2n+1}$ $\qquad \boxed{\xi^{-1} = \ln\frac{t}{\mu}}$

The same is true on the right side :

$$\Gamma_R = \sum_{n=0}\gamma_{2N-2n}\left(\frac{\mu}{t}\right)^n$$

$\cancel{\{\Gamma_L, \Gamma_R\}} \propto$ $\qquad \{\Gamma_L, \Gamma_R\} = 0$ once normalized

$\rightarrow$ In the thermodynamic limit they are zero-energy Majorana modes

Effective $H = i e^{-\frac{N}{\xi}}\Gamma_L\Gamma_R$

DISORDER in the TOPOLOGICAL PHASE :

$\langle \mu \rangle < \langle t \rangle$

$-i\frac{\mu_1}{2}\gamma_1\gamma_2 - i\frac{t_1}{2}\gamma_2\gamma_3 - i\mu_2\gamma_3\gamma_4 \dots$

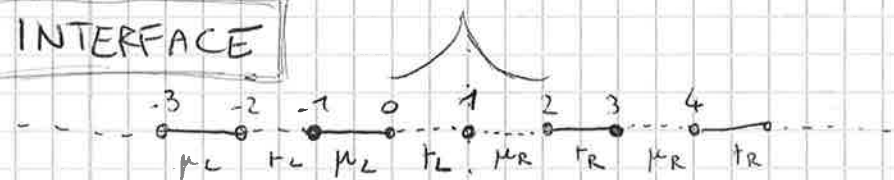
I am adding a disorder that conserves the fermionic parity and it is quadratic in $c, c^\dagger$

$$\Rightarrow \Gamma_L = \sum_{m=0} \left[\prod_{i=1}^{m} \left(\frac{\mu_i}{t_i}\right)\right]\gamma_{2m+1}$$

If $\Delta \neq t$ another two-Majorana term appears : $\propto \boxed{\Delta - t \; \gamma_{2x-1}\gamma_{2x+2}}$

→ still the modes survive

INTERFACE



$t_L < \mu_L$ TRIVIAL

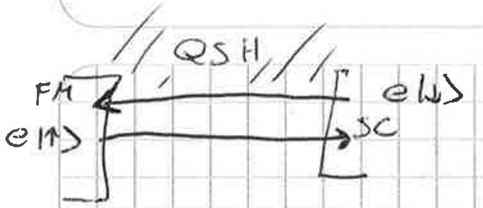
$t_R > \mu_R$ TOPOL.

$$[\gamma_1, H] = i t^L \gamma_0 - i \mu^R \gamma_2$$

$$\Gamma = \cancel{\gamma_0}^{\,?} + \sum_{\gamma < 1} \left(\frac{t^L}{\mu^L}\right)^{\gamma} \gamma_{2\gamma-1} + \sum_{\gamma \geq 1} \left(\frac{\mu^R}{t^R}\right)^{\gamma} \gamma_{2\gamma+1}$$

Two different length-scales : $\left(\ln \frac{\mu^R}{t^R}\right)^{-1}, \left(\ln \frac{t^L}{\mu^L}\right)^{-1}$

REALIZATION IN SC - QSH



$$\mathcal{H}_0 = v\left(i\psi_L^\dagger \partial_x \psi_L - i\psi_R^\dagger \partial_x \psi_R\right)$$

$T \to$ Backscattering : $+M\psi_R^\dagger \psi_L + h.c.$

$U(1)$ Charge Conserv $\to$ S.C : $-\Delta \psi_R^\dagger \psi_L^\dagger + h.c.$

Basis : $(\psi_\uparrow, \psi_\downarrow, -\psi_\downarrow^\dagger, \psi_\uparrow^\dagger)$ $\qquad [\Gamma, H] = H_{BdG}\Gamma$

$$H_{BdG} = iv\partial_x \sigma_z \tau_z + \Delta \tau_x + M\sigma_x \qquad [+, \tau_z]$$

$$H_{BdG} = \frac{1}{2}\begin{pmatrix} iv\partial_x & M & \Delta & 0 \\ M & -iv\partial_x & 0 & \Delta \\ \Delta & 0 & -iv\partial_x & M \\ 0 & \Delta & M & iv\partial_x \end{pmatrix}$$

- ~~Right~~: $M = 0$, $\Delta(x) = \Delta$ for $x > 0$

  Zero modes : $\begin{pmatrix} iv\partial_x & \Delta \\ \Delta & -iv\partial_x \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 0$

- $a = -i e^{-\int \frac{\Delta}{v} dx}$
  $b = e^{-\int \frac{\Delta}{v} dx}$ $\Rightarrow \Gamma_R = -i\psi_\uparrow e^{-\int} - \psi_\downarrow^\dagger e^{-\int}$

  $\Gamma_R^\dagger = i\psi_\uparrow^\dagger e^{-\int} - \psi_\downarrow e^{-\int}$

- ~~Right~~ : $M(x) = M \quad \Delta(x) = 0$

  $\begin{pmatrix} iv\partial_x & M \\ M & -iv\partial_x \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 0 \qquad a = i e^{\int M/v\, dx}$

  $b = e^{\int M/v\, dx}$

  $\Gamma_L = i\psi_\uparrow e^{\int} + \psi_\downarrow e^{\int}$

  $\Gamma_L^\dagger = -i\psi_\uparrow^\dagger e^{\int} + \psi_\downarrow^\dagger e^{\int}$

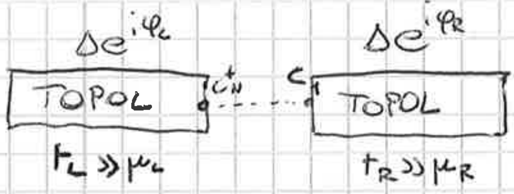
- In $x = 0$

$$\alpha \Gamma_R + \beta \Gamma_R^\dagger = \gamma \Gamma_L + \delta \Gamma_L^\dagger$$

$$\alpha\begin{pmatrix} -i \\ 0 \\ 1 \\ 0 \end{pmatrix} + \beta\begin{pmatrix} 0 \\ -1 \\ 0 \\ i \end{pmatrix} = \gamma\begin{pmatrix} i \\ 1 \\ 0 \\ 0 \end{pmatrix} + \delta\begin{pmatrix} 0 \\ 0 \\ -1 \\ -i \end{pmatrix}$$

Only one mode $\quad \Gamma_R + \Gamma_R^\dagger + \Gamma_L + \Gamma_L^\dagger = \Gamma = \Gamma^\dagger$

Localized Majorana zero mode



$H_{int} = \frac{J}{2} C^+_{N,L} C_{1,R} + h.c.$

$J \ll t_L, t_R$

- In the usual case the tunneling of e is suppressed by Δ
- In this case, beside the Cooper Pair tunneling, we have:

$$C_N^+ = \frac{e^{i\varphi_L/2}}{2}\left(\gamma_{2N-1} + i\gamma_{2N}\right) \qquad C_1 = \frac{e^{-i\varphi_R/2}}{2}\left(\gamma_1 - i\gamma_2\right)$$

Perturbatively $[J \ll t]$

$\delta\varphi_{SC} = \Delta B/\phi_0 \qquad \phi_0 = \frac{hc}{2e}$

$$\frac{J}{2} C_N^+ C_1 + h.c. = \frac{J}{4}\cos\Big(\overbrace{\frac{\varphi_L - \varphi_R}{2}}^{\delta\varphi_{SC}}\Big)\ \underbrace{i\gamma_{2N}\gamma_1}_{\text{Parity of the junction!}}$$

$\rightarrow 4\pi$ periodicity

- H is 2π periodic in φ_L and φ_R but the GS is not!

The same can be seen with bosonization

$\psi_L =: e^{i(\varphi - \theta)}: \qquad \psi_R =: e^{i(\varphi+\theta)}: \qquad \mathcal{H}_0 = \frac{v}{2\pi}\int dx\, (\partial_x\varphi)^2 + (\partial_x\theta)^2$

$-\Delta\, \psi_R^+\psi_L^+ + h.c. = -\Delta\cos 2\varphi$

$-\Delta e^{i\varphi_L}\psi_R^+\psi_L^+ + h.c. = -\Delta\cos(2\varphi - \varphi_L)$

$\Rightarrow \varphi_1 \approx \frac{\varphi_L}{2} \qquad \varphi_2 \approx \frac{\varphi_R}{2}$

If I vary $\varphi_R \to \varphi_R + 2\pi \Rightarrow \varphi_2 \to \varphi_2 + \pi$ and $\mathcal{H}_0$ changes

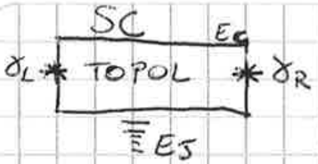
$\Rightarrow 4\pi$ periodicity

CHARGING ENERGY

$\tau - i\gamma_L\gamma_R$

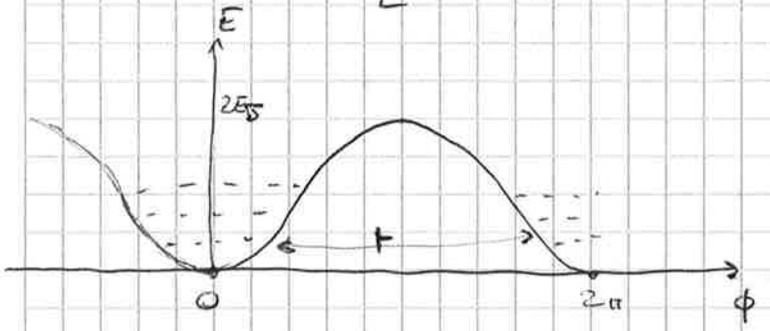
$c^\dagger c$ Majorana

N° Cooper pairs



$$H = E_c\left(2q_{cc} - q_M - q_{ind}\right)^2 + E_J(1-\cos\phi)$$

$$[q_{cc}, \phi] = i$$



$$t \overset{WKB}{=} e^{i\int_0^{2\pi} q_{cc}(\phi)\, d\phi}$$

$$q_{cc}(\phi) = \sqrt{\frac{E_J}{4E_c}(\cos\phi - 1)} + \frac{q_k}{2} = i\sqrt{\frac{E_J}{4E_c}(1-\cos\phi)} + \frac{q_k}{2}$$

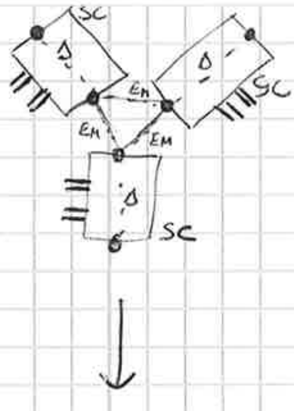
$$t = \mathrm{Exp}\left[-\int_0^{2\pi}\frac{E_J}{4E_c}(1-\cos\phi)\, d\phi + i\frac{q_k}{2}\right] =$$

$$= \mathrm{Exp}\left[-\sqrt{\frac{8E_J}{E_c}} + i\frac{q_k}{2}2\pi\right]$$

$$\Rightarrow H_{eff} \propto t + t^\dagger \propto e^{-\sqrt{\frac{8E_J}{E_c}}}\cos\pi q_k = e^{-\sqrt{\frac{8E_J}{E_c}}}\, i\gamma_L\gamma_R$$

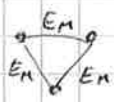
Split J.J : $E_J = E_{J,0}\cos\Phi_B \rightarrow$ tunable

$\rightarrow$ Mesoscopic Kitaev Chain



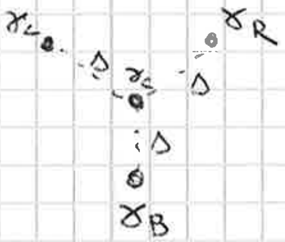
If $E_M \gg \Delta$ we do perturbation theory.

In particular :



$\longrightarrow \quad \propto H \propto E_M c^+ c + O \cdot \gamma_c$

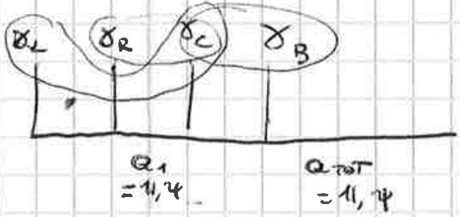
$$\gamma_c = \frac{\gamma_1 + \gamma_2 + \gamma_3}{\sqrt{3}}$$



$$H_{tri} = -\left(\Delta_L i\gamma_L\gamma_c + \Delta_R i\gamma_R\gamma_c + \Delta_B i\gamma_B\gamma_c\right)$$

$\Delta_i(\Phi_i)$ tunable exponentially

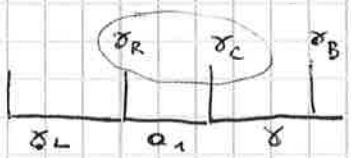
This H_{tri} encode the braiding : [we must choose a good basis]



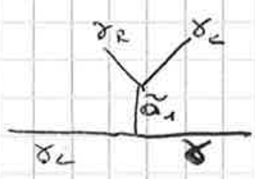
$$\boxed{\gamma \times \gamma = 1\!\!1 + \psi}$$

We write H_{tri} in terms of projectors over $1\!\!1$

$$\frac{-\Delta_R i\gamma_R\gamma_c + \Delta_R}{2} = \Delta_R c_R^+ c_R = \Delta_R \Pi_\psi^{RC} = \Delta_R(1 - \Pi_{1\!\!1}^{RC})$$



$= \left(F_\gamma^{\gamma_L\gamma_R\gamma_c}\right)_{\tilde{a}_1 a_1}$



$$\Pi_{1\!\!1}^{RC} = \left(F_\gamma^{\gamma_L\gamma_R\gamma_c}\right)^{-1}_{a_1'\tilde{a}_1} \delta_{\tilde{a}_1 1\!\!1} \left(F_\gamma^{\gamma_L\gamma_R\gamma_c}\right)_{\tilde{a}_1 a_1} = \left(F_\gamma^{\gamma\gamma\gamma}\right)^{-1}_{a_1', 1\!\!1} \left(F_\gamma^{\gamma\gamma\gamma}\right)_{1\!\!1, a_1}$$

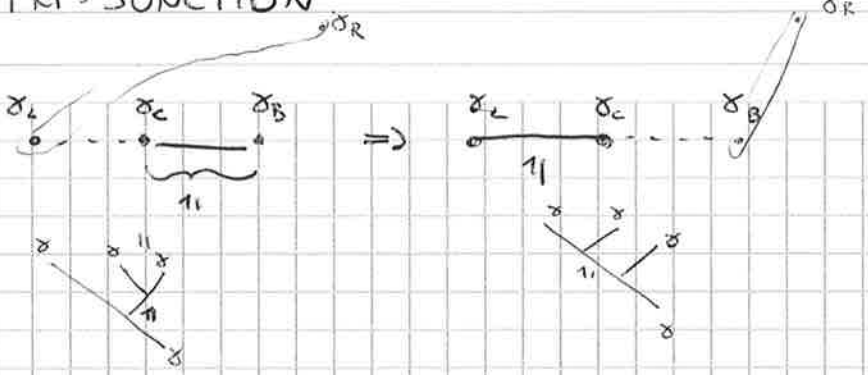
$$\Pi_{1\!\!1}^{LC} = \left(R_{LR}^{a_1'}\right)^{-1} \Pi^{RC}_{1\!\!1\, a_1', a_1} \left(R_{LR}^{a_1}\right)$$

$\Pi_{1\!\!1}^{BC} = \delta_{a_1, a_{TOT}} \quad \rightarrow$ it commutes with R_{LR}

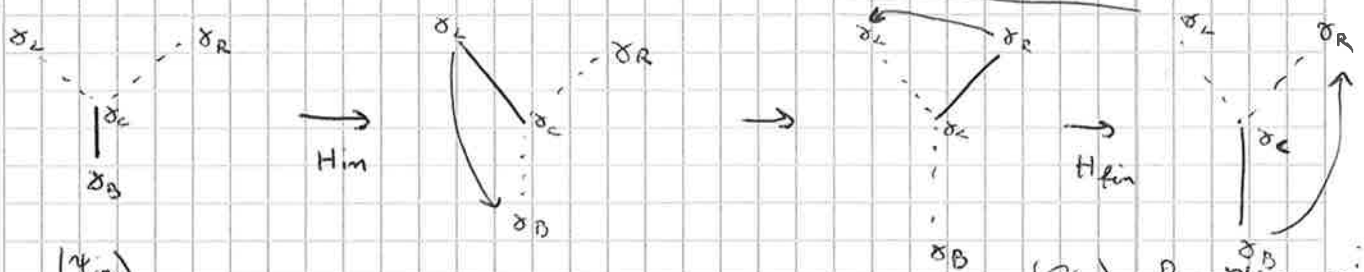
$$\left.\begin{aligned} H_{in} &= -\Delta_L \Pi_{1\!\!1}^{LC} - \Delta_B \Pi_{1\!\!1}^{CB} \\ H_{fin} &= -\Delta_R \Pi_{1\!\!1}^{RC} - \Delta_B \Pi_{1\!\!1}^{CB} \end{aligned}\right\} \quad H_{in} = R_{LR}^{-1} H_{fin} R_{LR}$$

If $|\psi\rangle_{in}$ is a GS of H_{in}
then $|\psi\rangle_{out} = R|\psi_{in}\rangle$ is a GS of H_{fin}

TRI - JUNCTION

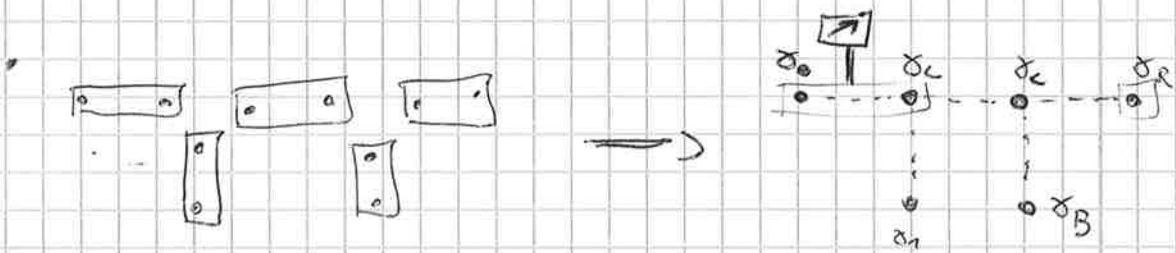


A Majorana was transferred from L to B without moving them

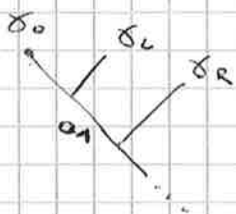


$|\psi_{in}\rangle$ $\qquad$ $|\psi_{out}\rangle = R_{LR}|\psi_{in}\rangle e^{i\theta}$

DETECTION : Π - JUNCTION



All tunable



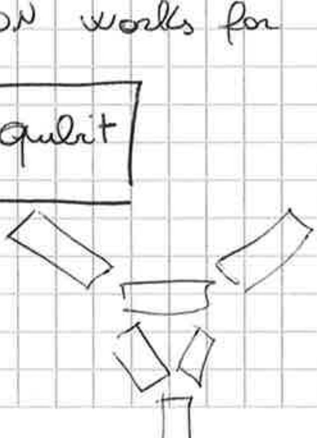
$$R_{LR} = e^{i\frac{\pi}{4}\sigma_x}$$

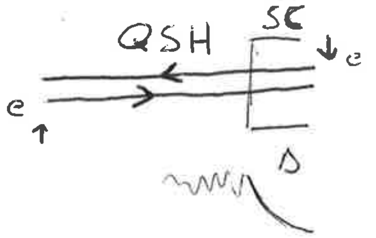
$$R_{LR}^2 \propto \sigma_x$$

$$|a_1 = 0\rangle \xrightarrow{R^{LR}} \frac{|0\rangle - i|1\rangle}{\sqrt{2}} \xrightarrow{R^{LR}} -i|1\rangle \xrightarrow{R^{LR}} \frac{-|0\rangle - i|1\rangle}{\sqrt{2}} \xrightarrow{R^{LR}} |0\rangle$$

The Π - JUNCTION works for U_{LR} only ...

$\boxed{\text{Topological qubit}}$





$H_0 = v\left(i\, \psi_L^\dagger \partial_x \psi_L - i\, \psi_R^\dagger \partial_x \psi_R\right)$

T $\rightarrow$ Backscattering $-M\, \psi_R^\dagger \psi_L + h.c.$

Charge Conservation $\rightarrow$ S.C. $-\Delta\, \psi_R^\dagger \psi_L^\dagger + h.c.$

$(\psi_\uparrow, \psi_\downarrow, -\psi_\downarrow, \psi_\uparrow)$

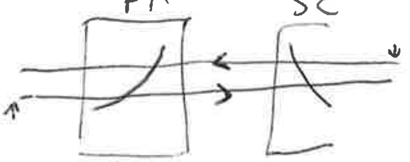
$H_{BdG} = iv\partial_x \cdot \sigma_z \cdot \tau_z + \Delta \tau_x + M\sigma_x$ $\quad H = \begin{pmatrix} +iv\partial_x & & \Delta & \\ & -iv\partial_x & & \Delta \\ \Delta & & -iv\partial_x & \\ & \Delta & & +iv\partial_x \end{pmatrix} \rightarrow \begin{pmatrix} iv\partial_x & \Delta \\ \Delta & -iv\partial_x \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}$

$H_{BDG}\,\Gamma_0 = 0$ $\rightarrow$ Zero mode localized in SC [but not out of it] $\Rightarrow$ $a = i e^{-\int \frac{\Delta}{v} dx}$, $b = e^{-\int \frac{\Delta}{v} dx}$

$H_{BDG}\,\Gamma_0^\dagger = 0$

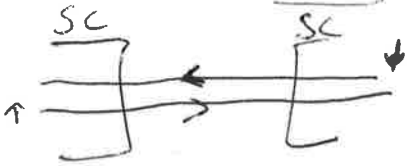
To fully localize:

$H_2 = \left[iv\partial_x \cdot \sigma_z + M\sigma_x\right]$



$\rightarrow$ Majorana mode!

Fu - Kane



$\rightarrow$ responsible for fractional J.

Experimental: Knez, Du, Sullivan PRL 2012 [Rui Rui Du, Houston...]

$$H_0 = \frac{v}{2\pi}\int (\partial_x\varphi)^2 + (\partial_x\theta)^2$$

φ, θ dual $\quad \partial_x\varphi = \partial_t\theta$, $\quad \partial_t\varphi = \partial_x\theta$

$[\varphi_{x_1}, \theta_{x_2}] = i\pi\,\Theta(x_2 - x_1)$

$\boxed{\text{INTEGER}}$

φ: massless scalar field $\quad (\partial_t^2\varphi) - \partial_x^2\varphi = 0$

H invariant for $\varphi \rightarrow \varphi + a$

$\rho = \dfrac{\partial_t\varphi}{\pi} = \dfrac{\partial_x\theta}{\pi}$ $\qquad j = S = \dfrac{\partial_x\varphi}{\pi}$

$\varphi_L = \varphi - \theta$

$\varphi_R = \varphi + \theta$

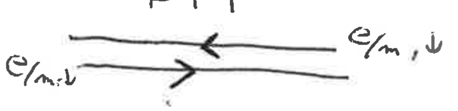
$\psi_L = :e^{i(\varphi-\theta)}:$ $\qquad \psi_R = :e^{i(\varphi+\theta)}:$

$\{\psi_{L/R}, \psi_{L/R}\} = 0$

$-M\,\psi_R^\dagger\psi_L + h.c. \rightarrow -M\cos 2\theta$

$-\Delta\,\psi_R^\dagger\psi_L^\dagger + h.c. \rightarrow -\Delta\cos 2\varphi$

FTI



FTI $\rightarrow$ 2 copies of Laughlin $\nu = 1/m$

m integer odd!!

$\rho = \dfrac{\partial_x\theta}{\pi}$

$$H_0 = \frac{mv}{2\pi}\int (\partial_x\varphi)^2 + (\partial_x\theta)^2 \qquad [\varphi(x_1), \theta(x_2)] = i\frac{\pi}{m}\,\Theta(x_2 - x_1)$$

$\psi_L = e^{im(\varphi-\theta)}$ $\qquad V_L = e^{i(\varphi-\theta)}$

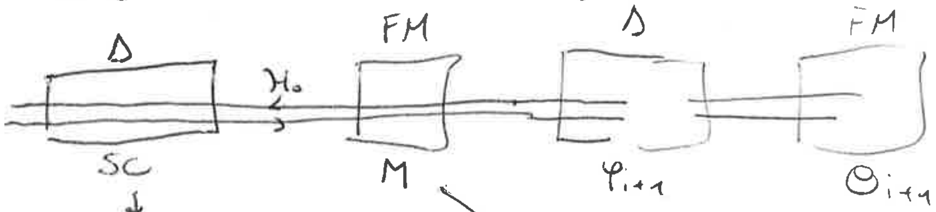
$\psi_R = e^{im(\varphi+\theta)}$ $\qquad V_R = e^{i(\varphi+\theta)}$

$$V_L(x_1)\, V_L(x_2) = V_L(x_2) V_L(x_1)\, e^{\,i\frac{\pi}{m}\,\mathrm{sign}(x_2-x_1)}$$

$$V_R(x_1)\, V_R(x_2) = V_R(x_2) V_R(x_1)\, e^{-i\frac{\pi}{m}\,\mathrm{sign}(x_2-x_1)}$$

2

$$\mathcal{H}_I = -\int \left(\Delta \cos 2m\varphi + M\cos 2m\Theta\right)$$



pinning φ pinning Θ

$\varphi_i = 0, \frac{\pi}{m} \dots \frac{2m-1}{m}\pi$ $\Theta_i = 0, \frac{\pi}{m} \dots \frac{2m-1}{m}\pi$

→ Shtengel etc...

$$\alpha_L \simeq e^{i(\varphi_i - \Theta_i)} \qquad \alpha_R \simeq e^{i(\varphi_i + \Theta_i)}$$

$$\alpha^{2m} = 1 \qquad \alpha^{+} = \alpha^{-1}$$

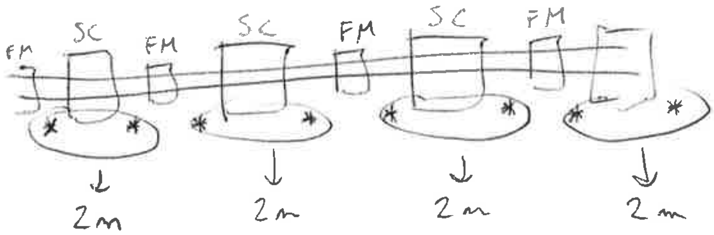
$$\alpha_{L,i}\, \alpha_{L,j} = e^{+i\frac{\pi}{m}\,\mathrm{sign}(j-i)}\, \alpha_{L,j}\, \alpha_{L,i}$$

⟶ Parafermionic zero modes !!

$$e^{-i\frac{\pi}{2m}}\, \alpha^{+}_{L,2i+1}\, \alpha_{L,2i} = e^{i(\Theta_{i+1}-\Theta_i)} = e^{i\pi q_i} \qquad \boxed{SC}$$

$$e^{-i\frac{\pi}{2m}}\, \alpha^{+}_{2i}\, \alpha_{2i-1} = e^{i(\varphi_{i+1}-\varphi_i)} = e^{i\pi S_i} \qquad \text{FM (JJ)}$$

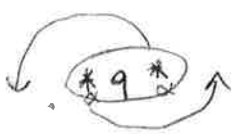
$$\left[\alpha^{+}_{L,2i+1}\, \alpha_{L,2i} = \alpha^{+}_{R,2i}\, \alpha_{R\,2i+1}\right]$$



$$\dim \mathcal{H} = (2m)^{N/2}$$

$$d_\alpha = \sqrt{2m}$$

Braiding



$$= R_{\pi,q}\, R^{-2}_{\pi,\alpha}$$

ind on q

$\frac{qe}{m}$ $\phi = q\frac{hc}{e}$ $q^2 \frac{\pi}{2m}$

$$R^{q}_{\alpha\alpha} = e^{iq^2\frac{\pi}{2m}} = e^{i(q-q_0)^2\frac{\pi}{2m}}$$

$$\alpha \times \alpha = \psi_0 + \dots + \psi_{2m-1}$$

$$\psi_0 = 1$$

$$\psi_i \times \psi_j = \psi_{i+j \bmod 2m}$$

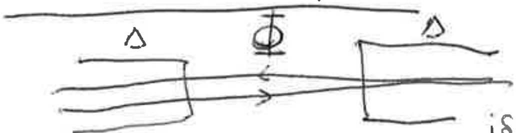
$$\alpha \times \psi_i = \alpha$$

③

① Overall Berry Abelian phase is not defined

② Objects bound to domain walls $\rightarrow$ Projective Non-Abelian Stat. [genons]

Fractional Josephson



$- \Delta \cos 2m\varphi_i$

$\Delta e^{i\delta_{es}} \psi^+ \psi^+ + h.c.$

$- \Delta \cos(2m\varphi_{i+1} - \delta_{sc})$

$\varphi_{i+1} = \frac{\delta_{sc}}{2m}$

If I adiabatically vary δ_{sc} I acquire energy

$\delta \rightarrow \delta + 2\pi$

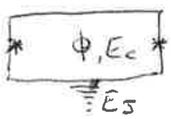
$\Rightarrow \varphi_{i+1} \rightarrow \varphi_{i+1} + \frac{\pi}{m}$

periodicity $4\pi m$ $(2\pi \cdot 2m)$

This is the tunneling of a fractional charge e/m

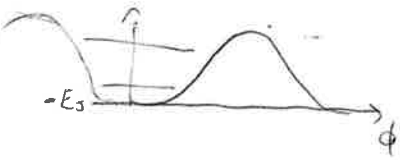
$H_J = \cancel{e^{i\frac{\pi}{2m}+} \alpha_{2i} \alpha_{2i-1}} - \frac{J}{2} e^{-i\left(\frac{\pi - \delta}{2m}\right)} \alpha^+_{2i} \alpha_{2i-1} + h.c.$

Charging Energy



$H = (N - q_K - q_{ind})^2 E_c - E_J \cos\phi$ $\qquad [N, \phi] = i$

$E_J \gg E_c$ $\qquad \Delta \propto e^{-\sqrt{\frac{8E_J}{E_c}}}$



$\updownarrow \sqrt{8E_cE_J}$ $\longrightarrow$ $H_c = -\Delta \cos \pi (q_K + \cancel{q_{ind}})$

$H_c = -\frac{\Delta}{2} e^{-i\frac{\pi}{2m}} \alpha^+_{2i+1} \alpha_{2i} + h.c.$

- Fendley's Chain

$H = -\frac{\Delta}{2} \sum_{sc} e^{-i\frac{\pi}{2m}} \alpha^+_{2K+1} \alpha_{2K} - \frac{J}{2} \sum_J e^{-i\frac{\pi}{2m}} \alpha^+_{2K} \alpha_{2K-1} + h.c.$

JW : $\alpha_{2K} = \sigma_{K+1} \prod \tau_K$

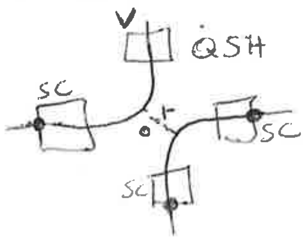
$\alpha_{2K+1} = e^{i\frac{\pi}{2m}} \tau_{K+1} \sigma_{K+1} \prod \tau_K$

4

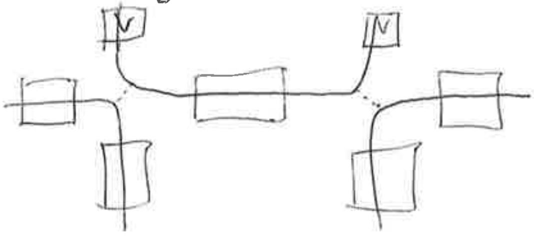
Computation	Majorana	PF
Single qbit	$e^{i\pi/4\,\sigma_z}$, $e^{i\pi/4\,\sigma_x}$ ↳ Clifford group We miss phase gate $\begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix}$	$U = e^{i\frac{\pi}{2m}q^2} \rightarrow$ reducible $= U_{Majorana} \otimes U_{something\ else}$ $q = m q_\sigma + 2 q_v$
CNOT	Not scalable with braidings only We require multi-qubit measurements (4-Majorana)	Possible with braidings only?
	Braiding + measurements + distillation is universal	Braidings + Measurements?

Braiding : Shuo, Pikulin .. PRB

Braider



Π junction



Artificial Non-Abelian Gauge Potentials

① Potentials ≠ not fields

② Non Abelian gauge : we aim at studying a minimal coupling $H = (p+A)^2$ [...]

③ Artificial : how to engineer that ? Berry matrices !

example spin orbit

Starting point $H = p^2 + V(x)$

↳ $n \times n$ potential for the inner degree of freedom

$$V(x) = \begin{pmatrix} 0 & \Omega_1 & \Omega_2 & \Omega_3 \\ \Omega_1^* & & & \\ \Omega_2^* & & 0 & \\ \Omega_3^* & & & \end{pmatrix}$$

$$V(x)\begin{pmatrix} 0 \\ U_1 \\ U_2 \\ U_3 \end{pmatrix} = 0$$

→ two dark states

For each position x we consider four eigenstates of V $|U^a(x)\rangle$ as a basis

$$\Psi(x) = \sum_{a=1}^{4} \alpha_a(x)\,|U^a(x)\rangle \quad / \quad V(x)\,|U^a(x)\rangle = \varepsilon^a(x)\,|U^a(x)\rangle$$

$$\Rightarrow \exists\, R(x) \;/\; R^\dagger V R = \begin{pmatrix} \varepsilon_1 & & & \\ & \varepsilon_2 & & \\ & & \varepsilon_3 & \\ & & & \varepsilon_4 \end{pmatrix}(x)$$

Kinetic energy

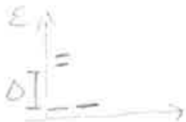
$$p\,\Psi(x) = -i\partial_x \left[\sum_{a=1}^{4} \varphi_a(x)\,|U^a(x)\rangle\right]$$

$$\langle U^b|p|\Psi(x)\rangle = \sum_a \langle U^b| \left[(-i\partial_x\varphi_a(x))|U_a\rangle + \varphi_a(x)(-i\partial_x|U_a\rangle)\right] =$$

$$= \sum_a \delta_{ba}\left(-i\partial_x\varphi_a(x)\right) + \varphi_a(x)\underbrace{\langle U_b|-i\partial_x U_a\rangle}_{-A_{ba}} = (p-A)\varphi(x)$$

$$H = p^2 + V(x) \;\rightarrow\; R^\dagger H R = (p-A)^2 + \begin{pmatrix} \varepsilon_1(x) & \\ & \varepsilon_4(x) \end{pmatrix} \qquad R\text{: unitary}$$

Adiabatic Approximation :



$\Rightarrow$ I project into the 2×2 [...] span at energy

$$(p-A)^2 + \Phi \qquad \Phi_{mn} = -\sum_{e=q+1}^{N} \langle U_m|\vec{\partial}_x U_e\rangle\langle U_e|\vec{\partial}_x U_n\rangle$$

$$\vec{A}_{me}\cdot\vec{A}_{en}$$

$$\begin{cases} \Omega_1 = \sin\theta\cos\phi\, e^{iS_1} \\ \Omega_2 = \sin\theta\sin\phi\, e^{iS_2} \\ \Omega_3 = \cos\theta\, e^{iS_3} \end{cases} \quad \text{Supposing} \quad \Rightarrow \quad \begin{aligned} A_{11} &= \cos^2\phi\,\nabla S_{23} + \sin^2\phi\,\nabla S_{13} \\ A_{12} &= \cos\theta\left[\tfrac{1}{2}\sin 2\phi\,\nabla S_{12} - i\nabla\phi\right] \\ A_{22} &= \cos^2\theta\left(\cos^2\phi\,\nabla S_{12} + \sin^2\phi\,\nabla S_{13}\right) \end{aligned}$$

Magnetic field through rotation

$$U(t) = e^{-it\Omega\cdot L}$$

$$H_0 \to H_0 - \Omega L \quad \text{with} \quad H_0 = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 r^2 \quad \Rightarrow \frac{1}{2m}(p_x - \omega m y)^2 + (p_y + [\ldots])^2 + \frac{1}{2}m[\ldots]$$

Gauge transformation $U(x)$ $\psi \to U\psi$

$\Rightarrow$ $A \to UAU^{+} + i\,U(\nabla U^{+})$

$\Phi \to U \Phi U^{+}$

Criterion for non-Abelianity:

① $[A_\mu(x), A_\nu(x')] \neq 0$ $\to$ not gauge invariant

$A = 0$ $\to$ crazy U

② Field. $F^{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\nu, A_\mu]$

$B = \vec{\nabla} \times \vec{A} + i\,\vec{A} \times \vec{A}$

a) It does not define the dynamics

b) $B \to U B U^{+}$!!

c) "Stokes theorem is a disaster"

$A = \sigma_z \frac{B}{2}(y, -x)$ $\downarrow$ $F = -B\sigma_z$	$A = \sqrt{\frac{B}{2}}\,(\sigma_x, \sigma_y)$ $\downarrow$ $F = -B\sigma_z$
σ_z is conserved by $(p-A)^2$ $\to$ 2 species ... $U(1) \times U(1)$	Truly $SU(2)$ [Rashba]

F is translational invariant: only these 2 potentials in $SU(2)$

③ I need to understand the physics

$= W$

Both have W translationally invariant

$W_1 = \begin{pmatrix} e^{i\Phi} & \\ & e^{-i\Phi} \end{pmatrix}$

$W_2 = e^{iq\sigma_x} e^{iq\sigma_y} e^{-iq\sigma_x} e^{-iq\sigma_y}$

$|\mathrm{Tr}\, W| < 2$ [For $U(1)$ $|\mathrm{Tr}| = 2$]

④ Commutation of the Wilson loops

$\forall x, \forall C_1 C_2$, $[W(C_1), W(C_2)] = 0$ as the theory is Abelian!

$$H = (p_x + A_x)^2 + (p_y + A_y)^2 \qquad A_x = -\frac{B}{2}y + q\sigma_x$$

$$A_y = \frac{B}{2}x + q\sigma_y$$

$B_z = B$ - [illegible]

$$\Pi_x = p_x - \frac{B}{2}y \qquad [\Pi_x, \Pi_y] = iB \qquad d = 2\Pi_y + 2i\Pi_x$$

$$\Pi_y = p_y + \frac{B}{2}x \qquad\qquad d^\dagger = 2\Pi_y - 2i\Pi_x$$

$$[d, d^\dagger] = 8B$$

$$H = \left(\frac{1}{4}d^\dagger d + B + 2q^2\right)\mathbb{1} + q\begin{pmatrix} & -id \\ id^\dagger & \end{pmatrix} \quad \to \text{Jaynes-Cummings}$$

$$\psi_{nm} = \frac{i^n d^{\dagger n} z^m \psi_0}{\sqrt{n!\, m!\, (2B)^m}} \qquad \psi_0 = e^{-\frac{B}{4}|z|^2}$$

$$H_{m} \begin{pmatrix} \psi_{m,m} \\ \psi_{m+1,m} \end{pmatrix}$$

$$H_{nm} = 2B\left(n + \frac{1}{2}\right) + 2q^2 + \begin{pmatrix} & q\sqrt{8Bn} \\ q\sqrt{8Bn} & 2B \end{pmatrix}$$

$$E_{nm} = 2Bn + 2q^2 \pm \sqrt{B^2 + 8Bnq^2}$$

$$\chi^{\pm}_{n,m} = \alpha_{\pm}(q^2/B)\, \psi_{n,m}|\uparrow\rangle + \beta_{\pm}(q^2/B)\, \psi_{n+1,m}|\downarrow\rangle$$

$$\psi_{0,m}|\downarrow\rangle$$