4

Maxwell

4.1 Introduction

War es ein Gott, der diese Zeichen schrieb?[1] So asks Boltzmann, quoting from Goethe, in an epigraph to his lectures on Maxwell's theory. In the nineteenth century section of the physicists' pantheon, Maxwell's rank remains the highest. The tribute is well deserved. Maxwell wrote the field equations which still form the basis of our understanding of electromagnetism. He subsumed optics under electromagnetism. He founded statistical physics. He created a new style of theoretical physics. As the first director of the Cavendish Laboratory, he contributed to the increasing sophistication of British experimental physics.

Glorification, however, tends to obscure the true nature of Maxwell's achievements. It was not a god who wrote these signs, but a man who had gone through two of the best British universities and had carefully studied Faraday and Thomson for himself. His electromagnetism and his style of physics, innovative though they were, owed much to Thomson, who had already transformed British physics in an even more significant manner and had defined basic concepts and new perspectives of electromagnetism. The heroic account also deforms Maxwell's results. His electrodynamics differed from today's 'Maxwell's theory' in several respects, as basic as the distinction between source and field. It was not a closed system, and it included suggestions for future electromagnetic research. In the present chapter, we will approach this more authentic Maxwell.

4.1.1 Scot and wrangler

James Clerk Maxwell was, like Thomson, a Cambridge graduate first trained in a Scottish university. Despite a seven-year difference in age, the two men's approaches to physics had deep similarities. They both lent a central role to geometry in the expression of mathematical and physical ideas. Following their Scottish professors (John Nichol for Thomson, and James Forbes for Maxwell), they held a broad view of physics, including the full range of experimental subjects and technical

[1] 'Was it a god who wrote these signs?' Boltzmann 1891–1893, Vol. 1: 96, from the introductory monologue of Goethe's *Faust*.

engineering problems. At the same time, they shared the mathematical virtuosity cultivated in the Cambridge Tripos, and had an eye for deeper theory as promoted by John Herschel and William Whewell. By drawing formal analogies between various branches of physics, they combined Baconian diversity and Newtonian unity.[2]

There were, however, perceptible nuances between Thomson's and Maxwell's research styles. Maxwell's involvement in technical, practical matters was less than Thomson's, while his interest in geometry was more sustained and diverse than Thomson's. Following the Clerk family's artistic bent, Maxwell was fascinated by the beauty of geometrical figures. After William Hamilton and Immanuel Kant, he regarded space and time as necessary forms of our intuition of phenomena. His interests and skills in philosophy and literature were exceptionally high for a British scientist. Unlike Thomson, he accompanied his use of dynamical analogies with sophisticated philosophical comment. He wrote good poetry, and brilliantly discussed moral philosophy for the Cambridge Apostles. Lastly, there was an essential psychological difference between Maxwell and Thomson. As an enthusiastic prodigy, Thomson launched essential ideas in numerous concise papers, but rarely found time for their full exploitation or for global syntheses. Maxwell was slower and more dependent on other physicists' innovations, but he could persevere several years on the same subject and erect lofty monuments.[3]

Maxwell first learned electricity and magnetism from James Forbes at Edinburgh University. Forbes adopted an empirical approach, and ignored French or German mathematical fluid theories. Maxwell was still free of theoretical prejudice when in February 1854 he asked his pen-friend William Thomson: 'Suppose a man to have a popular knowledge of electrical show experiments and a little antipathy to Murphy's Electricity [the British rendering of Poisson's electrostatics], how ought he to proceed in reading & working so as to get a little insight into the subject which may be of use in further reading?' Thomson's reply is lost. We know, however, that Maxwell read Faraday and Thomson first, then Ampère and Kirchhoff, and lastly Neumann and Weber. Thus, the young Maxwell assimilated Faraday's field conceptions and developed a distaste for continental theories. He later explained to Faraday: 'It is because I put off reading about electricity till I could do without prejudice, that I think I have been able to get hold of some of your ideas, such as the electro-tonic state, action of contiguous parts &c.'[4]

[2] On Maxwell's biography, cf. Campbell and Garnett 1882; Everitt 1975. For the relative effects of Maxwell's Scottish and Cambridge backgrounds, cf. Wilson 1985, Siegel 1991, and Harman 1998. On Cambridge's Mathematical Tripos, cf. Wilson 1982; Warwick [1999].

[3] On Maxwell and geometry, cf. Harman 1990: 2–3; Harman 1995a: 20–2, 28–9; Harman 1998: 13–15. On Maxwell, Scottish common sense, and Kant, cf. Harman 1985b, 1998: 27–36, and Hendry 1986. For the psychological comparison between Maxwell and Thomson, cf. Everitt 1975: 59–60.

[4] Maxwell to Thomson, 20 February 1854, *MSLP* 1: 237; Maxwell to Faraday, 19 October 1861, *MSLP* 1: 688. When he wrote to Thomson on 13 November 1854 (*MSLP* 1: 262), Maxwell had read Ampère and Kirchhoff, but not Neumann and Weber. He had read Thomson 1849–1850 (mathematical theory of magnetism) before his letter of February 1854 (cf. Harman 1998: 72–3). His interest in electricity was unusual for a Cambridge student, for this subject had been excluded from the Tripos curriculum some years before.

4.2 On Faraday's lines of force

4.2.1 Gridding the field

Before the end of 1854, Maxwell reported substantial progress to Thomson. Following Faraday, he defined the lines of force as the lines everywhere tangent to the force acting on a pole or point charge. Following Gauss and Thomson, he also introduced the surfaces normal to these lines, that is, the equipotentials. His first innovation was to consider simultaneously the lines and the surfaces and to regulate their spacing, in order to allow quantitative geometrical reasoning (Fig. 4.1). He had used similar space-gridding a few months earlier in a discussion of surface folding, and all his previous works involved the geometry of lines or surfaces.[5] In the electric or magnetic context, he required that the potential difference between two successive equipotentials should be a constant. On a given equipotential surface he drew two systems of curves defining cells with a size inversely proportional to the intensity of the electric or magnetic force, and then traced the tubes of force passing through these cells. The tubes played the same role as Faraday's unit lines of force.[6]

Maxwell expressed Faraday's law of electromagnetic induction in terms of the tubes. In the case of a closed circuit, the induced electromotive force depends on the decrease of the number of tubes passing through it. In mathematically precise terms, *the induced electromotive force around a circuit is equal to the decrease of the surface integral of magnetic force across any surface bounded by the circuit.* Maxwell immediately applied the law to a simple analytical case, the induction of currents in a conducting sphere rotating in the magnetic field of the Earth. Twenty years after the discovery of electromagnetic induction, he was the first theorist to take Faraday so seriously as to give a mathematical expression of his induction law.[7]

Maxwell's geometrical representation also helped him reformulate the relation between an electric current and the resulting magnetic field. In his vision, a current in a closed circuit determined a series of equipotentials bounded by the circuit (Fig. 4.2). The number of these equipotentials was a natural geometrical characteristic

[5] Maxwell to Thomson, 13 November 1854, *MSLP* 1: 258; Maxwell [1854a]: 252. Maxwell was aware of Thomson's theory of magnetization (1849–50), which introduced lamellar and tubular analysis of the distributions of magnetism in magnets. In my reconstruction, I assume that Maxwell had the line–surface gridding before he considered the relation between current and magnetic force. The essential point, however, is that he *simultaneously* considered the potential theory of magnetism and Faraday's lines of force.

[6] Originally, Maxwell spoke of lines of polarization instead of tubes of force. On the genesis and meaning of 'On Faraday's lines of force' I have found much inspiration in Norton Wise's insightful paper on 'the mutual embrace' (Wise 1979). Particularly important are his comments on Maxwell's field-geometrical method and on the role of the intensity/quantity distinction.

[7] Maxwell to Thomson, 13 November 1854, *MSLP* 1: 260; *ibid.*: 260–1, and Maxwell 1862: 226–9 for the rotating sphere. At that stage Maxwell did not have yet the distinction between force and flux (intensity and quantity). He used the term 'polarization' (which I have replaced with 'magnetic force') 'to express the fact that at a point of space the south pole of a small magnet is attracted in a certain direction with a certain force' (*MSLP* 1: 256).

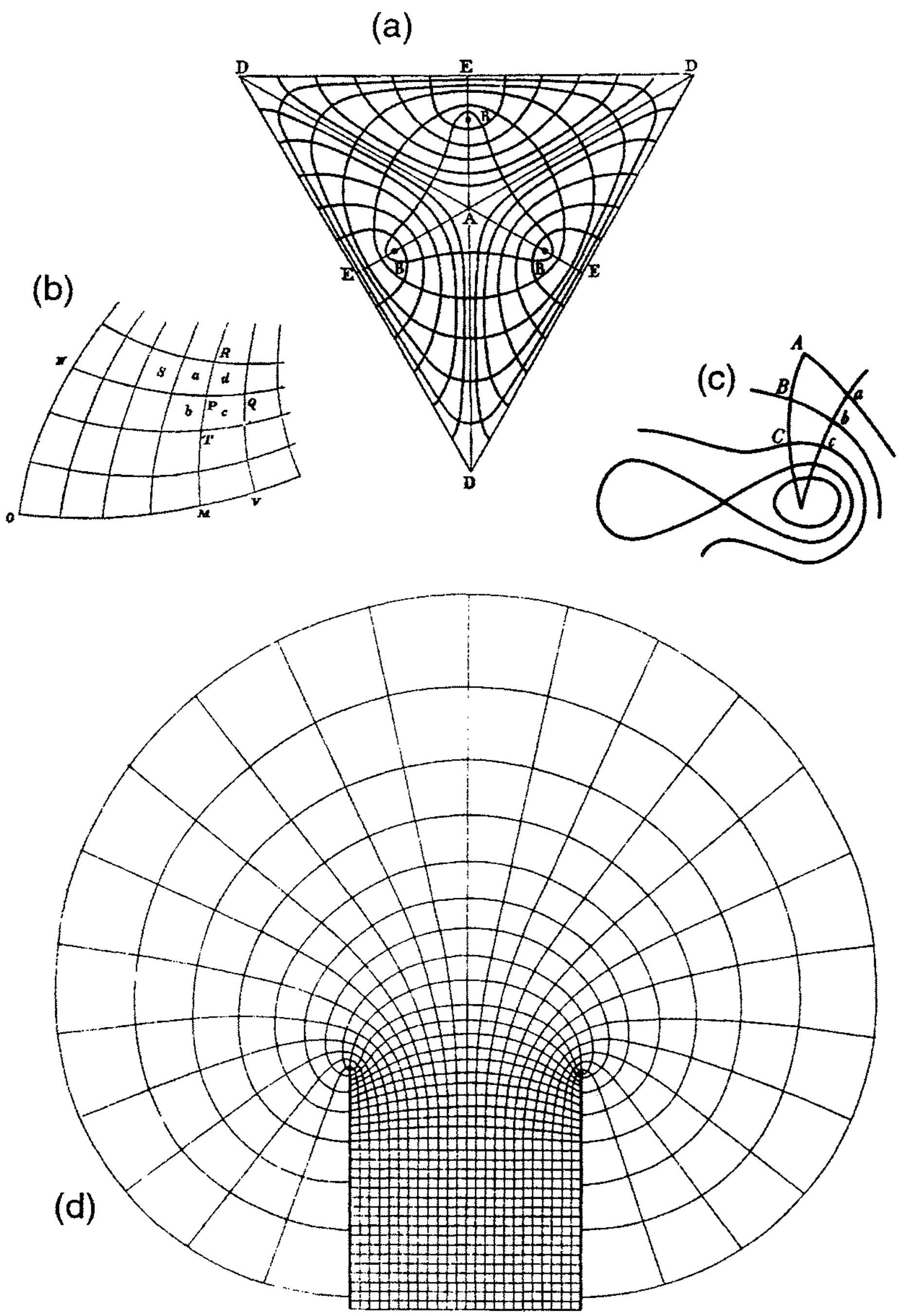

FIG. 4.1. Some of Maxwell's geometrical grids: (a) compression and dilation lines of a glass triangle (Maxwell 1850: 68), (b) lines of surface bending (Maxwell 1854b: 99), (c) electric lines of force and equipotentials (Maxwell [1854]: 252, used by permission of Cambridge University Press), (d) *idem* for a two-plate condensor (Maxwell 1873a: plate 12).

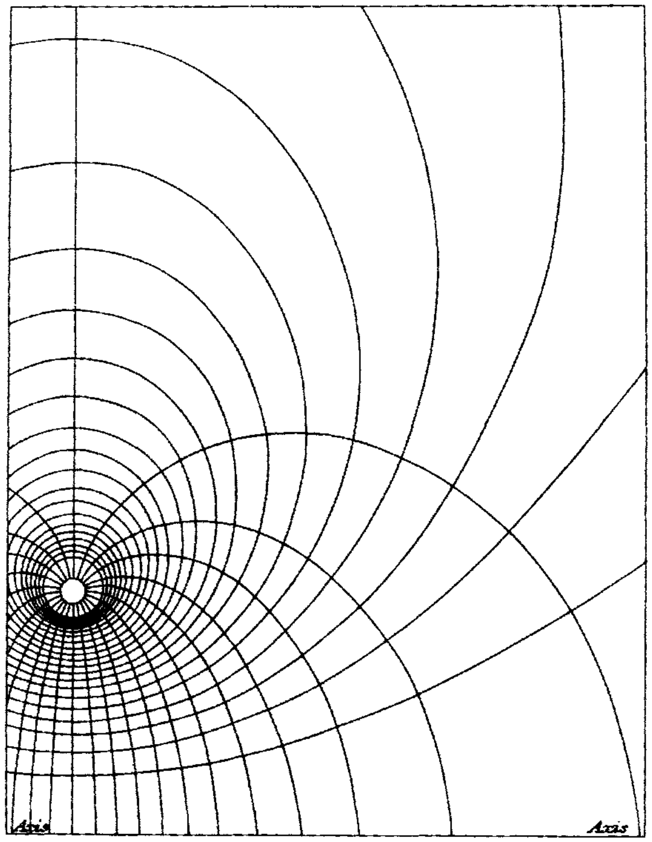

FIG. 4.2. Magnetic lines of force and equipotential surfaces of a circular current, in a half-plane delimited by the axis of the circle (Maxwell 1873a: plate 18).

that obviously depended on the intensity of the current.[8] It also had an energetic meaning, as the work performed by a unit magnetic pole on a curve γ embracing the circuit. In order to determine this number, Maxwell resorted to Ampère's equivalence between a circuit C and a net of contiguous loops (Fig. 4.3) and reasoned as follows.

If the small shaded loop embracing the curve γ were removed, the remaining loops would be equivalent to a double magnetic sheet with a hole at the place of the shaded loop. The corresponding potential would be single-valued, and its total variation on γ would be zero. Consequently, the line integral of the magnetic force, or the number of equipotentials, depends only on the current circulating in the shaded loop, which is equal to the current in C.[9] With a Gaussian eye for topological relations, Maxwell insisted that the integration curve and the current curve had to embrace each other.

[8] This number is well-defined for a proper choice of the potential unit.

[9] The numerial coefficient is determined by considering a particular case, for instance a circular current and its axis regarded as a curve closed at infinity.

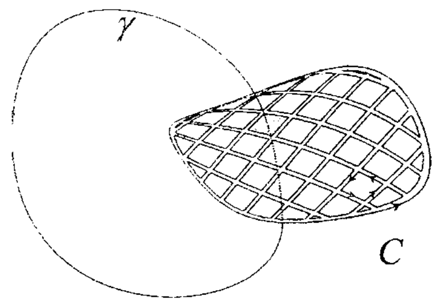

FIG. 4.3. Amperean net and mutually embracing curves for Maxwell's first proof of the magnetic circuital law.

In general, *the line integral of the magnetic force on any closed curve is measured by the sum of the intensities of the embraced currents.*[10]

Maxwell was first to enunciate this result, which is improperly called the Ampère law (or theorem).[11] Together with the induction law, it formed the basis of his own field theory of magnetism. William Thomson was no doubt aware of these two laws.[12] However, they did not appear as explicit, central statements in his papers. Having started from action-at-a-distance theory and energetic considerations, he gave central importance to the potential concept. In contrast, Maxwell started with Faraday's lines of force and expressed fundamental laws directly in terms of the field of force. He regarded the equipotentials as derivative constructs, defined as the surfaces orthogonal to the lines of force, even though they played a role in his derivation of the Ampère law and in his discussion of field energy.

4.2.2 *The resisted-flow analogy*

In the same letter to Thomson, Maxwell applied his line–surface geometry to conduction currents: here the lines refer to electric motion, and the surfaces to equal tension. He also suggested an analogous treatment of induced magnetism, based on Faraday's notion of conductive power for the magnetic lines of force. In the general

[10] Maxwell to Thomson, 13 November 1854, *MSLP* 1: 256–7. Maxwell also stated another theorem: the integral of the magnetic force across a surface bounded by the circuit only depends on the intensity of the current (and on the shape of the circuit, Maxwell should have added), not on the shape of the surface (*MSLP* 1: 257). This results from the equivalence of the circuit with a double magnetic sheet and from the fact that the integral of the magnetic force produced by magnetic masses is zero over any closed surface that does not contain masses, as Maxwell noted a little earlier in his letter. Cf. Wise 1979 and Hendry 1986: 126–30.

[11] I avoid the expression 'Ampère's law,' which is even more misleading.

[12] Thomson was certainly aware of the Ampère law, as appears from his discussion of the potential of a closed current (Thomson 1850b: 426n). However, he did not state it formally, presumably because a true field formulation of electromagnetism was not on his agenda. Regarding electromagnetic induction, Thomson had used Faraday's law and had given its mathematical expression in a particular case (Thomson 1851c: 484). Maxwell knew these papers very well, so he asked Thomson whether he had not 'the whole draught of the thing [Maxwell's "On Faraday's lines of force"] lying in loose papers' (Maxwell to Thomson, 13 September 1855, *SLMP* 1: 322).

case of variable conductivity, however, Maxwell did not know how to prove the existence of the potential. In 1848 Thomson had published a strict but enigmatic proof, based on minimizing a certain positive integral (see p. 129). Maxwell wondered whether his correspondent had a general theory based on the theorem. The answer is lost. In any case, by the spring of 1855 Maxwell was elaborating on the flow analogy that Thomson had so successfully applied to existence theorems. 'Have you patented that notion with all its applications?, for I intend to borrow it for a season,' he wrote Thomson.[13]

Maxwell's resulting analogy, published in the first part of 'On Faraday's line of force,' departed from Thomson's original heat analogy in several respects. Maxwell replaced heat with an 'imaginary incompressible fluid,' arguing that it would provide a more concrete analogy, since heat was no longer regarded as a substance. He treated the most general case of heterogenous and anisotropic conduction, whereas Thomson had mostly confined himself to the homogenous case. Most important, Maxwell integrated his tubes-and-cells geometry in the analogy and thus increased its intuitive appeal and demonstrative power. His aim was to produce a method that 'required attention and imagination but no calculation.'[14]

Maxwell first described the uniform motion of an incompressible and imponderable fluid through a resisting medium with sources and sinks. He parted the fluid into unit tubes, in which one unit of volume passes in a unit of time. The configuration of the tubes completely defines the flow, since their direction gives that of the fluid motion, and their inverse section determines the velocity. Maxwell further assumed the resistance of the medium (a porous body) to be proportional to the fluid velocity. Since the motion is uniform and the fluid has no mass, this implies that the velocity is proportional to the gradient of pressure, as Fourier's heat flux is proportional to the gradient of temperature.[15]

With this illustration, Maxwell proved essentially the same theorems as Thomson had done with the heat-flow analogy. He did not quite meet his aim to prove the existence of the potential—or pressure—in the case of a heterogenous medium.[16] But his reasonings, being based on the geometry of the tubes of flow, were more direct and vivid than Thomson's. For example, he obtained the surface-replacement theorem by the following simple consideration: the flow outside an imaginary closed surface is unchanged if we substitute for the fluid inside the surface a system of sources and sinks on the surface that maintain the flow in each intersecting tube.[17]

[13] Maxwell to Thomson, 13 November 1854, *MSLP* 1: 259–61; Maxwell to Thomson, 15 May 1855, *MSLP* 1: 307. Maxwell also thought of relating the work of electrodynamic forces to the number of cells in the field (*MSLP* 1: 259).

[14] Maxwell [1855]: 306; Maxwell to Stokes, 22 February 1856, *SLMP* 1: 403; Maxwell 1856b: Part I. Cf. Rosenfeld 1956: 1652–5; Heimann 1970; Everitt 1975: 87–93; Moyer 1978; Wise 1979; Hendry 1986: 133–8; Harman 1990: 12–15; Siegel 1991: 30–3.

[15] Maxwell 1856b: 160–4.

[16] He only proved that if the potential flow exists in a heterogenous medium, then it may be regarded as created by an imaginary system of sources spread in a homogenous medium (Maxwell 1856b: 168–71).

[17] Maxwell 1856b: 168 (#20).

Maxwell also introduced 'surfaces of equal pressure' such that a unit pressure difference exists between two consecutive surfaces. He used the cells determined by the intersection of these surface with the tubes of flow to express the energy spent by the fluid to overcome the resistance of the porous medium. In a given cell, a unit mass of fluid experiences a pressure decrease of one unit. Therefore, one unit of energy is spent in each cell, and the total amount of dissipated energy is equal to the total number of cells. This amount must be equal to the work produced or received by the sources and sinks, which is the sum of the products of their rate of flow times the pressure under which they are working. In this picturesque manner, Maxwell justified the interchange of a field integral with a sum over sources, which Gauss and Thomson had obtained by purely analytical means.[18]

Next, Maxwell explained the analogy of the imaginary flow with various domains of electricity and magnetism. For electrostatics, the tubes of flow correspond to Faraday's lines of electric induction, the pressure to the potential, and the resistance of the medium to the inductive capacity of the dieletric. For magnetism, the tubes of flow correspond to Faraday's magnetic lines of force,[19] the pressure gradient to 'the resultant force of magnetism,' and the resistance of the medium to the inverse of Faraday's 'conducting power' for the lines of force. For electrokinetics, the tubes of flow correspond to the lines of current, the pressure to the electrostatic potential or tension, and the resistance of the medium to the electric resistance.[20]

The total number of cells also has a counterpart in each of the three analogies. Clearly, it is equal to the electrostatic energy in the electrostatic case and to the Joule heat in the electrokinetic case. Maxwell only discussed the case of para- and diamagnetism, for it justified Faraday's rule of least resistance to the passage of the lines of force: the total number of cells, or resistance overcome by the flow, is then equal to the total magnetic potential from which mechanical forces are derived. Note, however, that the analogy could not help Maxwell locate the magnetic energy in the field: the number of fluid cells (corresponding to the later $\int \mathbf{B} \cdot \mathbf{H}\, \mathrm{d}\tau$) did not measure an energy stored in space, but the energy dissipated by the flow.[21]

4.2.3 *Intensity/quantity*

A more relevant aspect of the analogy was the distinction between force and flux implied in the idea of a resisted flow. Maxwell knew that for electric conduction and electrostatic induction Faraday distinguished between electric intensity and quantity. Intensity meant tension causing the current or the electroscopic effect. Quantity

[18] Maxwell 1856b: 161–2, 173–5. In electrostatic symbols, the interchange reads $\int \rho V \mathrm{d}\tau = \int \varepsilon E^2 \mathrm{d}\tau$.

[19] This is not quite true, because Faraday's magnetic lines of force have no source, whereas Maxwell's tubes of flow have sources corresponding to the magnetic masses.

[20] Maxwell 1856b: 175–83. In the draft of December 1855 (*MSLP* 1: 364), Maxwell did not introduce the electrostatic potential as a counterpart of the pressure. He did in the final version, as a consequence of his reading Kirchhoff 1849b.

[21] Maxwell 1856b: 178–80. Maxwell also combined this analogy with the equivalence between closed currents and double magnetic sheets, to derive the rule that circuits tend to move in such a way as to maximize the magnetic quantity (flux) across them (*ibid.*: 185).

referred to the strength of the electric current, or to the integral current that a charged condenser could produce. Faraday forcefully defended this usage, even though it departed from Ampère's and Thomson's. Shortly before Maxwell's elaboration of his lines of force, he wrote: 'The idea of intensity or the power of overcoming resistance [to induction or to conduction], is as necessary to that of electricity, either static or current, as the idea of pressure is to steam in a boiler, or to air passing through apertures or tubes; and we must have language competent to express these conditions and these ideas.' Maxwell used his flow analogy to systematize the distinction.[22]

In Maxwell's frame of tubes and surfaces, quantity referred to the number of tubes crossing a surface and intensity to the number of surfaces crossed by a given tube. In terms reminiscent of Faraday's, Maxwell wrote: 'The amount of fluid passing through any area in a unit of time measures the *quantity* of action over this area; and the moving force which acts on any element in order to overcome the resistance, represents the total *intensity* of action within the element.' This distinction immediately became central to Maxwell's field theory. An essential virtue of formal analogies accorging to Maxwell was to provide a classification of physico-mathematical quantities that guided theory construction.[23]

Maxwell's first use of the quantity/intensity distinction was unfortunate. To a given intensity, he surmised, there should correspond one and only one quantity. Therefore, the quantity corresponding to the electric potential differences should be the same in electrostatics and in electrokinetics, and a dielectric should be nothing but a very bad conductor in which the electric quantity or current were too small to be detected. Maxwell believed that he could find support for this idea in Faraday's assertion that 'insulation and ordinary conduction cannot be properly separated when we are examining into their nature,' whereas Faraday only meant that conduction always involved the build up and breakdown of electrostatic induction.[24]

Maxwell made a more felicitous use of quantities and intensities in further reflections on electromagnetic induction. He had already been able to express Faraday's law in mathematical terms, and he had learned from Helmholtz how to derive it by an energetic argument. Yet he was no more satisfied with the form of Faraday's law than Faraday himself was:

> This law, though it is sufficiently simple and general to render intelligible all the phaenomena of induction in closed circuits, contains the somewhat artificial conception of the number of lines *passing through* the circuit, exerting a physical influence on it. It would be better if we could avoid, in the enunciation of the law, making the electromotive force in a conductor depend upon lines of force external to the conductor.

Maxwell wanted to express the electromotive force as the variation of some 'intensity' representing the electrotonic state of the conductor. The quantity/intensity

[22] Faraday 1854: 519

[23] Maxwell 1856a: 371; Maxwell 1856b: 182, 189–92. *Ibid.* on 182 Maxwell referred to the distinction of Faraday 1854: 519. Cf. Wise 1979; Everitt 1975: 89–90; Moyer 1978; Hendry 1986: 136–42.

[24] Maxwell 1856b: 181, including a reference to Faraday 1854: 513n.

distinction, a derived symbolism, and some theorems by Thomson and Stokes provided the answer.[25]

In symbols, the fluid quantity across a surface element $\mathbf{dS}$ is $\mathbf{a}\cdot\mathbf{dS}$, where $\mathbf{a}$ denotes the fluid current. The intensity (pressure difference) along the length element $\mathbf{dl}$ is $\boldsymbol{\alpha}\cdot\mathbf{dl}$, where $\boldsymbol{\alpha}$ denotes the moving force. The incompressibility of the fluid gives $\nabla\cdot\mathbf{a} = 0$ (in the absence of sources). The resistance k of the medium implies $\boldsymbol{\alpha} = k\mathbf{a}$. To specify the magnetic and electric cases, Maxwell inserted the suffixes 1 and 2. Then the Ampère law applied to an infinitesimal closed curve yields

$$\nabla\times\boldsymbol{\alpha}_1 = \mathbf{a}_2 \quad (\nabla\times\mathbf{H} = \mathbf{j}). \tag{4.1}$$

Conversely, this relation implies the Ampère law, because, as Maxwell had learned from Stokes,

$$\int\boldsymbol{\alpha}\cdot\mathbf{dl} = \iint(\nabla\times\boldsymbol{\alpha})\cdot\mathbf{dS}, \tag{4.2}$$

if the first integration is performed over a curve bounding the surface of the second.[26]

In the same notation, Faraday's law reads:

$$\int\boldsymbol{\alpha}_2\cdot\mathbf{dl} = -\frac{\mathrm{d}}{\mathrm{d}t}\iint\mathbf{a}_1\cdot\mathbf{dS} \quad \left(\int\mathbf{E}\cdot\mathbf{dl} = -\frac{\mathrm{d}}{\mathrm{d}t}\iint\mathbf{B}\cdot\mathbf{dS}\right). \tag{4.3}$$

Maxwell wanted to reformulate this law in terms of a state of the circuit itself. From Thomson he knew that any divergenceless vector could be regarded as the curl of another vector. He therefore introduced the intensity $\boldsymbol{\alpha}_0$ such that[27]

$$\mathbf{a}_1 = \nabla\times\boldsymbol{\alpha}_0 \quad (\mathbf{B} = \nabla\times\mathbf{A}). \tag{4.4}$$

According to theorem (4.2), the line integral of this intensity is equal to the magnetic quantity passing through the curve. Consequently, the induced electromotive force is simply given by

$$\boldsymbol{\alpha}_2 = -\frac{\partial\boldsymbol{\alpha}_0}{\partial t} \quad \left(\mathbf{E} = -\frac{\partial\mathbf{A}}{\partial t}\right). \tag{4.5}$$

[25] Maxwell 1856a: 373. On Maxwell's reference to Faraday's electrotonic state, cf. Doncel and Lorenzo 1996.

[26] Maxwell 1856b: 203–5; *ibid.*: 206, with proof of Stokes' theorem based on the equivalence between the curves and a net of infinitesimal loops. This theorem was first stated by Thomson in a letter to Stokes of 2 July 1850 (Wilson 1990: 96–7; Stokes 1880–1905, Vol. 5: 320–1), and published by Stokes in the Smith prize examination for 1854, which Maxwell took.

[27] When there are magnetic masses (magnets), $\mathbf{a}_1$ is not divergenceless; Maxwell extracted the divergence-free part in a manner found in Stokes memoir on diffraction (Stokes 1849: 254–7): Maxwell 1856b: 200–1, 203–4.

Maxwell called $\boldsymbol{\alpha}_0$ the 'electro-tonic intensity,' for he believed that he had found the mathematical expression of Faraday's long-sought electro-tonic state.[28]

More generally, Maxwell professed to have reached the 'mathematical foundation of the modes of thought indicated in the *Experimental Researches*.' His success depended on a geometric deployment of the resisted-flow analogy, followed by a more symbolic approach in which the quantity/intensity distinction played a crucial guiding role. Unlike Thomson, Maxwell accompanied his use of analogy with philosophical comments. He explained that 'physical analogies' offered 'a method of investigation which allows the mind at every step to lay hold of a clear physical conception, without being committed to any theory founded on the physical science from which that conception is borrowed, so that it is never drawn aside from the subject in pursuit of analytical subtleties, nor carried beyond the truth by a favorite hypothesis.' Weber's theory, elegant though it was, depended on a questionable physical hypothesis. In contrast, Maxwell's own theory did not contain 'even the shadow of a true physical theory; in fact,' Maxwell went on, 'its chief merit as a temporary instrument of research is that it does not, even in appearance, *account for* anything.' The fluid analogy applied indifferently to separate compartments of electric science; it did not account for mechanical forces among charged bodies, currents, or magnets; and it ignored the relation between electricity and magnetism. The incompressible fluid was purely imaginary, the electro-tonic intensity purely symbolic. Nevertheless, 'by a careful study of the laws of elastic solids and of the motions of viscous fluids' Maxwell hoped 'to discover a method of forming a mechanical conception of the electro-tonic state adapted to general reasoning.'[29]

4.3 On physical lines of force

4.3.1 *Molecular vortices*

In May 1857, after reading Thomson's 'new lights' on the Faraday effect and molecular vortices, Maxwell wrote to his friend Cecil Monro: 'This was a wet day & I have been grinding at many things and lately during this letter at a Vortical theory of magnetism and electricity which is very crude but has some merits, so I spin & spin.' In a letter to Thomson written a few months later he described a gyromagnetic device that would confirm the existence of vortices in magnetized iron, if only the rotating fluid had enough inertia. Three years later, in the first part of 'On

[28] Maxwell 1856a: 374. In the final paper (1856b), instead of assuming Faraday's law, Maxwell used a flawed energetic reasoning inspired by Helmholtz's 'derivation' of electromagnetic induction: cf. Knudsen 1995.

[29] Maxwell 1856b: 207, 156, 207, 188. For Maxwell's reaction to Weber's theory, see also Maxwell to Thomson, 15 May 1855, *MSLP* 1: 305–6. On the analogies of 'On Faraday's lines of force,' cf. Moyer 1978; Wise 1979, 1981a; Hendry 1986: 143–55; Siegel 1991: 30–3, 38–9. On Maxwell's use of analogy in general, cf. Turner 1955; Hesse 1961, 1966, 1973; Kargon 1969; Chalmers 1973a; Hendry 1986; Siegel 1991; Cat 1995.

physical lines of force' he proposed a theory of magnetism based on molecular vortices.[30]

In his 1856 paper on the Faraday effect, Thomson had written: 'The explanation of all phenomena of electro-magnetic attraction or repulsion, and of electro-magnetic induction, is to be looked simply in the inertia and pressure of the matter of which the motions constitute heat.' He then assumed heat to consist of Rankine's molecular vortices and magnetism in the alignment of these vortices. In 1860 Maxwell supported Clausius's kinetic theory, and therefore could not follow the whole of Thomson's suggestion. He did not doubt, however, that magnetism involved vortical motion, as a consequence of Thomson's analysis of the Faraday effect. And he could precisely see why Thomson believed that the pressure and inertia of the revolving matter determined magnetic forces and electromagnetic induction.[31]

If there exist fluid vortices along the lines of force, he reasoned, then the centrifugal force of the vortices implies a larger pressure in the directions perpendicular to the lines of force than along the lines of force. This is equivalent to an isotropic pressure combined with a tension along the lines of force. Maxwell thus retrieved Faraday's intuition of a mutual repulsion of the lines of force and a tension along them. He only had to verify that this stress system implied the known magnetic attractions and repulsions.[32]

Calling p the isotropic pressure, μ the density of the medium, and $\mathbf{H}$ a vector giving the direction of the vortices and the average linear velocity of the fluid, the stress system is

$$\sigma_{ij} = -p\delta_{ij} + \mu H_i H_j, \tag{4.6}$$

in anachronistic tensor notation. From the net effect of these stresses on the sides of an infinitesimal cube, Maxwell derived the force

$$f_i = \partial_j \sigma_{ij} = -\partial_i p + H_i \partial_j (\mu H_j) + \mu H_j (\partial_j H_i - \partial_i H_j) + \mu \partial_i \left(\frac{1}{2} H^2\right), \tag{4.7}$$

or

$$\mathbf{f} = (\nabla \cdot \mu\mathbf{H})\mathbf{H} + (\nabla \times \mathbf{H}) \times \mu\mathbf{H} + \mu\nabla\left(\frac{1}{2} H^2\right) - \nabla p. \tag{4.8}$$

[30] Maxwell to Faraday, 9 November 1857, *MSLP* 1: 552: 'But there are questions relating to the connexion between magneto-electricity and a possible confirmation of the physical nature of magnetic lines of force. Professor W. Thomson seems to have some new lights on this subject'; Maxwell to Monro, 20 May 1857, *MSLP* 1: 507; Maxwell to Thomson, 30 January 1858, *MSLP* 1: 579–80. Cf. Siegel 1991: 33–7; Harman 1990: 30–1; Everitt 1975: 93–5; Everitt 1983: 132–4.

[31] Thomson 1856: 571. Cf. Chapter 3, pp. 133.

[32] Maxwell 1861: 452–5. Cf. Siegel 1991: 56–65.

Maxwell was now on the grounds of his 'On Faraday's lines of force.' Identifying **H** and $\mu\mathbf{H}$ with the magnetic intensity and quantity defined there, in the successive terms of eqn. (4.8) he recognized the force acting on the imaginary magnetic masses $\nabla \cdot \mu\mathbf{H}$, the force acting on the current $\nabla \times \mathbf{H}$, and the force responsible for the tendency of paramagnetic (diamagnetic) bodies to move toward places of stronger (weaker) magnetic intensity. Hence Thomson's molecular vortices and the resulting stresses accounted for all known magnetic and electromagnetic forces, with striking mathematical exactitude.[33]

4.3.2 *The idle wheels*

Maxwell next wondered why a distribution of vortices for which $\nabla \times \mathbf{H}$ did not vanish indicated an electric current. His answer came with the resolution of the following puzzle:

I have found great difficulty in conceiving of the existence of vortices in a medium, side by side, revolving in the same direction about parallel axes. The contiguous portions of consecutive vortices must be moving in opposite directions; and it is difficult to understand how the motion of one part of the medium can coexist with, and even produce, an opposite motion of a part in contact with it.

Being aware of electromagnetic induction, Maxwell expected the system of vortices to act as a connected mechanism, able to transfer electric motion from one conductor to another. Like his father and his Scottish professors, he was highly interested in practical mechanics. He had read several treatises on this subject, and taught his students the rudiments of kinematics with toothed wheels and cranks. He was surely familiar with the use of 'idle wheels' for transmitting rotation between two toothed wheels without change in the sense of rotation. Accordingly, he somewhat rigidified his fluid vortices and introduced between them a layer of small, round particles that rolled without sliding (Fig. 4.4(a)).[34]

Whenever two contiguous vortices do not rotate at the same speed, the particles between them must shift laterally (Fig. 4.4(b)). For example, if the vortices are parallel to the axis Oz, and if the rotation velocity H_z grows in the direction Ox, the shift occurs in the direction Oy at the rate $-\partial_x H_z$. In general, the shift is given by $\nabla \times \mathbf{H}$, which is equal to the electric current. Maxwell therefore identified the stream of particles with the electric current.[35]

After this purely kinematical analysis, Maxwell examined the dynamics of the new model. As a result of the tangential action **T** of the particles on the cells, there

[33] Maxwell 1861: 456–64. Note that the 'quantity' $\mu\mathbf{H}$ differs from the **B** of Maxwell's *Treatise* when there are magnets.

[34] Maxwell 1861: 468. Maxwell attended Robert Willis's lectures on mechanism: cf. Maxwell to John Clerk Maxwell, 12 November 1855, *MSLP* 1: 333; and he read a few books on this topic, including Goodeve's *Elements of mechanism* and Rankine's *Applied mechanics* to which he referred in Maxwell 1861: 469n, 458n. On Maxwell's teaching of kinematics, cf. Maxwell to William Thomson, 30 January 1858, *MSLP* 1: 580. On the kinematics of the vortex model, cf. Siegel 1991: 65–9.

[35] Maxwell 1861: 469–71.

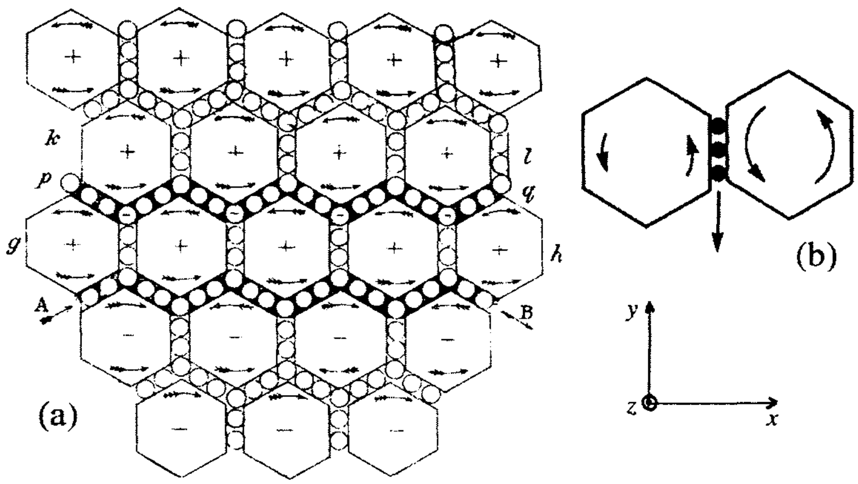

FIG. 4.4. Maxwell's cells and idle wheels (Maxwell 1861: 488 for (a) with mistakes in the arrows from the *MCP* reprint; Siegel 1991: 69 for (b), used by permission of Cambridge University Press).

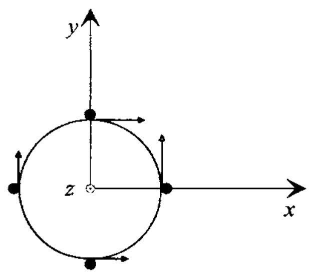

FIG. 4.5. Tangential actions of four idle wheels on a cell.

is a torque acting on each cell. For example, the torque around Oz is proportional to $\partial_x T_y - \partial_y T_x$ (see Fig. 4.5). According to a well-known theorem of dynamics, this torque must be equal to the time derivative of the angular momentum of the cell, which is proportional to $\mu\mathbf{H}$. According to the equality of action and reaction, the force $\mathbf{T}$ must be equal and opposite to the tangential action of the cell on the particles. Maxwell interpreted the latter action as the electromotive force $\mathbf{E}$ of magnetic origin acting on the current. In sum, the curl of $\mathbf{E}$ is found to be proportional to the time derivative of $\mu\mathbf{H}$. The condition that the work of the force $\mathbf{E}$ on the particles should be globally equal to the decrease of the kinetic energy of the cells determines the coefficient. The final equation of motion is

$$\nabla\times\mathbf{E} = -\frac{\partial\mu\mathbf{H}}{\partial t}, \tag{4.9}$$

in conformity with Maxwell's earlier expression of Faraday's induction law.[36]

Maxwell accompanied his derivation of the fundamental field equations with an intuitive explanation of electromagnetic induction. Consider two conducting circuits separated by an insulator, and let a current be started in one of the circuits. The corresponding flow of particles induces a rotation of the cells immediately outside the conductor. Since in the insulator the particles cannot circulate, they transmit the rotation to the next layer of cells, and so forth until the surface of the second conducting circuit is reached. At this surface the particles are again able to circulate. If there were no electric resistance, they would circulate for ever, and the cells within the conductor would remain at rest. In actual conductors, a frictional force gradually checks the circulation of the particles, and the cells of the conductor are set into rotation. Hence the induced current is only temporary, and the magnetic field in the second conductor is soon the same as it would be in an insulator.[37]

With his wonderful model Maxwell demonstrated the possibility of reducing electromagnetic actions to contiguous mechanical actions. He published his reasoning in the spring of 1861, with a few comments on the awkwardness of the model and on his ignorance of the true nature of electricity. At that time he did not seem to forecast any extension of the model. The obvious limitation to closed currents could not worry him much, since the electrodynamic properties of open currents were experimentally inaccessible.[38]

4.3.3 Electrostatics and light!

A few months elapsed before Maxwell realized that the elasticity of the vortices, which was necessary to their mechanical linking, offered an opportunity to connect electrodynamics with optics and electrostatics. Perhaps a transverse vibration of the substance of the cells could represent light. Perhaps an elastic yielding of the cells under the pressure of the particles could represent dielectric polarization. Specifically, Maxwell imagined that the tangential action of the particles on the cells, which is opposed to the electromotive force by Newton's third law, induced an elastic deformation of the kind represented in Fig. 4.6. Owing to this deformation, the particles in contact with the cells are displaced in a direction opposite to the electromotive force. Calling δ the average displacement, we have

$$\boldsymbol{\delta} = -\varepsilon \mathbf{E}, \tag{4.10}$$

where ε is a constant depending on the elastic constants and on the shape of the cells. The kinematic relation between the flux of particles and the rotation of the cells becomes:

[36] Maxwell 1861: 472–6. Instead of using the theorem of angular momentum, Maxwell used an imperfect energetic reasoning (cf. Darrigol 1993b: note 47). On pp. 479–82 he treated the case of a moving conductor (cf. Darrigol 1993b: 277–9).

[37] Maxwell 1861: 477–8. Cf. Everitt 1975: 96–7.

[38] Cf. Siegel 1991: 75–7; Bromberg 1967: 227; Harman 1970: 191; Everitt 1975: 98–9.

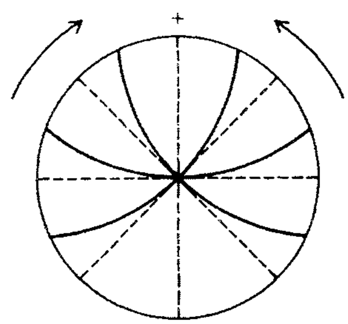

FIG. 4.6. Tangential distortion of a spherical cell (Maxwell to Faraday, 19 October 1861, *MSLP* 1: 684).

$$\mathbf{j} = \nabla \times \mathbf{H} + \frac{\partial \boldsymbol{\delta}}{\partial t}. \tag{4.11}$$

Consequently, the divergence of the current is

$$\nabla \cdot \mathbf{j} = \frac{\partial}{\partial t} \nabla \cdot \boldsymbol{\delta}. \tag{4.12}$$

This equation agrees with the conservation of electricity if the charge density is given by

$$\rho = -\nabla \cdot \boldsymbol{\delta}. \tag{4.13}$$

Although Maxwell does not explictly say so, we may note that ρ also represents an excess of particles, typically occurring at the limit between a conductor and a non-conductor.[39]

Next, Maxwell proceeded to derive the usual electrostatic forces. To this end he considered the elastic energy of the medium,

$$U = \frac{1}{2}\int(-\mathbf{E})\cdot\boldsymbol{\delta}\,\mathrm{d}\tau = \frac{1}{2}\int \varepsilon E^2 \mathrm{d}\tau, \tag{4.14}$$

computed it for two point charges q and q', and derived this quantity with respect to the distance d between the charges. The result, $qq'/4\pi\varepsilon d^2$, agreed with Coulomb's

[39] Maxwell 1862: 489–96. For a detailed analysis of the workings of the model, cf. Boltzmann 1898, and Siegel 1986, 1991: 77–119. Most other commentators have misunderstood the mechanics of the model and treated Maxwell's negative sign in the relation between displacement and electromotive force as a mistake. Siegel clarifies this point, and shows how the model accounts for basic electrostatic effects. Some of Maxwell's phrases suggest that he wanted to interpret $\boldsymbol{\delta}$ as a polarization in the Poisson–Mossotti sense. However, in an insulator the displacement of the particles due to the distorsion of the cells must be exactly compensated by a differential rotation of these cells so that the net current $\mathbf{j}$ is zero. As Boltzmann and Siegel argue, the fixity of the particles is essential to the transmission of strain from cell to cell.

law and gave the value of the absolute electrostatic unit of electric charge as $(4\pi\varepsilon_0)^{1/2}$ (the index 0 referring to a vacuum). Consequently, the ratio c of the electromagnetic to the electrostatic charge unit had to be $(\varepsilon_0\mu_0)^{-1/2}$.[40]

At that stage Maxwell had a consistent mechanical model that unified electrostatics and electrodynamics, and he could write the corresponding system of field equations, now called 'the Maxwell equations.' This is not all. He considered transverse waves in the elastic medium. Their velocity is $(k/m)^{1/2}$ if k denotes the transverse elasticity and m the density of the medium. The constant k is inversely proportional to ε, and m is proportional to μ. In order to determine the proportionality coefficients, Maxwell assumed that the cells were spherical and that their elasticity was due to forces between pairs of molecules. He found $k = 1/4\pi^2\varepsilon$, and $m = \mu/4\pi^2$. Then the velocity of transverse waves in a vacuum had to be identical to the ratio c.[41]

Comparing Fizeau's value for the velocity of light and the value of c from Weber and Kohlrausch, Maxwell found agreement within 1% and concluded: 'We can scarcely avoid the inference that *light consists in the transverse undulations of the same medium which is the cause of electric and magnetic phenomena.*' In the same stroke, Maxwell explained the strange proximity of the electromagnetic constant c with the velocity of light and realized Faraday's dream of unifying optics and electromagnetism. Yet the quality of the numerical agreement was accidental. As Duhem pointed out many years later, Maxwell had overlooked a factor of 2 in the transverse elasticity of the cells' substance. In any case, the cells could not be spherical. Moreover, Weber and Kohlrausch's and Fizeau's measurements later proved to be both wrong by 3%. What Maxwell truly had was a rough magneto-mechanical theory of light, based on the elasticity of the substance whose rotation represented the magnetic field.[42]

In the last part of his memoir Maxwell returned to the very phenomenon that had inspired his vortex model, the Faraday effect. The rotation of the cells implied a rotation of the polarization of light in the same direction, by an amount proportional to the radius of the cells. Faraday's observations could be explained if the cells were much smaller in a vacuum than in transparent matter, and if their size depended on the kind of matter. However, Maxwell's model implied that the optical rotation should always be in the direction defined by the magnetic field, whereas Emile Verdet had recently observed an opposite rotation for solutions of iron salts. Maxwell briefly suggested that a proper combination of his cellular model with Weber's molecular currents would explain the anomaly.[43]

[40] Maxwell 1862: 497–9. The stress of the cells, which is linear in **E**, cannot be directly responsible for the electrostatic forces, which require a quadratic stress (see Appendix 6). Maxwell never found a mechanical representation of Faraday's electric stresses (cf. Siegel 1991: 83). Maxwell's notation for c was v. This constant is related to that of Weber's theory by $c = C/\sqrt{2}$.

[41] Maxwell 1862: 499.

[42] Maxwell 1862: 500 (Maxwell's emphasis); Duhem 1902: 208–9, 211–12. Cf. Siegel 1991: 136–41. Bromberg 1967 called Maxwell's theory of light of 1862 'electro-mechanical.' I prefer 'magneto-mechanical' because magnetic vortices were the starting point.

[43] Maxwell 1862: 502–13; Verdet 1854–1863. Cf. Knudsen 1976: 255–8.

4.3.4 An orrery

Maxwell had more to say on the status of his mechanical assumptions. His previous analogies with resisted flow, he recalled, were intended to provide a clear geometrical conception of the lines of force. They did not involve any hypothesis on the deeper nature of electric and magnetic actions. In contrast, his new approach assumed the existence of stresses from which observed mechanical actions derived. The lines of force now referred to these stresses and were therefore as physical as Faraday wanted them to be. Maxwell further adopted Thomson's assumption that the stresses in the magnetic field were due to molecular vortices. These physical hypotheses permitted a unified, dynamical understanding of magnetism and electromagnetism; and they were anchored on the rock of Thomson's argument on the Faraday effect. They remained in the core of Maxwell's theory until his death.[44]

However, Maxwell did not believe in the literal truth of his more specific assumptions regarding the constitution and interconnection of the molecular vortices:

> The conception of a particle having its motion connected with that of a vortex by perfect rolling contact may appear somewhat awkward. I do not bring it forward as a mode of connexion existing in nature, or even as that which I would willingly assent to as an electrical hypothesis. It is however, a mode of connexion wich is mechanically conceivable, and easily investigated, and it serves to bring out the actual mechanical connexions between the known electro-magnetic phenomena; so that I venture to say that any one who understands the provisional and temporary character of this hypothesis, will find himself rather helped than hindered by it in his search after the true interpretation of the phenomena.

Maxwell did not doubt the truth of the relations he had obtained between the electric and magnetic fields, and he believed that these relations derived from the laws of mechanics. But a peculiar combination of vortices and idle wheels could not meet his idea of the simplicity of nature. As he explained to Tait: 'The nature of this mechanism is to the true mechanism what an orrery is to the solar system.'[45]

4.4 The dynamical field

After the publication of 'On physical lines of force,' Maxwell's agenda included the experimental verification of three predictions of his theory. He planned to renew his attempts at detecting gyromagnetic effects. He envisioned precise measurements of the inductive capacity ε of various transparent substances in order to verify the theoretical relation with the optical index ($\varepsilon = n^2$). Most importantly, he intended to verify the identity of the velocity of light with the ratio of absolute electromagnetic and electrostatic charge units by improving on Weber and Kohlrausch's measurement. His enrollment in the British project for electric standards eased this task. In 1864 he imagined an arrangement based on the direct comparison between an elec-

[44] Maxwell 1862: 451–3. Cf. Knudsen 1976: 248–55; Siegel 1991: 39–55.
[45] Maxwell 1861: 486; Maxwell to Tait, 23 December 1867, *MSLP* 2: 337.

trodynamic and an electrostatic force. Four year laters he published the results of a more sophisticated experiment based on the same principle.[46]

Considering that the electromagnetic derivation of the velocity of light was his most important result, Maxwell tried to 'clear the electromagnetic theory of light of all unwarranted assumptions.' The velocity of light could not possibly depend on the shape of vortices or on their kind of elasticity. In 1864 Maxwell managed to reformulate his theory without any specific mechanism and to describe wave propagation in purely electromagnetic terms. In order to understand how he accomplished this, we must return to the electrotonic state.[47]

4.4.1 The reduced momentum

When Maxwell designed the vortex model, he was still looking for a mechanical interpretation of the electrotonic state. He found one of an unexpected sort. Having rewritten the induction law (4.9) as

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} \quad \text{with} \quad \mu\mathbf{H} = \nabla\times\mathbf{A}, \tag{4.15}$$

he noted that **A** played the role of a 'reduced momentum' for the mechanism driven by the flow of particles. He thereby meant a generalization of Newton's second law, in which the force $-\mathbf{E}$ served to increase the reduced momentum. More concretely, he compared **A** with 'the *impulse* which would act on the axle of a wheel in a machine if the actual velocity were suddenly given to the driving wheel, the machine being previously at rest.' Impulse and momentum were prominent notions in the treatises on mechanism he had been reading, especially Rankine's. Also, impulsive forces played a central role in Stokes's and Thomson's considerations of irrotational flow.[48]

Two simple examples will illustrate what Maxwell had in mind. In the case of a single linear circuit, the current i sets the surrounding cells into rotary motion, as a rack pulled between toothed wheels (Fig. 4.7). If the mass of the axle is negligible, a finite force is still necessary to set it into motion because of the inertia of the connected wheels. By Maxwell's definition, the reduced momentum is the impulse

[46] Maxwell to Thomson, 10 December 1861, *MSLP* 1: 694–8; Maxwell to Thomson, 15 October 1864, *MSLP* 2: 176; Maxwell 1868a. On the gyromagnetic experiments, cf. Maxwell to Faraday, 19 October 1861, *MSLP* 1: 688–9; Maxwell 1861: 485n–6n; Maxwell 1873a: ##574–5; and Galison 1982 for the later history of such effects. For a classification of the devices to measure the units ratio, cf. Jenkin and Maxwell 1863. In Maxwell's 1864 device, the repulsion of two current-fed coils is balanced by the attraction between two electrified disks; the current feeding the coils passes through a resistance of known absolute value; and the potential difference at the ends of this resistance is applied to the disks. On the ensuing project, cf. Schaffer 1995; d'Agostino 1996: 31–6; Simpson 1997: 347–63; Harman 1998: 65–8.

[47] Maxwell to Hockin, 7 September 1864, *MSLP* 2: 164; Maxwell 1865.

[48] Maxwell 1861: 478. I have changed the sign of **A** for consistency with Maxwell's later papers. Reference to Rankine's *Applied mechanics* is found *ibid.*: 458n (for the definition of stresses). On impulsive forces, cf. Moyer 1977: 257–8.

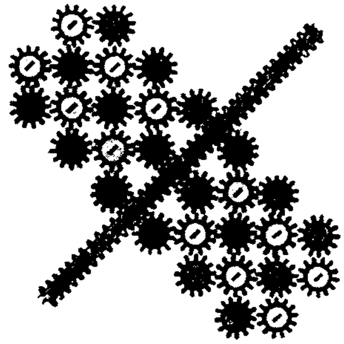

FIG. 4.7. Illustration of self-induction in a linear circuit (from Lodge 1889: 186). The + and – signs indicate the sense of the rotation.

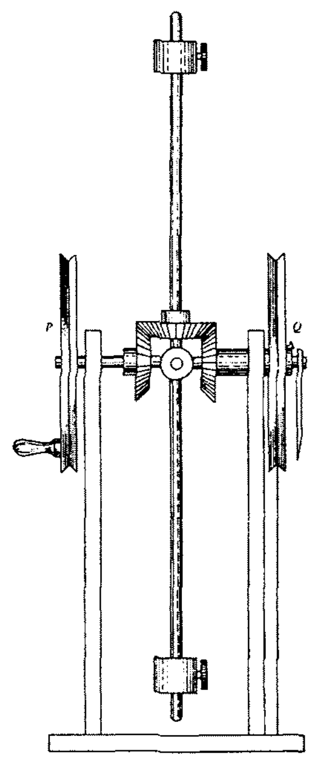

FIG. 4.8. Maxwell's model for mutual induction (Maxwell 1891, Vol. 2: 228).

necessary to obtain a given velocity i. This impulse is proportional to the velocity and to the inertia of the wheels. In electric language, it is equal to Li, where L is the self-inductance of the circuit, and it measures the electro-tonic state.

In the case of two linear cicuits, the rotation of the cells is in a one-to-one correspondence with the two currents i_1 and i_2. This situation is analogous to the mechanism of Fig. 4.8, in which the rotations of the wheels P and Q play the role of the two currents and the rotation of the fly-weights plays the role of the vortex rotation in the magnetic field. The wheels P and Q have a negligible inertia. However, a finite force is in general necessary to set them into motion, because of the inertia of the fly-weights. The reduced momenta at P and Q are the impulses necessary to impart on them the velocities i_1 and i_2. These impulses have the linear forms

$$p_1 = L_1 i_1 + M i_2, \quad p_2 = L_2 i_2 + M i_1. \tag{4.16}$$

They measure the electro-tonic states of the two circuits. Generalizing to a three-dimensional current distribution **J**, the electromotive force necessary to start this current impulsively must be a certain linear function of **J**, to be identified with the electrotonic state **A**.[49]

4.4.2 Hidden mechanism

Maxwell reached this mechanical interpretation of the electrotonic state in 1861, on the basis of the vortex model. Three years later, he realized that the interpretation was essentially independent of any specific mechanism and could serve as a more abstract foundation for the dynamics of the magnetic field. He simply admitted that through an unspecified connected mechanism the existence of an electric current implied a motion in the surrounding field. Then, the force necessary to communicate this motion had to be the time derivative of a generalized momentum **A**, which he now called the 'electromagnetic momentum.' In the case of two circuits, this yields the usual equations for inductive coupling (Neumann's)

$$\begin{aligned} e_1' - R_1 i_1 &= \frac{\mathrm{d}}{\mathrm{d}t}(L_1 i_1 + M i_2) \\ e_2' - R_2 i_2 &= \frac{\mathrm{d}}{\mathrm{d}t}(L_2 i_2 + M i_1) \end{aligned} \tag{4.17}$$

where e_1' and e_2' are the impressed electromotive forces and R_1 and R_2 the resistances.[50]

Maxwell next considered the energy brought by the electromotive sources according to Thomson:

$$\begin{aligned} e_1' i_1 + e_1' i_2 = R_1 i_1^2 + R_2 i_2^2 &+ \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2} L_1 i_1^2 + \frac{1}{2} L_2 i_2^2 + M i_1 i_2\right) \\ &+ \frac{1}{2}\frac{\mathrm{d}L_1}{\mathrm{d}t} i_1^2 + \frac{1}{2}\frac{\mathrm{d}L_2}{\mathrm{d}t} i_2^2 + \frac{\mathrm{d}M}{\mathrm{d}t} i_1 i_2. \end{aligned} \tag{4.18}$$

[49] Maxwell constructed this model in 1874. Cf. Maxwell 1891: 228, and Everitt 1975: 103–4.

[50] Maxwell 1865: 536–40. Cf. Simpson 1970; Topper 1971; Chalmers 1973; Moyer 1977; Siegel 1981; Buchwald 1985a: 20–3; Hendry 1986: 191–206; Siegel 1991.

The two first terms represent the Joule heat. The third represents the variation of the energy

$$T = \frac{1}{2}(p_1 i_1 + p_2 i_2) \tag{4.19}$$

stored in the hidden mechanism. The three last terms exist only if the geometrical configuration of the circuits varies: they represent the work of electrodynamic forces during this motion. Maxwell thus inverted the prodedure followed by Helmholtz and Thomson; that is, he derived the expression of electrodynamic forces from the laws of induction.[51]

4.4.3 *Lagrangian dynamics*

Maxwell's reasoning appeared in his 'dynamical theory of the electromagnetic field,' published in 1865. With this title he meant to announce a reduction of electrodynamics to hidden motion in the field. In the *Treatise*, published in 1873, he improved his presentation by a recourse to the Lagrange equations. A system of two currents according to Maxwell is a connected system the motion of which is completely defined by two generalized velocities i_1 and i_2. Following Lagrange, the motion of this system is completely determined by the form of its kinetic energy, which is

$$T = \frac{1}{2}L_1 i_1^2 + \frac{1}{2}L_2 i_2^2 + M i_1 i_2, \tag{4.20}$$

for the electrodynamic part. The Lagrange equations with respect to the generalized velocities i_1 and i_2 give

$$e_1' - R_1 i_1 = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial i_1}, \quad e_2' - R_2 i_2 = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial i_2}, \tag{4.21}$$

in conformity with eqns (4.17). If the geometric configuration of the circuits depends on the coordinate ξ, the corresponding Lagrange equation yields the electrodynamic force $\partial T/\partial\xi$ (see Appendix 9 for later three-dimensional generalizations).[52]

Originally, Lagrange designed his analytical method as a way of eliminating the quantities that pertain to the internal connections of a connected mechanical system. Laplacian physicists had little use of the method, since they always started with molecular forces. British physicists were first to appreciate the great power of the method: it gave the equations of motion of a mechanical system by an automatic

[51] Maxwell 1865: 541–2.
[52] Maxwell 1873a: ##578–83. Save for their Lagrangian justification, Maxwell's circuit equations are exactly identical to those of Neumann's theory of induction.

prescription, directly in terms of the controllable elements. For example, in 1837 George Green derived the equations of motion of an elastic solid by expressing its kinetic and potential energy in terms of the local displacements and writing the corresponding Lagrange equations. William Thomson adopted the method, for it shared the virtue of the energy principle of dealing with controllable inputs and outputs. He tried to make it less abstract by combining it with the more physical notions of work and impulse. In his and Tait's *Treatise of Natural Philosophy* (known as TT′), first published in 1867 and proof-read by Maxwell, he defined generalized forces through the work they brought to the system ($\Sigma f_i \mathrm{d}q_i$, for a variation $\mathrm{d}q_i$ of the generalized coordinates), and the generalized 'momenta' p_i as the impulses necessary to suddenly start the motion of the system from rest. The Lagrange equations, $f_i = \mathrm{d}p_i/\mathrm{d}t - \partial T/\partial q_i$, thus took a physically transparent form.[53]

Maxwell was very sympathetic to Thomson and Tait's presentation. He developed it in a chapter of his *Treatise*, with the comment: 'We avail ourselves of the labours of the mathematicians [Lagrange and Hamilton], and retranslate their results from the language of the calculus into the language of dynamics, so that our words may call up the mental image, not of some algebraical process, but of some property of moving bodies.' In this process Maxwell was less careful than Thomson and Tait, and erred in a pseudo-derivation of the Lagrange equations based on energy conservation. However, thanks to the new dynamical language he perceived an essential advantage of Lagrange's method: that the motion of the driving points of a connected mechanism could be studied without any knowledge of the internal connections, as some kind of black box. Maxwell used the metaphor of a belfry, the machinery of which is controlled by a number of ropes. The machinery being originally at rest, finite velocities are impressed impulsively on the ropes. If the necessary impulses are measured for every possible value of the positions and final velocities of the ropes, the kinetic energy of the system can be computed as a function of generalized coordinates and velocities (the homogeneity of T implies that $2T = \Sigma\, p_i \mathrm{d}q_i/\mathrm{d}t$). Then the motion of the ropes for any applied force is given by the corresponding Lagrange equations.[54]

4.4.4 *The electromagnetic momentum*

With the momentum interpretation, the vector potential became the central dynamical concept of Maxwell's theory. The induced electromotive force in a circuit was

[53] Green 1838: 246: 'One of the great advantages of this method [of the *Mécanique analytique*], of great importance, is, that we are necessarily led by the mere process of the calculation, and with little care on our part, to all the equations and conditions which are *requisite* and *sufficient* for the complete solution of any problem to which it may be applied'; Thomson to Stokes, 20 October 1847, in Wilson 1990: 32 (for least action applied to impulsively started fluid motion); Thomson and Tait 1867: 217–35. Cf. Siegel 1981: 259–63; Everitt 1975: 105–6; Everitt 1983: 128–9; Buchwald 1985: 60–1; Harman 1987: 287–88; Smith and Wise 1989: 270–3, 390–5 (on TT′).

[54] Maxwell 1873a: #554; Maxwell 1879: 783–84 for the belfry metaphor (for simplicity, I have excluded potential energy). Cf. Moyer 1977; Siegel 1981; Simpson 1970; Topper 1971; Buchwald 1985: 20–3.

just the time derivative of its reduced momentum. Maxwell further assumed that the circuit momentum was the line integral of the 'electromagnetic momentum' **A**. This gives

$$\int \mathbf{E}\cdot d\mathbf{l} = -\frac{d}{dt}\int \mathbf{A}\cdot d\mathbf{l}, \tag{4.22}$$

or

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} + \mathbf{v}\times(\nabla\times\mathbf{A}) - \nabla\phi \tag{4.23}$$

for the electromotive force **E** at a point of the conductor moving with the velocity **v**. Maxwell called ϕ the 'electric potential' and mentioned that it was determined by other conditions of the problem.[55]

In order to relate **A** to the magnetic field, Maxwell followed Faraday's suggestion of defining the magnetic lines of force by the electromotive force induced during their cutting by a linear conductor. Hence the magnetic quantity **B** must be identified with the curl of the electromagnetic momentum **A**. For the determination of **B** in terms of the current **J**, Maxwell used the reasoning of his 'On Faraday lines of force,' which leads to

$$\nabla \times \mathbf{H} = \mathbf{J} \tag{4.24}$$

for the intensity $\mathbf{H} = \mathbf{B}/\mu$.[56]

Maxwell then generalized the expression (4.20) for the kinetic energy of two currents, which gives

$$T = \frac{1}{2}\int \mathbf{J}\cdot\mathbf{A}\, d\tau. \tag{4.25}$$

This expression was most important in the new dynamical theory, for $\frac{1}{2}\mathbf{J}\cdot\mathbf{A}\,d\tau$ meant the energy *controlled* by the current in the volume element $d\tau$. Using $\mathbf{J} = \nabla \times \mathbf{H}$ and a partial integration, it could be transformed back into the expression given by the vortex model,

$$T = \int \frac{1}{2}\mathbf{B}\cdot\mathbf{H}\, d\tau, \tag{4.26}$$

in which $\frac{1}{2}\mathbf{B}\cdot\mathbf{H}\,d\tau$ referred to the energy *stored* in the element $d\tau$.[57]

[55] Maxwell 1865: 555–60.
[56] Maxwell 1865: 550–54, 556–57.
[57] Maxwell 1865: 562–63.

4.4.5 Closing the circuit

The Ampère law (4.24) only applies to divergenceless or closed current. More fundamentally, Maxwell's dynamical reasoning implies the restriction to closed currents, because only in this case is the magnetic field motion completely determined by the currents. If there is any elastic yielding of the field mechanism, as Maxwell assumed in his vortex model, then the motion also depends on the deformation of this mechanism. Maxwell's solution to this difficulty was to change the definition of the electric current. In the vortex model he had defined the current as the flux of particles between the vortices. In his 'dynamical theory,' he tried to follow Faraday's notion that the electric current was a variation or transfer of polarization.[58]

Maxwell first defined the polarization or 'electric displacement' **D** as a displacement of electricity in the molecules of the dielectric, referring here to Mossotti's theory of electrostatic induction. Being elastically resisted, the displacement required an electromotive force $\mathbf{E} = \mathbf{D}/\varepsilon$, and implied a potential energy of the medium

$$U = \frac{1}{2}\int \mathbf{E}\cdot\mathbf{D}\mathrm{d}\tau. \tag{4.27}$$

A variation of displacement implied an electric current $\partial\mathbf{D}/\partial t$. Electric conduction occurred when electricity was allowed to pass from one molecule to the next at the rate **j**. Hence, in a medium presenting both inductive capacity and conductivity, the total current was

$$\mathbf{J} = \frac{\partial\mathbf{D}}{\partial t} + \mathbf{j}. \tag{4.28}$$

The resulting expression of the Ampère law was the same as that given in 'On physical lines of force':

$$\nabla\times\mathbf{H} = \mathbf{j} + \frac{\partial\varepsilon\mathbf{E}}{\partial t}. \tag{4.29}$$

We must note, however, an important difference of interpretation. In the old theory, what Maxwell called the 'displacement current' was $-\partial\varepsilon\mathbf{E}/\partial t$ and contributed to the conduction current. In the new theory, the displacement current became a contribution to a divergenceless total current.[59]

In conformity with Mossotti's picture of polarization Maxwell took

$$\rho = -\nabla\cdot\mathbf{D} \tag{4.30}$$

[58] Maxwell 1865: 531. [59] Maxwell 1865: 554, 560. Cf. Siegel 1991: 145–52.

to represent the density of 'free electricity.' This brought him into grave difficulties, part of which he solved by reversing the sign in Ohm's law (he took $\mathbf{j} = -\sigma\mathbf{E}$). In fact, his equations were not compatible with the conservation of electricity, as is easily seen by taking the divergence of eqn. (4.28). He was here a victim of his well-known plus-minus dyslexia. He tended to place signs in his equations according to the underlying physical idea, not according to algebraic compatibility. Unfortunately, the physical idea under eqn. (4.30) was incompatible with Faraday's concept of electric charge, as Maxwell later realized.[60]

4.4.6 *Electromagnetic light waves*

Fortunately, the most important application of the new theory, the derivation of the equation of electromagnetic disturbances in a non-conducting medium, did not depend on the sign of electric charge. Maxwell combined eqns. (4.23), (4.24), and (4.28), and reached, for the magnetic induction,

$$\varepsilon\mu\frac{\partial^2\mathbf{B}}{\partial t^2} = \Delta\mathbf{B}. \tag{4.31}$$

This is a wave equation with the propagation velocity $(\varepsilon\mu)^{-1/2}$. Maxwell also treated the case of a crystalline medium, and determined how conductivity affected transparency. He now had an electromagnetic theory of light *sensu stricto*, since he could describe the waves directly in terms of the electric and magnetic fields. Moreover, his derivation of the velocity of the waves became independent of any assumption on the underlying mechanism.[61]

The electromagnetic momentum $\mathbf{A}$ being central to his new approach, Maxwell tried to determine how it propagated. Today's physicist knows that $\mathbf{A}$ is ambiguous: any gradient can be added to it without changing the measurable fields $\mathbf{E}$ and $\mathbf{H}$, provided that a compensating change of the scalar potential is performed. In 1862 Maxwell thought differently. He believed that $\mathbf{A}$ was unambiguously defined as the impulse necessary to start a given current. Also, he believed that he could maintain the general validity of Poisson's equation ($\Delta\phi + \rho = 0$). On this erroneous assumption he found that the longitudinal part of $\mathbf{A}$ could not propagate as a wave, in conformity with the transverse character of light waves.[62]

To summarize, by 1865 Maxwell had all the elements of a powerful dynamical theory of the electromagnetic field based on the following principles:

[60] Maxwell 1865: 561. Maxwell reversed the sign in Ohm's law, presumably to mend his theory of electric absorption (*ibid.*: 573–6); but he kept the plus sign in his study of wave absorption by conductors! On the problem of the sign of charge, cf. Siegel 1991: 148–52.

[61] Maxwell 1865: 577–88. Cf. Bork 1966a; Bromberg 1967; Chalmers 1973; Siegel 1991: 152–7.

[62] Maxwell 1865: 580–2. Maxwell was unaware that as the conjugate momentum of the 'velocity' $\mathbf{J}$, the potential $\mathbf{A}$ is ambiguous, because of the constraint $\nabla\cdot\mathbf{J} = 0$ (see Appendix 9). On Maxwell's confusions about the potentials, cf. Bork 1966a: 847–8; Bork 1967; Anderson 1991; Hunt 1991a: 116–17; Cat 1995.

1. Closed currents control a hidden motion in the field.
2. All current are closed.
3. Charge and current derive from polarization, which is an elastic deformation of the medium under electromotive force.

However, the theory was still hampered by confusions regarding the concepts of electromagnetic momentum and dielectric polarization.

4.4.7 Electromagnetic momentum, revised

When in 1868 Maxwell published the results of his new measurement of the ratio c of the electromagnetic to the electrostatic charge unit, he restated the electromagnetic theory of light 'in the simplest form, deducing it from admitted facts, and shewing the connexion between the experiments already described [for the measurement of c] and those which determine the velocity of light.' The 'admitted facts' were Oersted's electromagnetism, Faraday's law of electromagnetic induction, and Faraday's doctrine of polarization. From them Maxwell extracted four simple 'theorems' expressing in words the integrals of the magnetic and electric intensities on closed curves, the relation between electric intensity and displacement, and the displacement current. All reference to the electromagnetic momentum was gone, and the deduction of electromagnetic plane waves became quite elementary.[63]

Maxwell could not, however, renounce the dynamical foundation of his theory. It was an essential part of his later *Treatise*, in the Lagrangian form already described. There he acknowledged the gap in the definition of the electromagnetic momentum $\mathbf{A}$, and introduced the condition $\nabla\cdot\mathbf{A}=0$ as a convenient way to remove the ambiguity. With this choice the momentum of a given current in a medium of uniform permeability μ became

$$\mathbf{A}(\mathbf{r})=\frac{\mu}{4\pi}\int\frac{\mathbf{J}(\mathbf{r}')\mathrm{d}\tau'}{|\mathbf{r}-\mathbf{r}'|}. \tag{4.32}$$

The analogy with the scalar potential in a uniform dielectric,

$$\phi(\mathbf{r})=\frac{1}{4\pi\varepsilon}\int\frac{\rho(\mathbf{r}')\mathrm{d}\tau'}{|\mathbf{r}-\mathbf{r}'|}, \tag{4.33}$$

justified the alternative name 'vector potential' for $\mathbf{A}$.[64]

Unfortunately, in his derivation of the equation for the propagation of electromagnetic disturbances, Maxwell repeated the error of considering the scalar potential formula (4.33) as generally valid, independently of the choice of $\nabla\cdot\mathbf{A}$. In fact

[63] Maxwell 1868: 138. Cf. Everitt 1975: 108–9; Hendry 1986: 220–6; Siegel 1991: 153–4. This simple formulation of Maxwell's theory was largely unnoticed until its reprint by Niven in *MSP* in 1890 (I thank Bruce Hunt for this remark).

[64] Maxwell 1873a: #617.

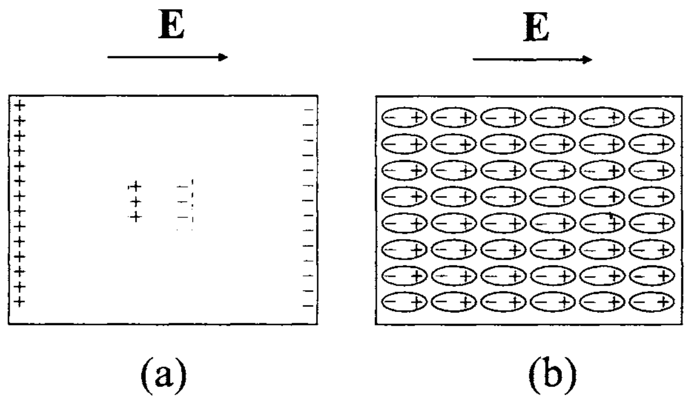

FIG. 4.9. Polarization: (a) according to Maxwell, (b) according to Mossotti.

there is no evidence that he ever combined the equation $\nabla \cdot \varepsilon \mathbf{E} = \rho$ with the electromotive force formula (4.23), except in the electrostatic case. He does not seem to have fully realized that his general assumptions on the electromagnetic field implied a much deeper interconnection of electrostatic and electrodynamic actions than was assumed in continental theories.[65]

4.4.8 Displacement, revised

However, Maxwell managed to clear up his concept of polarization. In the *Treatise* he adopted the positive sign in the relation $\rho = \nabla \cdot \mathbf{D}$, which means that a portion of polarized dielectric is charged positively where the polarization starts and negatively where the polarization ends (Fig. 4.9(a)). This choice agrees with Faraday's definition of positive charge as the starting point of electric lines of force, but contradicts Mossotti's picture of displaced electric charge (Fig. 4.9(b)). Maxwell, like Faraday, avoided the contradiction by considering the concept of polarization as more primitive than the concept of charge. If anything was displaced in the elements of a polarized dielectric, it could not be electric charge. This is an essential point, which must always be kept in mind when reading Maxwell's difficult sections on charge and current.[66]

Maxwell's views then appear to be very similar to Faraday's. Polarization (Faraday's 'induction') is defined as a state of constraint of the dielectric, such that each portion of it acquires equal and opposite properties on two opposite sides. By definition, electric charge is a spatial discontinuity of polarization. Typically, charge

[65] Maxwell 1873a: #783. Several of Maxwell's early readers, including Larmor and J. J. Thomson, inherited Maxwell's confusion on the definition of ϕ. In the same vein, Maxwell gave $-\rho\nabla\phi$ for the force acting on electrified matter (#619), which could be true only in the electrostatic case and contradicted his expression of electric stresses (#108) (see Appendix 6). This mistake is corrected in FitzGerald 1883b, and in the third edition of the *Treatise*.

[66] Maxwell 1873a: ##60–2, #111. See also Maxwell to Thomson, 5 June 1869, *MLSP* 2: 485–6. For a lucid account, cf. Buchwald 1985: 23–34; also Knudsen 1978.

occurs at the limit between a polarized dielectric and a conductor, because by definition a conductor is a body that cannot sustain polarization. As Maxwell explains, 'the electrification at the bounding surface of a conductor and the surrounding dielectric, which on the old theory was called the electrification of the conductor, must be called in the theory of induction the superficial electrification of the surrounding dielectric.'[67]

Conductors cannot sustain polarization. However, they may transfer polarization. This transfer, according to Faraday and Maxwell, results from a competition between polarization build-up and decay in the conductor. A conductor thus appears to be a yielding dielectric: 'If the medium is not a perfect insulator,' Maxwell writes, 'the state of constraint, which we have called electric polarization, is continually giving way. The medium yields to the electromotive force, the electric stress is relaxed, and the potential energy of the state of constraint is converted into heat.' By definition, the electric current is the rate of transfer of polarization. In a dielectric, it is simply measured by the time derivative of the polarization. In a conductor, it also depends on the decay mechanism, the microscopic details of which are unknown. Its expression must therefore be determined empirically (by Ohm's law). Thus defined, the electric current is always closed, for the current in an open conducting circuit is continued through the dielectric.[68]

All of this is quite consistent, and does not involve any of the absurdities later denounced by Maxwell's continental readers. Yet Maxwell's terminology was truly misleading. He called the polarization of a portion of dielectric 'a displacement of electricity.' By this phrase he only meant that a portion of the dielectric, if separated in thought from the rest of the dielectric, would present opposite charges at two opposite extremities. He certainly did not mean that an electrically charged substance was displaced. However, many of his readers understood just that. To make it worse, Maxwell asserted that 'the motions of electricity are like those of an *incompressible* fluid.' Here he only meant that the closed character of the total current made it analogous to the flow of an incompressible fluid. But he was often lent the opinion that electricity *was* an incompressible fluid.[69]

As long as it is used with proper care, the fluid analogy is useful to illustrate the relations between displacement, charge, and conduction. Suppose an incompressible fluid to pervade a space in which a rigid scaffolding has been erected. In 'insulating' parts of this space, the portions of the fluids are elastically linked to the scaffolding. In a 'conducting' part, such links also exist, but when under tension they tend to break down and dissipate their energy into heat; every breaking link is immediately replaced by a fresh, relaxed link. In this illustration, the extension of the links corresponds to Maxwell's displacement (or polarization); the pressure gradient of the fluid to the electromotive force; the flow of the fluid to the electric current; and the discontinuity of the average extension of the links when crossing the limit between conductor and insulator corresponds to electric charge. The analogy

[67] Maxwell 1873a: #60, #111. [68] Maxwell 1873a: #111. Cf. Buchwald 1985a: 28–9.
[69] Maxwell 1873a: #61. See also the fluid-piston illustration of a dielectric, *ibid.* in #334.

properly illustrates the equations $\mathbf{D} = \varepsilon\mathbf{E}$, $\nabla \cdot \mathbf{J} = 0$, $\mathbf{J} = \mathbf{j} + \partial\mathbf{D}/\partial t$, and $\rho = \nabla \cdot \mathbf{D}$. It is, however, misleading when one comes to propagation problems and energy flow, as we will later see.

4.5 *Exegi monumentum*

Around 1867 Maxwell set himself to work on a major treatise on electricity and magnetism. His intention was partly to propel his new theory and Faraday's underlying views. There also was an urgent need for that kind of book. Although the field of electricity and magnetism had grown enormously since Oersted and Ampère, there was as yet no unified presentation of all its experimental, technical, and mathematical aspects. The gap had widened between the practical electricity of telegraphists and the mathematical electricity of learned professors. There was a growing multiplicity of terms, conventions, and theories; and little attempt at uniformization and comparison, despite the high intellectual and economical stakes.[70]

Maxwell was especially sensitive to the neglect of the quantitative aspects of the subject. He believed that the mathematical theories of electricity and magnetism were ripe to be taught in the university, and pressed the Cambridge authorities to introduce them in the Mathematical Tripos. Only the proper reference book was missing, as Maxwell himself judged:[71]

> There are several treatises in which electrical and magnetic phenomena are described in popular way. These, however, are not what is wanted by those who have been brought face to face with quantities to be measured, and whose minds do not rest satisfied with lecture-room experiments.—There is also a considerable mass of mathematical memoirs which are of great importance in electrical science, but they lie concealed in the bulky Transactions of learned societies; they do not form a connected system; they are of very unequal merit, and they are for the most part beyond the comprehension of any but professed mathematicians.—I have therefore thought that a treatise would be useful which should also indicate how each part of the subject is brought within the reach to methods of verification by actual measurement.

Books on electricity were indeed few, and failed to provide a full, systematic introduction to the subject. Auguste de la Rive's *Traité d'électricité* of 1853 was very empirical, had almost no mathematics, and ignored or misrepresented Faraday's theoretical views. Gustav Wiedemann's *Lehre vom Galvanismus* of 1863 gave precise and clear accounts of nearly all works published on the subject, with a fair share of the British views; but its encyclopedic scope and structure made it unsuited to the guidance of students. Maxwell's *Treatise*, published in 1873, filled a major gap in the existing literature.[72]

[70] On the gap between practical and academic electricity, cf. the introduction of Jenkin 1873.

[71] Maxwell 1873a: ix. On the 1867 reform of the Cambridge Mathematical Tripos and on the editorial circumstances of Maxwell's project, cf. Achard 1998.

[72] On the genesis of the *Treatise*, cf. Harman 1995a: 26–33.

4.5.1 Mathematical and empirical foundations

Maxwell's challenge was to expound a new doctrine and at the same time to establish new standards in the treatment of current problems. In order to meet these two conflicting requirements, he carefully separated the basic mathematical and empirical foundations of the subject from more speculative theory. In a preliminary 'on the measurement of quantities' he expounded Fourier's doctrine of dimensions, Hamilton's distinction between scalar and vector, the notions of force and flux corresponding to his older 'intensity' and 'quantity,' various theorems relating the integrals of force and flux, and related topological questions. He regarded the classification of physico-mathematical quantities as a way to short-circuit formal analogies and organize the field of knowledge: 'It is evident that [. . .] if we had a true mathematical classification of quantities,' he had earlier explained, 'we should be able at once to detect the analogy between any system of quantities presented to us and other systems of quantities in known sciences, so that we should lose no time in availing ourselves of the mathematical labours of those who had already solved problems essentially the same.'[73]

Maxwell then defined the basic physical quantities in a neutral manner that could be accepted both by fluid and field theorists. For example, he introduced the quantity of electric charge of a body by means of Faraday's hollow conductors: two charges could be added by bringing their carriers into a hollow conducting vessel and noting the charge of the vessel. He defined the electric potential in Thomson's manner, as the work done on a unit point charge to bring it at a given place. Lastly, he defined the magnetic force **H** and flux **B** in a polarizable substance as the forces acting on a magnetic unit pole (end of uniformly magnetized needle) placed in a small cylindrical cavity, elongated for **H**, and flat for **B**.[74]

With these neutral definitions, Maxwell could conduct much of the mathematical analysis without deciding the nature of electricity and magnetism. This can be seen in his Thomsonian presentation of the potential theories of electrostatics and magnetism. The *Treatise* was in part meant as a source book for computational and experimental techniques for competent electricians, whatever they might think of the essence of electricity. The originators of these techniques were as diverse as their potential users. They could be Laplace on spherical harmonics, Gauss on geomagnetism, Weber on galvanometric measurements, Kirchhoff on circuit theory, Thomson on electrometers, or Maxwell himself on the calculation of inductance.

[73] Maxwell 1873a: ##1–26; Maxwell 1870: 258 (quote). Cf. Harman 1987: 278–87. On dimensions, cf. Jenkin and Maxwell 1863; Everitt 1975: 100–1; d'Agostino 1996: 37–41. On topology, cf. Epple 1998; Harman 1998: 153–6.

[74] Maxwell 1873a: #34 and #63 (for charge), #70 (for potential), ##398–400 (for **B** and **H**). Maxwell and Thomson disagreed on the definition of the electrostatic potential when contact between different metals was involved: cf. Hong 1994a. Maxwell's **B** and **H** corresponded to Thomson 'electromagnetic' and 'polar' definitions of the magnetic field (see Chapter 3, p. 130); however, they referred to two different physical concepts (flux and force), whereas Thomson only meant two different ways of characterizing the same physical entity: cf. Wise 1981a.

These techniques could serve the Mathematical Tripos, German seminars, and telegraphists in the whole industrialized world.[75]

4.5.2 Tolerance

Once equipped with operational definitions and phenomenologico-mathematical theories, the reader of the *Treatise* could enter the realm of higher theory. Maxwell presented the field view, the fluid view, and the relations between the two. Of course, he preferred Faraday's field conception. Compared with the fluid conception, he wrote, it is 'no less fitted to explain the phenomena, and [. . .] though in some parts it may appear less definite, corresponds, as I think, more faithfully with our actual knowledge, both in what it affirms and in what it leaves undecided.' In private, he made fun of the 'learned Germans,' the 'heavy German writers,' or Ampère's 'kind of ostensive demonstration.'[76]

Yet the *Treatise* paid due respect to 'the Newton of electricity' (Ampère) and to the 'eminent' Germans who cultivated action at a distance; and it expounded their theories in sufficient details. This was not only diplomacy: as we will later see, Maxwell integrated some of Ampère's and Weber's atomistics into his own theory. Also, he believed that much could be learn from the comparison between the two kinds of theory:

> In a philosophical point of view, moreover, it is exceedingly important that two methods should be compared, both of which have succeeded in explaining the principal electromagnetic phenomena, and both of which have attempted to explain the propagation of light as an electromagnetic phenomenon [more on this later], and have actually calculated its velocity, while at the same time the fundamental conceptions of what actually takes place, as well as most of the secondary conceptions of the quantities concerned, are radically different.[77]

4.5.3 Field basics

For the essentials of field theory, Maxwell followed Faraday closer than he had ever done. As we have seen, he adopted the field-based definitions of electric charge and current, the concept of conduction as the competition between polarization build up and decay, and the reduction of all electric and magnetic actions to stresses in the field. Even the idea that all currents are closed can be traced back to Faraday's idea of the indivisibility of the electric current (cf. Chapter 3, p. 91). Lastly, Maxwell renounced his earlier theory of magnetism, in which the 'quantity' **B** had sources in

[75] Cf. Maxwell 1873a: Vol. 1, Part 1, Ch. 4 ('General theorems' of potential theory); 2.3.3 (on Thomson's 'Magnetic solenoids and shells'); 1.1.9 ('Spherical harmonics'); 2.3.8 ('Terrestrial magnetism'); 2.4.15 ('Electromagnetic instruments'); 1.2.6 ('Mathematical theory of the distribution of electric currents'); 1.1.13 ('Electrostatic instruments'); 2.4.13 ('Parallel currents').

[76] Maxwell 1873a, Vol. 1: xii; Maxwell to John Clerk Maxwell, 5 May 1855, *MSLP* 1: 294; Thomson to Tait, 1 December 1873, *MSLP* 2: 947; Maxwell to Thomson, 13 November 1854, *MSLP* 1: 255.

[77] Maxwell 1873a: #528; *ibid.*, Vol. 1: xii; Vol. 2, Part 4, Ch. 2 ('Ampère's investigation . . .'); Vol. 2, Part 4, Ch. 13 ('Theories of action at a distance'); Vol. 1: xii.

the magnetic masses of magnets as the electric quantity **D** had sources in charged bodies. In his new theory, **B** was always divergenceless, in conformity with Faraday's notion of magnetic lines of force and with Thomson's flat-cylinder-cavity definition.[78]

For the field equations, Maxwell also depended on Thomson's field mathematics, on the distinction between force and flux, and on the interpretation of Lagrange's equations in terms of energy, force, and momentum. In short, from Faraday's notion of dielectric polarization Maxwell derived the equations

$$\mathbf{D} = \varepsilon \mathbf{E}, \quad \nabla \cdot \mathbf{D} = \rho, \quad \mathbf{J} = \mathbf{j} + \frac{\partial \mathbf{D}}{\partial t} \tag{4.34}$$

for the electric force **E**, the electric displacement **D**, the total current **J**, and the conduction current **j**. From his own theory of magnetization and from the equivalence between an infinitesimal current loop and a magnetic dipole, he deduced

$$\mathbf{B} = \mathbf{H} + \mathbf{I}, \quad \nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{H} = \mathbf{J} \tag{4.35}$$

for the magnetic force **H**, the magnetic induction **B**, and the intensity of magnetization **I**. From the Lagrangian dynamics of closed currents he obtained

$$\mathbf{E} = \frac{\partial \mathbf{A}}{\partial t} + \mathbf{v} \times (\nabla \times \mathbf{A}) - \nabla \phi, \quad \mathbf{f} = \mathbf{J} \times \mathbf{B}, \tag{4.36}$$

where **A** is the electromagnetic momentum, **v** the velocity of the current carrier, and **f** the electrodynamic force acting on the current carrier. The first formula gives Faraday's induction law if **A** is the vector potential such that $\mathbf{B} = \nabla \times \mathbf{A}$. Maxwell further imposed $\nabla \cdot \mathbf{A} = 0$ in order to simplify the relation between **A** and the total current. Lastly, in the absence of a specific mechanism for the decay of displacement, he admitted Ohm's law $\mathbf{j} = \sigma \mathbf{E}$. In a separate chapter, he gave the formula

$$w = \frac{1}{2} \mathbf{E} \cdot \mathbf{D} + \frac{1}{2} \mathbf{B} \cdot \mathbf{H} \tag{4.37}$$

for the energy density of the field (in the absence of permanent magnetism) and the expression

$$\sigma_{ij} = D_i E_j - \frac{1}{2} \delta_{ij} \mathbf{E} \cdot \mathbf{D} + B_i H_j - \frac{1}{2} \delta_{ij} \mathbf{H} \cdot \mathbf{B} \tag{4.38}$$

[78] Maxwell 1873a: Part 3, Ch. 2: 'Magnetic force and magnetic induction.'

for the stresses in the field. In principle, all mechanical forces of electric or magnetic origin could be derived from these stresses (see Appendix 6).[79]

4.5.4 Physical ideas and equations

The sheer number of equations (especially in Cartesian notation) was likely to scare Maxwell's reader. So as to please 'the Chief Musician upon Nabla' (his friend Tait), and for the sake of mathematical power and beauty, Maxwell also wrote his equations in quaternion form.[80] This could hardly help the average reader, as Maxwell himself suspected. A more pedagogical step would have been to eliminate the potentials. Maxwell refused to do so in the *Treatise*, arguing that 'to eliminate a quantity which expresses a useful idea would be rather a loss than a gain in this stage of our enquiry.' He wanted to place the electromagnetic momentum at the forefront of his theory.[81]

In general, Maxwell's use of mathematical symbolism differed essentially from continental or modern practice. For him the equations were always subordinated to the physical picture. He sought consistency, completeness, and simplicity in the picture, not necessarily in the equations. The latter were symbolic transcriptions of partial aspects of the picture, and therefore could not be safely used without keeping the underlying picture in mind. This is quite visible in the way Maxwell treated the electrodynamics of moving bodies. His equations included electromagnetic induction in moving bodies, but not other effects of motion that resulted from his pictures of charge and current. For example, he knew that the convection of electrified bodies constituted an electric current because of the corresponding variation of displacement; but there was no convection current in his equations.

4.5.5 Microphysics

Maxwell had another reason not to seek algebraic completeness. He was aware that his theory was essentially incomplete in its treatment of the relation between ether and matter. The general pictures of dielectric and magnetic polarization, and also the idea of currents controlling a hidden motion, implied that ether and matter behaved as a single medium with variable inductive capacity, permeability, and conductivity. Maxwell admitted, however, that some phenomena required a closer look at the interaction between ether and matter. First of all, his picture of electric conduction left the mechanism of polarization decay in the dark. Like Faraday he hoped

[79] Maxwell 1873a: #68, #83, #610 (for eqns. 4.34); #400, #403, #607 (for eqns. 4.35); #598, #603 (for eqns. 4.36); #241 (for Ohm's law); #630, #634 (for eqn. 4.37); #108, #641 (for eqn. 4.38). Maxwell recapitulated the field equations in ##237–38.

[80] Maxwell 1873a: #17, #25, #619. 'Nabla' is an Assyrian harp, of the same shape as Hamilton's ∇; at the BA meeting of September 1871, Maxwell dedicated a poem to Tait, the 'Chief Musician upon Nabla': cf. Campbell and Garnett 1882: 634–6. On the history of quaternions, cf. Crowe 1967. On their use by Maxwell, cf. Harman 1987: 279–82, 1994: 29–30; 1998: 145–53; McDonald 1965; and the related manuscripts and letters in *SMLP* 2.

[81] Maxwell 1873a: #615.

that the study of electric glows, and especially that of electrolysis would shed light on the deeper nature of electricity: 'Of all electrical phenomena,' he declared, 'electrolysis appears the most likely to furnish us with a real insight into the true nature of the electric current, because we find currents of ordinary matter and currents of electricity forming essential parts of the same phenomnenon.'[82]

In his chapters on electrolysis, Maxwell did not follow Faraday's phenomenological approach. As a believer in atomistics, he found it

> extremely natural to suppose that the currents of the ions are convection currents of electricity, and, in particular, that every molecule of the cation is charged with a certain fixed quantity of positive electricity, which is the same for the molecules of all cations, and that every molecule of the anion is charged with an equal quantity of negative electricity.

This assumption accounted for Faraday's law, and could be perfected to explain electrode polarization. But the quantization of charge puzzled Maxwell. It seemed to suggest the existence of 'molecules of electricity,' as if electricity were a discrete fluid. Maxwell bore the contradiction, though not in silence: 'This phrase, gross as it is, and out of harmony with the rest of this treatise, will enable us at least to state clearly what is known about electrolysis, and to appreciate the outstanding difficulties.' He regarded the propounded theory as a provisional mnemonic aid: 'It is extremely improbable that when we come to understand the true nature of electrolysis we shall retain in any form the theory of molecular charges, for then we shall have obtained a secure basis on which to form a true theory of electric currents, and so become independent of these provisional theories.'[83]

Yet Maxwell did not doubt that molecular structure played a role in conduction. He also approved Weber's theory of induced magnetism, which required the existence of permanently magnetized molecules. And he took Ampère's and Weber's molecular currents quite seriously. In his opinion, Verdet's finding that magneto-optical rotation had opposite signs in diamagnetic and ferromagnetic bodies excluded Faraday's doctrine that a diamagnetic was nothing but a lesser conductor of magnetism than vacuum. The *Treatise* had a chapter devoted to the improvement of Weber's theory of ferromagnetism, and another to 'the electric theories of magnetism,' including Weber's induced molecular currents. Maxwell emphasized the simplification of the magnetic field equations when all magnetism was reduced to electromagnetism: molecular currents thus became the only sources, and the fields **B** and **H** became identical and divergenceless.[84]

Maxwell also believed that the molecular structure of matter played a role in the propagation of light. He did not trust his field equations for high-frequency vibrations in material bodies. In dielectrics, these equations did not include optical dispersion and implied a relation between optical index and inductive capacity ($n = \varepsilon^{1/2}$)

[82] Maxwell 1873a: #55 (glow), #255 (electrolysis). [83] Maxwell 1873a: #255, #260.

[84] Maxwell 1873a: Vol. 2, Part 3, Ch. 6 ('Weber's theory of magnetic induction,' with a modification explaining residual magnetization); Maxwell to Tait, 23 December 1867, in *SLPM* 2: 336, and Maxwell 1873a: #809 (Verdet excluding Faraday); Vol. 2, Part 4, Ch. 22 ('Electric theory of magnetism'); #835 (simplicity of Amperean view).

that seemed to hold only very roughly. In conductors, they predicted an absorption of light much larger than that measured on gold leaves. In such cases, Maxwell judged, 'our theories of the structure of bodies must be improved before we can deduce their optical properties from their electrical properties.'[85]

Maxwell's equations did not contain the Faraday effect either: their linearity excluded any action of an external magnetic field on the propagation of light. Remember, however, that Maxwell had earlier given a theory of the Faraday effect, based on his vortex model of the magnetic field. In the *Treatise*, he extracted from this model the basic idea of a magnetic vortex motion perturbing the optical vibrations, and cast it in Lagrangian form. He borrowed the unperturbed part of the Lagrangian from the elastic solid theory of light, and assumed the simple expression $k(\nabla \times \partial\boldsymbol{\xi}/\partial t)\cdot\mathbf{H}$ for the energy density of the magneto-optical interaction; $\boldsymbol{\xi}$ represents the elastic displacement of the medium, $\nabla \times \boldsymbol{\xi}$ twice the corresponding rotation, and $k\mathbf{H}$ the vortical motion implied by the magnetic force $\mathbf{H}$. The latter differs from the impressed magnetic force $\mathbf{H}_0$ by the amount $(\mathbf{H}_0\cdot\nabla)\boldsymbol{\xi}$ if the vorticity depends on the displacement $\boldsymbol{\xi}$ in the manner implied by Helmholtz's theory of vortex motion ($\mathbf{H}\cdot d\mathbf{S}$ invariant). Then the optical Lagrangian involves a new term combining $\boldsymbol{\xi}$ and $\partial\boldsymbol{\xi}/\partial t$, from which the magneto-optical rotation is easily deduced. With this semi-phenomenological reasoning, Maxwell avoided atomistic speculation and absorbed the whole effect of matter into one coupling constant, whose value and sign were to be drawn from Verdet's measurements.[86]

This theory of the Faraday effect, and all of Maxwell's attempts to specify the relation between ether and matter, were meant to be provisional. The macroscopic character of his unification of electrodynamics, electrostatic, and optics, conflicted with the empirical need to introduce the molecular structure of matter. Maxwell did not know to what extent his electromagnetic concepts applied at the molecular scale. He avoided microphysical considerations whenever the macroscopic approach proved sufficient.

4.6 Conclusions

Proceeding from Faraday's and Thomson's writings, Maxwell reached the essentials of his electromagnetic field theory stepwise, in three great memoirs. In 'On Faraday's lines of force' his aim was to obtain a mathematical expression of Faraday's field conception. He found the methods of Thomson's field mathematics particularly useful, but modified them substantially. Thomson gave the electric and magnetic (scalar) potentials a central role, as neutral mediators between the mathematics of action at a distance and Faraday's field reasonings. Instead Maxwell made

[85] Maxwell 1873a: ##788–9 ($\varepsilon \sim n^2$ and dispersion); #800 (gold sheets) and also transparency of electrolytes in #799; #789 (quote). Maxwell gave a molecular theory of anomalous dispersion in a Tripos question of 1868 (*SLMP* 2: 419–21, and Rayleigh 1899), also in an 1873 manuscript (*SLMP* 2: 461–2): see Whittaker 1951: 262; Buchwald 1985a: 236; Harman 1994: 11–12.

[86] Maxwell 1873a: #822–7 (Maxwell also included Cauchy's dispersion terms). Cf. Knudsen 1976: 278–81.

the lines of force the central concept of his theory. He threw a geometrical net of lines of force and orthogonal surfaces over Faraday's field, and caught the mathematical field laws directly in terms of the field quantities. He also used Thomson's flow analogy, and extracted from it an essential structural component of his theory: the distinction between intensity and quantity (force and flux). With these modifications of Thomson's methods, Maxwell invented a powerful field-gridding geometry and obtained two circuital laws $\nabla \times \mathbf{H} = \mathbf{j}$ and $\mathbf{E} = -\partial\mathbf{A}/\partial t$ that captured Faraday's intuition of the relations between electricity and magnetism.

In the first part of 'On physical lines of force' Maxwell exhibited a mechanical model of the magnetic field that closely followed Thomson's insights into the vortical nature of magnetism. Unlike the previous flow analogy, this model accounted for the mechanical forces of magnetic origin and for electromagnetic induction. Maxwell soon modified it to include electrostatics and optics, in a manner totally unforeseen by Thomson. This gave the displacement current, the full set of Maxwell's equations, and an expression of the velocity of light in terms of electromagnetic quantities. Although Maxwell acknowledged the artificiality of his model, he firmly believed in the reality of two features: the mutually connected vortical rotations, and the elastic yielding of the connecting mechanism. The rotations represented the magnetic field, and the elastic yielding the electric field (displacement).

In his 'dynamical theory of the electromagnetic field,' Maxwell replaced his vortex model with a dynamical justification of his field equations. He treated the magnetic field as a hidden mechanism, whose motion was controlled by the electric current. The potential **A** thus acquired a central importance as the reduced momentum of the field mechanism dragged by the electric current. Maxwell combined his field equations to obtain a wave equation, and reached a truly electromagnetic optics in which light became a waving electromagnetic field.

The dynamical approach required that the magnetic motion should be determined by the currents only. Accordingly, Maxwell made the displacement current part of the total current. This move brought him closer to Faraday's concepts of charge and current. In the vortex model, the electric current corresponded to the flow of the particles between the vortices and charge to their accumulation. In the new dynamical theory, and more definitely in the *Treatise*, Maxwell defined the electric current as a transfer of polarization, and charge as a discontinuity of polarization. Here polarization was a primitive concept: any attempt to interpret it as a microscopic displacement of electric charge led to absurdities. Maxwell's theory was a pure field theory, ignoring the modern dichotomy between electricity and field.

In the mature form of the *Treatise*, Maxwell's theory had a central core founding the general theory of the electromagnetic field, and a periphery dealing with less understood phenomena. The core contained the pure field theory of electricity with field-based concepts of charge and current, a dynamical derivation of the equations of motion by the Lagrangian method, and the essentials of the electromagnetic theory of light. The periphery included fragmentary mechanisms for the various kinds of electric conduction, and special theories of magnetization and magneto-optical rotation.

The core was essentially macroscopic, in the sense that the basic concepts of field, charge, and current had a macroscopic meaning. It treated matter and ether as a single continuous medium with variable macroscopic properties (specific inductive capacity, magnetic permeability, and conductivity), and avoided speculation on ether models and matter molecules. At the periphery, Maxwell recognized the need for a more detailed picture of the connection between ether and matter. He tried three different strategies. For magnetization, he modified his theory to integrate molecular assumptions; for electrolysis, he proposed a temporary ionic theory that contradicted his general concept of the electric current; for the Faraday effect, his method was essentially based on a phenomenological modification of the optical Lagrangian, although he invoked a deeper molecular mechanism.

By rejecting direct action at a distance and electric fluids, Maxwell distanced himself from continental physics. Whether he did so in a consistent manner has been a major question for Maxwell's commentators. Recent scholarship has established that Maxwell was far more consistent than has usually been admitted. As Siegel has shown in detail, Maxwell's vortex model holds together very well and accounts for all electrodynamic and electrostatic phenomena known to Maxwell. Most of the inconsistencies perceived by earlier commentators of this model can be traced to their failure to distinguish the relevant concepts of charge and current from those proposed in the *Treatise*.[87] Admittedly, there were genuine inconsistencies in the memoir on the dynamical theory due to the unwarranted mixture of Faraday's and Mossotti's concepts of polarization. In the form given in the *Treatise*, however, Maxwell's concepts of charge and current were quite consistent, as Buchwald has most clearly shown. Here Maxwell's readers were often misled by the metaphor of 'displacement of electricity,' which seems to indicate a shift of electric charge (as occurs in the continental concept of polarization), whereas Maxwell only meant something analogous to the shift of a *neutral* incompressible fluid. Charge is not what is displaced, it is a spatial discontinuity in the strain implied in the 'displacement.' As will be seen in the next chapter, the consistency of Maxwell's views comes out clearly in the more pedagogical presentations offered by Maxwell's followers.

Another logorrhea of Maxwellian scholarship has been about the origin of the displacement current. The excessive focus on this question has resulted in a misrepresentation of Maxwell's overall endeavors and achievements in electric topics. As Wise pointed out, Maxwell's first major innovations were an essentially new geometrization of Faraday's and Thomson's field concepts, and the important distinction between quantity and intensity. The former yielded Maxwell's form of the Ampère law ($\nabla \times \mathbf{H} = \mathbf{j}$), and the latter prepared the ground for the dynamical theory. As for Maxwell's path to the displacement current, it may be summarized as follows.

When Maxwell worked out Thomson's notion of a vortical motion in the mag-

[87] Also, some of them were unable to understand the mechanics of the model.

netic field, he introduced the idle wheels as a direct illustration of the current being the curl of the magnetic force. The original purpose of this mechanism was purely electrodynamic. Maxwell knew, however, that both Faraday's electrostatics and the wave theory of light required an elastic medium. He also knew that the mechanical consistency of his model required an elasticity of the rotating cells. When he took this elasticity into account, he found it to imply a new contribution $-\partial \varepsilon \mathbf{E}/\partial t$ to the current $\mathbf{j}$ of idle wheels. The corresponding modification of the Ampère law allowed for open currents.

In such a dense argument, it would be vain to single out a specific reason for Maxwell's introduction of the displacement current. He sought the most complete and consistent theory that would comply with a number of entangled conditions: expression in terms of Faraday's lines of force and the related intensity/quantity pairs (**E**, **D**) and (**H**, **B**), existence of vortical motion in the magnetic field, integration of the vortical motion in a mechanical model of the ether, possibility of dielectric polarization, identity of the electromagnetic and optical ethers.[88] To make the story even more complex, in his later dynamical theory and in his *Treatise* Maxwell provided a different justification of the displacement current, based on Faraday's concepts of charge and current. Every current became closed and the Ampère law no longer needed to be modified.

Maxwell's electromagnetic theory exemplified a powerful methodology. Important aspects of this methodology can be traced to other British authors. Maxwell praised Thomson and Tait's 'method of cultivating science, in which each department in turn is regarded, not merely as a collection of facts to be coordinated by means of the formulae laid up in store by the pure mathematicians, but as itself a new mathesis by which new ideas may be developed.' This approach included the dynamical ideas through which the 'two Northern wizzards' conducted their mathematical reasonings. It also provided the illustrations and analogies that Maxwell shared with Thomson. The basics of field mathematics were not born in the brains of pure mathematicians. They required the suggestive imagery of flowing liquids and strained solids.[89]

Maxwell's methodology had more original components. He developed the classification of mathematical quantities as a short-cut through the method of formal analogies. He gave more weight to geometrical reasoning than Thomson did, and filled his papers with beautiful figures of curving lines and surfaces. He had an eye for topological relations, as today's field theorists do. Lastly, he inaugurated a moderate kind of mechanical reductionism, in which the connecting mechanism was no longer exhibited. The mere assumption of the existence of such a mechanism implied the existence of a Lagrangian, from which the evolution of empirically controllable quantities could be deduced. Maxwell still hoped, however, for a more detailed mechanical understanding of field processes. For the time being he made sure that Lagrangian dynamics would not be too abstract. He fleshed it out with metaphors, illustrations, and energetics.[90]

Regarding consistency, economy, and pedagogy, Maxwell's *Treatise* was

[88] Cf. Siegel 1975. [89] Maxwell 1873c: 325.
[90] Cf. the penetrating analysis in Harman 1987.

imperfect, even in its core. For example, Maxwell did not fully realize the ambiguity of his potentials; he refused to eliminate them from the final equations; and he misled many of his readers with his metaphor of displacement. In the periphery, he tolerated the contradiction of quantized electric charge, and he occasionally regressed to the elastic solid theory of light. However, the system of the *Treatise* was sufficiently definite to guide further improvements. Maxwell defined a new kind of theoretical physics in which the classification of mathematical quantities, vector symbolism, and Lagrangian dynamics became major construction tools. He also revealed a tension between field macrophysics and the atomic structure of matter, and inaugurated ways of dealing with this tension. His physics was an unended quest. He provided methods that kept theory open and alive.