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FUNDAMENTAL INVESTIGATIONS 'ON THE MOTION OF BODIES'(1) [Autumn 1684–Winter 1685/6]

Excerpts from the originals in the University Library, Cambridge

§1. THE FIRST TRACT 'DE MOTU CORPORUM' (AUTUMN 1684).(2)

De motu corporum in gyrum.(3)

Def. 1. Vim centripetam(4) appello qua corpus impellitur vel attrahitur versus aliquod punctum quod ut centrum spectatur.

Def. 2. Et vim corporis seu corpori insitam qua id conatur perseverare in motu suo secundum lineam rectam.

(1) We here reproduce Newton's original autograph draft of the (now lost) tract 'De motu Corporum' which he sent to London in November 1684, and the two successive versions of its major revision in 1685 which he subsequently deposited (in an incomplete state) in Cambridge University Library, purportedly as the text of lectures delivered from the Lucasian chair during 1684–5. As stated in the preceding introduction, it is our primary concern to stress internal mathematical aspects of these documents, seeking to pinpoint their place in the broad sequence of Newton's mathematical development. To that end we relatively—and injustly—neglect to give detailed examination of the often revolutionary dynamical principles which they embody. For this we refer to the many able and penetrating discussions of the quality, novelty and sequence of Newton's developing notions of space, motion and force which we have already listed.

(2) ULC. Add. 3965.7: 55^{r}–$62bis^{r}$, first published by J. W. Herivel in *The Background to Newton's Principia. A Study of Newton's Dynamical Researches in the Years 1664–84* (Oxford, 1966): 257–74. In the much corrected and overwritten manuscript as we now have it two principal layers in its composition may be identified. The text of its initial state agrees narrowly with those of the two known contemporary transcripts of the putative fair copy (by Newton's amanuensis, Humphrey Newton?) which was sent to London by way of Edward Paget in November 1684—that, namely, which was entered in (or shortly after) early December following in the Royal Society's Register Book (**6**: 218–34, first printed by S. P. Rigaud in his *Historical Essay on the first Publication of Sir Isaac Newton's 'Principia'* (Oxford, 1838): Appendix No. 1: 1–19 under the title 'Isaaci Newtoni Propositiones De Motu') and that, made perhaps a little earlier, by Edmond Halley (whose first five propositions, afterwards returned to Newton, are now ULC. Add. 3965.7: 63^{r}–70^{r}). The only significant deficiences in Newton's original

Translation

On the motion of bodies in an orbit.(3)

Definition 1. A 'centripetal' force(4) I name that by which a body is impelled or attracted towards some point regarded as its centre.

Definition 2. And the force of—that is, innate in—a body I call that by which it endeavours to persist in its motion following a straight line.

autograph as against the copy subsequently registered—namely, the explicit enunciation in the latter of 'Hyp[othesis] 4' and the addition there of two opening Lemmas already employed as riders in the body of the former draft—are here made good by appropriate editorial insertions in its text (see notes (12), (13) and (15) below). The changes and revisions which were afterwards effected in this primary state of the 'De motu Corporum in gyrum' merely convert it to be identical with corresponding portions of the augmented tract 'De motu sphæricorum Corporum in fluidis' (ULC. Add. 3965.7: 40^{r}–54^{r}; see Appendix 1 following) which Humphrey Newton penned from it a while later. (The main innovations in this latter text were first published by W. W. R. Ball in his *An Essay on Newton's 'Principia'* (London, 1893): 51–6 in sequel to his straightforward repeat (*ibid.*: 35–51) from Rigaud of the Royal Society transcript of the original fair copy sent by Newton to London; reproduction *in toto* of its text is made by A. R. and M. B. Hall in their *Unpublished Scientific Papers of Isaac Newton. A Selection from the Portsmouth Collection in the University Library, Cambridge* (Cambridge, 1962): 243–67.) For the historical background to Newton's composition of the present piece see the preceding introduction, and compare I.B. Cohen's *Introduction to Newton's Principia* (Cambridge, 1971): Chapter III, 'Steps towards the *Principia*': 47–81, especially 54–62.

(3) Literally a closed circuit, but understand any path which is everywhere convex round some internal point. While Newton has principally in view the minimally eccentric ellipses which the orbits of the solar planets narrowly approximate, in his 'Prob. 4' below he will not for instance, exclude the open parabolas and hyperbolas which are equally possible orbits under an inverse-square force directed to a focus. The revised manuscript (see note (2)) bears the more sophisticated title 'De motu sphæricorum Corporum in fluidis' (On the motion of spherical bodies in fluids) which better defines its theme. A Newtonian 'fluid' is, of course, any uniform medium—such as the terrestrial atmosphere— which may or may not offer appreciable resistance to the passage of a body through it, while the insistence that the latter be a spherical mass may just possibly suggest that Newton had by this time already achieved the insight that in an inverse-square force-field such a body behaves as though its mass were concentrated at its centre, as he was soon rigidly to demonstrate (see §2.3: note (188)).

(4) The first occurrence of this classical *terminus technicus*, contrived as the complement of the term 'vis centrifuga (ex motu circulari) used by Christiaan Huygens to denote the 'endeavour' outwards from the centre of a body constrained to rotate uniformly in a circle, and first published in his *Horologium Oscillatorium sive De Motu Pendulorum ad Horologia aptato* (Paris, 1673): 159. When in 1719 Newton wrote to Des Maizeaux regarding Leibniz' correpondence during 1715–16 'sur l'invention des Fluxions & du Calcul Differentiel' (as Des Maizeaux headed his gathering of it in Tome II of his *Recueil de Diverses Pieces*..., Amsterdam, $_{1}$1720), he observed at one point in a critique of Leibniz' celebrated 'Apostille' to his letter to Conti in December 1715 that 'M^{r} Hygens gave the name of *vis centrifuga* to the force by w^{ch} revol[v]ing bodies recede from the centre of their motion. M^{r} Newton in honour of that author retained the name & called the contrary force *vis centripeta*' (ULC. Add. 3968.28: 415^{v}; see also A. Koyré and I. B. Cohen, 'Newton & the Leibniz–Clarke Correspondence', *Archives Internationales d'Histoire des Sciences*, **15**, 1962: 63–126, especially 122–3).

Def. 3.(5) Et resistentiam(6) quæ est medij regulariter impedientis.

Hypoth 1.(7) Resistentiam in proximis novem propositionibus nullam esse, in sequentibus esse ut corporis celeritas et medij densitas conjunctim.(8)

Hypoth 2. Corpus omne sola vi insita uniformiter secundum rectam lineam in infinitum progredi nisi aliquid extrinsecus impediat.(9)

Hyp. 3. Corpus in dato tempore viribus conjunctis eo ferri quo viribus divisis in temporibus æqualibus successivè.(10)

Hyp. 4. [Spatium quod corpus urgente quacunq; vi centripeta ipso motus initio describit esse in duplicata ratione(11) temporis.](12)

[*Lem. 1.* Quantitates differentijs suis proportionales sunt continuè proportionales. Ponatur A ad $A-B$ ut B ad $B-C$ & C ad $C-D$ &c et dividendo fiet A ad B ut B ad C et C ad D &c.](13)

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(5) A late insertion in the manuscript, evidently added when (in afterthought?) Newton decided to append Problems 6 and 7 and their scholium on motion resisted as the instantaneous speed.

(6) Understood to be a Newtonian *vis* acting instantaneously in a direction contrary to that of the body's motion.

(7) Newton initially here made the blanket assumption that 'Corpora nec medio impediri nec alijs causis externis quo minus viribus insitæ et centripetæ exquisitè cedant' (Bodies are hindered neither by the medium nor by other external causes from yielding perfectly to their innate and to centripetal forces). The lack of reference to resisted motion in this preliminary supposition strongly supports our earlier suggestion (note (5)) that Newton's final Problems 6 and 7 below were added in afterthought.

(8) In later redraft (see Appendix 1, and compare note (3) above) Newton interjected 'et corporis moti sphærica superficies' (and the spherical surface of the moving body), but without further elaborating the addendum. In about the autumn of 1685 he returned to the topic, computing the total resistance to uniform translation of a hemispherical surface to be half that of its great-circle plane; see 2, §1, Appendix 1 below.

(9) In other words, it is supposed that 'natural' (force-free) motion of a body takes place at a uniform rate in an infinite straight line, in which state it is sustained by its (internal) *vis insita*. It is now well established that Newton arrived at this fundamental postulate of inertial rectilinearity by combining the *prima/altera leges naturæ* ('quod unaquæq; res quantum in se est, semper in eodem statu [*sc.* movendi] perseveret' and 'quod omnis motus ex se ipso sit rectus' respectively) which Descartes set down in his *Principia Philosophiæ* (Amsterdam, 1644): Pars II, §§ XXXVII/XXXIX: 51–4. In his own earliest notes on mechanics, penned by him in January 1665 in his Waste Book (ULC. Add. 4004: 10r–15r/38v, printed in Herivel's *Background* (note (2)): 132–82), Newton inserted (f. 12r) an 'Ax: 100. Every thing doth naturally persevere in yt state in wch it is unlesse it bee interrupted by some external cause, hence... A body once moved will always keep ye same celerity, quantity & determinacon of its motion'.

(10) This late addition in the manuscript's margin, given lemmatical status and formal proof in Newton's immediate revise (see Appendix 1: note (7) below) in effect enunciates the familiar 'parallelogram' rule for compounding uniform speeds, here generated 'instantaneously' by the single, simultaneous application of forces at a point: in Theorem 1 following, as a subtlety, one of these is taken to be a general *vis centripeta*, but the other the *vis insita* which, according to Hypothesis 2, sustains a given uniform motion in a given straight line.

(11) That is, the square. Below, similarly, we render 'triplicata ratio' (cube) as 'tripled ratio', and so on.

Definition 3.(5) While 'resistance'(6) is that which is the property of a regularly impeding medium.

Hypothesis 1.(7) In the ensuing nine propositions the resistance is nil; thereafter it is proportional jointly to the speed of the body and to the density of the medium.(8)

Hypothesis 2. Every body by its innate force alone proceeds uniformly into infinity following a straight line, unless it be impeded by something from without.(9)

Hypothesis 3. A body is carried in a given time by a combination of forces to the place where it is borne by the separate forces acting successively in equal times.(10)

Hypothesis 4. [The space which a body, urged by any centripetal force, describes at the very beginning of its motion is in the doubled ratio(11) of the time.](12)

[*Lemma 1.* Quantities proportional to their differences are in continued proportion. Set $A:(A-B) = B:(B-C) = C:(C-D) = \ldots$ and there will come, *dividendo*, to be $A:B = B:C = C:D = \ldots$.](13)

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(12) Only the opening phrase 'Hyp[othesis] 4' is present—as a late marginal addition—in the manuscript. The inserted text is that of the putative fair copy later sent to London (as settled by the Royal Society and Halley transcripts; see note (2)). It will be evident that Newton presupposes that the central force acting upon a body may, over a vanishingly small length of its orbital arc, be assumed not to vary significantly in magnitude or direction, and hence that that infinitesimal arc is approximated to sufficient accuracy by a parabola whose diameter passes through the force-centre, with its deviation from the inertial tangent-line accordingly proportional to the square of the time. The point is further explored in note (19) below.

(13) This necessary lemma is likewise (compare the preceding note) here inserted from the putative fair copy, as the Royal Society and Halley transcripts (see note (2)) establish its text. The corollary that, when A is 'prima & maxima' and the 'quantitates proportionales' are 'numero infinitæ', then 'erit $A-B$ ad A ut A ad summam omnium' (as James Gregory stated it in Propositio I of his 'N. Mercatoris Quadratura Hyperboles [*sc.* in his 1668 *Logarithmotechnia*; see II: 166] Geometrice Demonstrata' [=*Exercitationes Geometricæ* (London, 1668): 9–13, especially 9]) is all-important in Newton's application of the lemma in Problems 6 and 7 below. (Gregory's assertion that this limit-summation of a converging geometrical progression 'passim demonstratur apud Geometras' is considerably exaggerated: the result was widely used by the 'calculators' of early 14th century Oxford—Richard Swineshead and others—and was widely familiar by the early 16th century, while Archimedes in his *Quadrature of the Parabola* had given rigid proof of the particular case when the proportion factor is $\frac{1}{4}$ by a technique generalisable to instances where the factor is less than $\frac{1}{2}$, but the first completely general demonstration of Gregory's proposition appeared only in Grégoire de Saint-Vincent's *Opus Geometricum Quadraturæ Circuli et Sectionum Coni* (Antwerp, 1647): 51–177; see II: 246, note (146).) In effect Newton derives in each case the solution $\log(x_t/x_0) = -kt$ as the general solution of the fluxional equation $\dot{x}_t\ (= dx_t/dt) = -kx_t$ by setting (in the geometrical equivalents of his hyperbolic model) $A = x_0$, $B = x_{t/n}$, $C = x_{2t/n}$, $D = x_{3t/n}$, ... together with

$$x_0/(x_0 - x_{t/n}) = x_{t/n}/(x_{t/n} - x_{2t/n}) = \ldots = x_{(n-1)t/n}/(x_{(n-1)t/n} - x_t) = 1/(kt/n),$$

where $x_0/x_{t/n} = x_{t/n}/x_{2t/n} = \ldots = x_{(n-1)t/n}/x_t = 1/(1-kt/n)$ and therefore $x_t/x_0 = (1-kt/n)^n$: accordingly, in the limit as n becomes infinitely great there results $x_t/x_0 = e^{-kt}$.

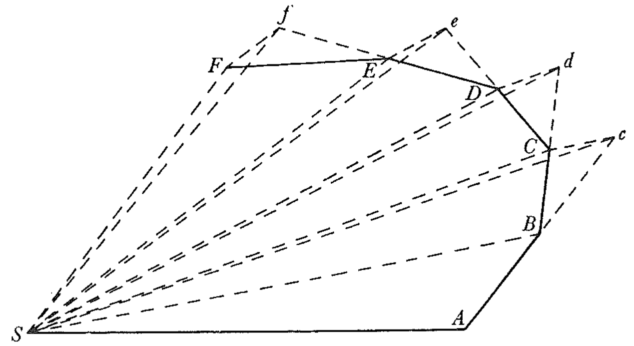
[*Lem. 2.* Parallelogramma omnia circa datam Ellipsin descripta,(14) esse inter se æqualia. Constat ex Conicis.](15)

Theorema 1. Gyrantia(16) omnia radijs ad centrum (17) ductis areas temporibus proportionales describere.

Dividatur tempus in partes æquales, et prima temporis parte describat corpus vi insita rectam *AB*. Idem secunda temporis parte si nil impediret *a*rectà pergeret ad *c* describens lineam *Bc* æqualem ipsi *AB* adeo ut radijs *AS*, *BS*, *cS* ad centrum actis confectæ forent æquales areæ *ASB*, *BSc*. Verum ubi corpus venit ad *B* agat vis centripeta impulsu unico sed magno, faciatq; corpus a recta *Bc* deflectere et pergere in recta *BC*. Ipsi *BS* parallela agatur c*C* occurrens *BC* in *C* et completa secunda temporis parte *b*corpus reperietur in *C*. Junge *SC* et triangulum *SBC* ob parallelas *SB*, *Cc* æquale erit triangulo *SBc* atq; adeo etiam triangulo *SAB*. Simili argumento si vis centripeta successivè agat in *C*, *D*, *E* &c, faciens corpus singulis temporis momentis singulas describere rectas *CD*, *DE*, *EF* &c[,] triangulum *SCD* triangulo *SBC* et *SDE* ipsi *SCD* et *SEF* ipsi *SDE* æquale erit. Æqualibus igitur temporibus æquales areæ describuntur. Sunto jam hæc triangula numero infinita et infinitè parva, sic, ut singulis temporis momentis singula respondeant triangula, agente vi centripeta sine intermissione,(18) & constabit propositio.(19)

*a*Hyp. 1.

*b*Hyp. 3.



(14) Understand so as to touch it at the end-points of a pair of conjugate diameters. The phrasing of the sequel suggests that Newton is unaware that the present lemma is Apollonius, *Conics*, VII, 31 (first printed by G. A. Borelli in his *Apollonii Pergæi Conicorum Lib. V. VI. VII* (Florence, 1661): 370–1). We have already, in demolishing Whiston's absurd claim (*Memoirs*, London, ₁1749: 39) that Newton's enunciation of this theorem, in the little modified form in which it was to appear in the *Principia*, manifested his mathematical ability to 'see almost by Intuition, even without Demonstration', remarked (see IV: 9, note (24)) that the property is all but self-evident when the ellipse is viewed as the orthogonal projection of a circle. It may be, however, that Newton by his 'Constat ex Conicis' makes oblique reference to the ingenious proof by area-dissection given by Grégoire de Saint-Vincent in his *Opus Geometricum* (note (12)): Liber IV, Propositio LXXII, 'Rectangulum sub dimidijs axibus æquale est parallelogrammo sub diametris conjugatis': 281.

(15) Like the preceding, this lemma (needed in the proof of Problems 2 and 3 below) is here inserted from the putative fair copy (as we know it from the Royal Society and Halley transcripts).

[*Lemma 2.* All parallelograms described about a given ellipse(14) are equal to one another. This is established from the *Conics*.](15)

Theorem 1. All orbiting bodies(16) describe, by radii drawn to their centre,(17) areas proportional to the times.

Let the time be divided into equal parts, and in the first part of the time let the body by its innate force describe the straight line *AB*. It would then in the second part of time, were nothing to impede it, proceed directly*a* to *c*, describing the line *Bc* equal to *AB* so as, when rays *AS*, *BS*, *cS* were drawn to the centre, to make the areas *ASB*, *BSc* equal. However, when the body comes to *B*, let the centripetal force act in one single but mighty impulse and cause the body to deflect from the straight line *Bc* and proceed in the straight line *BC*. Parallel to *BS* draw *cC* meeting *BC* in *C*, and when the second interval of time is finished the body will*b* be found at *C*. Join *SC* and the triangle *SBC* will then, because of the parallels *SB*, *Cc*, be equal to the triangle *SBc* and hence also to the triangle *SAB*. By a similar argument, if the centripetal force acts successively at *C*, *D*, *E*, ..., making the body in separate moments of time describe the separate straight lines *CD*, *DE*, *EF*, ..., the triangle *SCD* will be equal to the triangle *SBC*, *SDE* to *SCD*, *SEF* to *SDE* (and so on). In equal times, therefore, equal areas are described. Now let these triangles be infinitely small and infinite in number, such that to each individual moment of time there corresponds an individual triangle (the centripetal force acting now without interruption(18)), and the proposition will be established.(19)

*a*Hypoth. 1.

*b*Hypoth. 3.

(16) Understand 'in plano', as Newton will later make explicit in revised enunciation of this theorem (see §2, Appendix 2.3).

(17) Of force, that is, and not (necessarily) of their orbit.

(18) The Royal Society transcript here reads 'remissione' (abatement).

(19) Newton's proof of this fundamental generalisation of Kepler's areal law (contrived by the latter—more than a little shakily—to regulate the varying speeds of the solar planets in their minimally eccentric elliptical orbits and on a purely kinematic basis; compare E. J. Aiton, 'Kepler's Second Law of Planetary Motion', *Isis*, **60**, 1969: 75–90) is more subtle and considerably less cogent than it may at first appear. His unanalysed procedure of breaking down the action of a continuous central force directed instantaneously towards the centre *S* as a body covers the orbital arc $\widehat{BF}$ (it will be obvious that the initial, vanishingly small segment *AB* merely serves to define the direction of force-free motion at *B*) by splitting it into the compound of an infinity of infinitesimal force-impulses, equal one to the other and each directed to the centre *S*, applied at equal (likewise infinitesimal) intervals of time at a corresponding infinity of intervening points *C*, *D*, *E*, ... of the orbit does indeed guarantee that the focal triangles *BSC*, *CSD*, *DSE*, ... are equal in area to each other, and hence that the time in which a body under the continual bombardment of such force-impulses traverses the limit-polygonal arc *BCDE* ... is, at any point, proportional to the related focal segment (*SBCDE* ...). But he ignores whether significant error is introduced in the total action by supposing at each stage that a continuous force instantaneously directed to the centre *S* (and so varying infinitesimally in direction over the vanishingly small continuous arc in question) is adequately approxi-

Theorem. 2. Corporibus in circumferentijs circulorum uniformiter gyrantibus vires centripetas esse ut[20] *arcuum simul*[21] *descriptorum quadrata applicata ad radios circulorum.*

Corpora B, b in circumferentijs circulorum BD, bd gyrantia simul[21] describant

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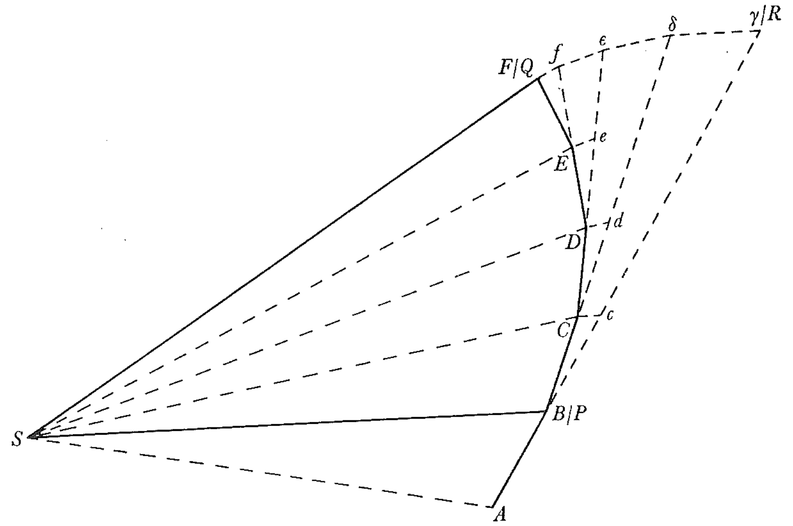
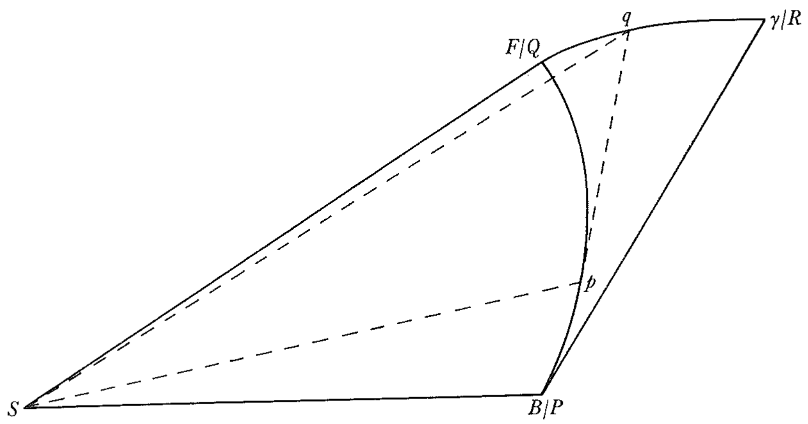
mated by an equivalent impulse of force striking instantaneously at just one point; nor does he justify his assumption that the limit-polygonal arc $BCDE \ldots F$ passes into a unique orbital arc $\widehat{BF}$ as the number of bombarding force-impulses increases to infinity. Furthermore, he will in sequel at once suppose that the direction and length of the total 'spatium superatum' (deviation from the initial rectilinear path AB) is also uniquely—and intuitively—given.



However, in terms of Newton's model, this deviation in the total time from B to F of the impulse (applied at B) which in the time of passage from B to C generates the infinitesimal segment cC is measured in length and direction by a line $\gamma\delta$, parallel to BS, where $B\gamma$ is the force-free path travelled from B in that total time; similarly, the deviation in the time from C to F of the impulse (applied at C) which in the time from C to D generates the segment dD is



Theorem 2. Where bodies orbit uniformly in the circumferences of circles, the centripetal forces are as the squares[20] *of arcs simultaneously*[21] *described, divided by the radii of their circles.*

Let the bodies B, b orbiting in the circumferences of the circles BD, bd simultaneously[21] describe the arcs $\widehat{BD}$, $\widehat{bd}$. By their innate force alone they

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measured in length and direction by the line $\delta\epsilon$, parallel to CS; and so on: whence the deviation effected by the totality of impulses is representable by the polygonal arc $\gamma\delta\epsilon \ldots F$, whose limiting form (as may readily be proved) is the *curvilinear* arc $\widehat{\gamma qF}$, any point q in which is such that, if pq is drawn tangent (at p) to the orbital arc $\widehat{BpF}$, then the triangle pSq is equal in area to the orbital sector (pSF). The necessary requirement for these unconsidered subtleties to be (in present context) negligible is, in fact, that the total orbital arc $\widehat{BF}$ be itself infinitesimally small, in which case the general distance Sp of the orbiting body does not vary appreciably in magnitude, and therefore the central force, f say, acting over the whole arc $\widehat{BF}$ may be considered to be constant. It follows that, if the time of orbit over that infinitesimal arc be dt, and n equal impulses of force act (at B, C, D, E, ...) at intervals of time dt/n, then each (second-order infinitesimal) deviation $cC = dD = eE = \ldots = f.(dt/n)^2$, so that

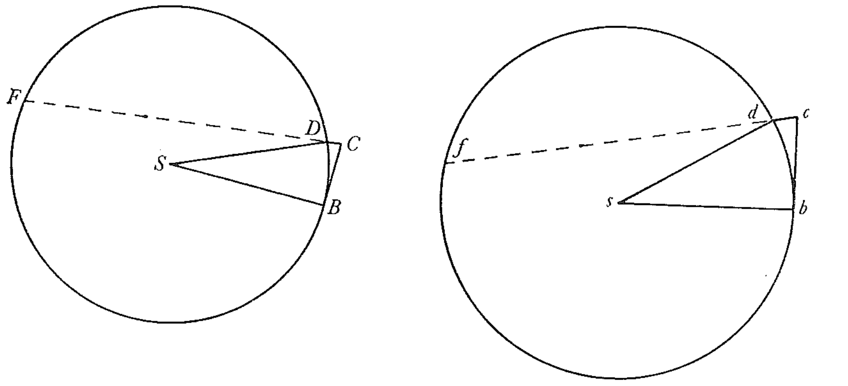
$$\gamma\delta = nf.(dt/n)^2, \quad \delta\epsilon = (n-1)f.(dt/n)^2, \quad \ldots$$

and hence the total polygonal arc $\gamma\delta\epsilon\ldots F$ is $\sum_{0\leqslant i\leqslant n-1}(n-i)f.(dt/n)^2 = \frac{1}{2}(1+1/n)f.dt^2$. In the limit, accordingly, as $n\to\infty$ the deviation arc $\widehat{\gamma F} = \frac{1}{2}f.dt^2 \propto dt^2$ (compare note (9)), and has, moreover, a maximum slope—at F—which is that of SF, where $\widehat{BSF}$ is of infinitesimal size, so that it is adequately approximated in length and direction by any line drawn through F to $B\gamma$ whose slope does not exceed that of SF. In the sequel Newton will, as it suits him, take the deviation 'line' γF—or rather CD in Theorem 2 following and RQ thereafter—to be either the chord γF or, most often, the parallel through F to SB. Since in the dynamical contexts where the generalised areal law receives its Newtonian application the central angle $\widehat{BSF}$—or, correspondingly below, $\widehat{BSD}$ and $\widehat{PSQ}$—is invariably infinitesimal, such sophisticated complexities do not there bedevil the logical cogency of its present proof. Whether Newton himself fully appreciated these underlying subtleties and the validity of neglecting them is not clear. Certainly, neither in subsequent redraft nor in any of the editions of his *Principia* (where the theorem is rightly given pride of place as Proposition I of Book 1) did he ever insert any modification which would suggest that he later did other than continue to believe in its superficial simplicities, but its is only fair to add that none of his contemporaries and immediate successors—even Johann Bernoulli, his arch critic and an equal grand master of the infinitely small—saw fit to impugn the adequacy of Newton's demonstration. On the historical questions, finally, of how early and directly Newton became aware of Kepler's prior statement of the areal law in the case of the solar planets (and planetary satellites), and when he came to realise the fundamental rôle which its generalisation plays in the creation of a mathematical theory of central-force orbital dynamics, see the preceding introduction and compare D. T. Whiteside, 'Newton's Early Thoughts on Planetary Motion: A Fresh Look' (*British Journal for the History of Science*, **2**, 1964: 117–37): 120–2, 128–31.

(20) Newton has deleted a tentative insertion 'celeritatum sive' (of the speeds or), evidently because this generalisation to the case where the arcs $\widehat{BD}$ and $\widehat{bd}$ are described in differing times is delayed to be his prime corollary below.

(21) Or, of course, 'æqualibus temporibus' (in equal times).

arcus BD, bd. Sola vi insita describerent tangentes BC, bc his arcubus æquales.(22) Vires centripetæ sunt quæ perpetuò retrahunt corpora de tangentibus ad circumferentias, atq adeo hæ sunt ad invicem ut spatia ipsis superata CD, cd, id est productis CD, cd ad F et f(23) ut $\frac{BS\text{quad}}{CF}$ ad $\frac{bc\text{quad}}{cf}$ sive ut $\frac{BD\text{quad}}{\frac{1}{2}CF}$ ad $\frac{bd\text{quad}}{\frac{1}{2}cf}$. Loquor de spatijs BD, bd minutissimis inq infinitum diminuendis sic ut pro $\frac{1}{2}CF$, $\frac{1}{2}cf$ scribere liceat circulorum radios SB, sb. Quo facto constat Propositio.



Cor 1. Hinc vires centripetæ sunt ut celeritatum(24) quadrata applicata ad radios circulorum.

Cor 2. Et reciprocè ut quadrata temporum periodicorum applicata ad radios.

Cor 3. Unde si quadrata temporum periodicorum sunt ut radij circulorum vires centripetæ sunt æquales. Et vice versa.

Cor 4. Si quadrata temporum periodicorum sunt ut quadrata radiorum vires centripetæ sunt reciprocè ut radij. Et vice versa.

(22) In the terms of the second diagram in note (19) above, if the orbit $\widehat{BF}$ (here $\widehat{BD}$) is a circle-arc of centre S, then the tangent pq at its general point p (extended to its meet with the deviation arc $\widehat{F\gamma}$, here named $\widehat{DC}$) is perpendicular to its corresponding radius Sp, so that, since the circle-sector ($\widehat{pSF}$) is equal in area to the triangle pSq, in every case $\widehat{pF} = pq$, and in particular $\widehat{BF}$ (that is, $\widehat{BD}$) $= B\gamma(BC)$. The infinitesimal arc $\widehat{F\gamma}(\widehat{DC})$ is evidently a circle-involute, a curve already identified in parallel circumstances by Huygens in his yet unpublished tract 'De Vi Centrifuga [ex Motu Circulari]' (*Œuvres complètes*, **16**, 1929: 255–301, especially 265/7). It is interesting that Huygens there approximates the arc $\widehat{DC}$ by its tangent at D (extended to its meet with BC), while Newton here (see next note) makes equivalent use of its chord DC.

(23) In effect Newton approximates the infinitesimal involute arcs $\widehat{CD}$, $\widehat{cd}$ by their chords CD, cd, and accurately sets the points F, f a little above the extensions of the radii BS, bs. Indeed, from his researches in 1676 (compare IV: 670, note (44)) into the approximate rectification of the arc of a central conic he was aware that, on taking $BS = DS = 1$ and $\widehat{BSD} = \phi$ (so that, where $D\alpha$ is drawn perpendicular to BS, at once $D\alpha = \sin\phi$ and $S\alpha = \cos\phi$), the near-equality $\phi/3 \gtrsim \sin\phi/(2+\cos\phi)$ determines that CD meets BS in β such that $B\beta \lesssim 3BS$, whence

would describe the tangent lines BC, bc equal to these arcs.(22) The centripetal forces are those which perpetually drag the bodies back from the tangents to the circumferences and hence are to each other as the distances CD, cd surmounted by them, that is, on producing CD, cd to F and f,(23) as BC^2/CF to bc^2/cf or as $\widehat{BD}^2/\frac{1}{2}CF$ to $\widehat{bd}^2/\frac{1}{2}cf$. I speak here of distances $\widehat{BD}$, $\widehat{bd}$ which are very minute and indefinitely to be diminished, so that in place of $\frac{1}{2}CF$, $\frac{1}{2}cf$ it is allowable to write the radii SB, sb of the circles. And once this is done the proposition is established.

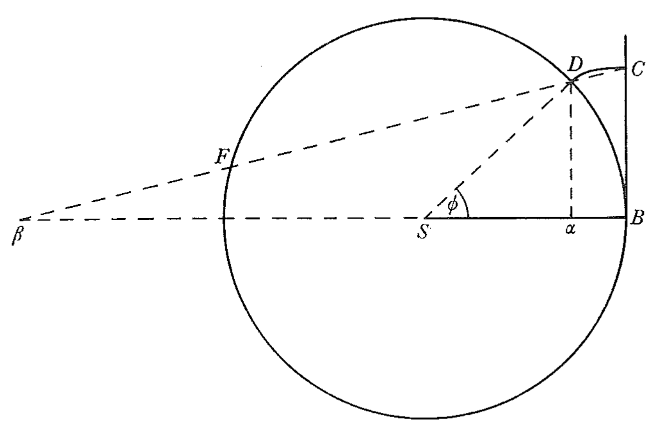
Corollary 1. Hence the centripetal forces are as the squares of the speeds(24) divided by the radii of the circles.

Corollary 2. And reciprocally as the squares of the periodic times divided by the radii.

Corollary 3. Whence, if the squares of the periodic times are as the radii of the circles, the centripetal forces are equal. And conversely so.

Corollary 4. If the squares of the periodic times are as the squares of the radii, the centripetal forces are reciprocally as the radii. And conversely so.

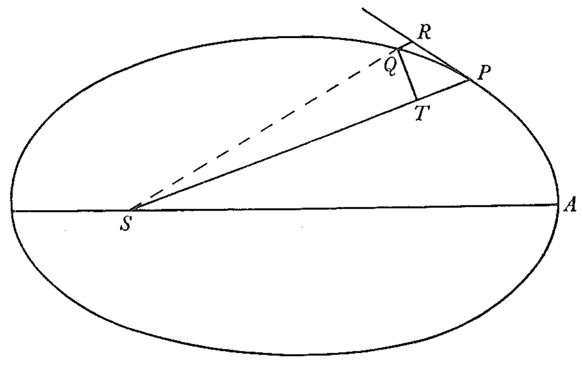
$\widehat{B\beta C} \approx \frac{1}{3}\widehat{BSD}$. (This familiar inequality, publicly enunciated by Willebrord Snell in Propositio XXVIII of his *Cyclometricus. De Circuli Dimensione...ad Mechanicem accuratissima* (Leyden, 1621): 42, was later given rigid proof by Huygens in Theorema XIII of his widely read *De Circuli Magnitudine Inventa* (Leyden, 1654) [= *Œuvres complètes*, **13**, 1910: 113–81, especially 159–63].)



(24) That is, the length of the previous infinitesimal arcs ($\widehat{BD}$, $\widehat{bd}$) divided by the equal times in which they are uniformly traversed. This fundamental Huygenian result, here after some thought (see note (20)) delayed to be but a corollary to the preceding main theorem had initially been attained by Newton about January 1665 (ULC. Add. 4004: 1r) by, much as in his present demonstration of Theorem 1, breaking down the action of a constant, continuously impressed *conatus recedendi a centro* into the aggregate of an infinity of infinitesimal impulses spaced at equal, correspondingly infinitesimal periods of time. (See J. W. Herivel, 'Newton's Discovery of the Law of Centrifugal Force', *Isis*, **51**, 1960: 546–53 and his *Background to Newton's Principia* (note (2)): 129–30; compare also D. T. Whiteside, 'Newtonian Dynamics' (*History of Science*, **5**, 1966: 104–17): 108–10.) The revised argument here offered, one making a direct appeal to the notion of a *vis centrifuga* acting continuously over an infinitesimal time-interval exactly as Huygens' earlier (but yet unpublished) demonstration in his 1659 manuscript 'De Vi Centrifuga' (note (22)), was subsequently evolved by Newton in an untitled paper of about 1670 (ULC. Add. 3958.5: 87r/87v, first published by A. R. Hall in 'Newton on the Calculation of Central Forces', *Annals of Science*, **13**, 1957: 62–71, especially 64–6; see also Herivel's *Background*...: 192–8). Newton's derivation further parallels Huygens' prior investigation in there approximating the deviation arc $\widehat{DC}$ by the tangent to the involute at D, but the

Cor 5. Si quadrata temporum periodicorum sunt ut cubi radiorum vires centripetæ sunt reciprocè ut quadrata radiorum. Et vice versa.[25]

Schol. Casus Corollarij quinti obtinet in corporibus cœlestibus. Quadrata temporum periodicorum sunt ut cubi distantiarum a communi centro circum quod volvuntur. Id obtinere in Planetis majoribus circa Solem gyrantibus inqꝫ minoribus circa Jovem et Saturnum[26] jam statuunt Astronomi.



Theor. 3. Si corpus P circa centrum S gyrando, describat lineam quamvis curvam APQ, et si tangat recta PR curvam illam in puncto quovis P et ad tangentem ab alio quovis curvæ puncto Q agatur QR distantiæ SP parallela[27] *ac demittatur QT perpendicularis ad distantiam SP: dico quod*[28] *vis centripeta sit reciprocè ut solidum* $\frac{SP^{\text{quad}} \times QT^{\text{quad}}}{QR}$, *si modò solidi illius ea semper sumatur quantitas quæ ultimò fit ubi coeunt puncta P et Q.*

Namqꝫ in figura indefinitè parva $QRPT$ lineola[29] QR dato tempore est ut vis
[a]Hyp 4. centripeta et data vi ut [a]quadratum temporis atqꝫ adeo neutro dato ut vis centripeta et quadratum temporis conjunctim, id est ut vis centripeta semel et

possibility that Newton in 1670 could even have been aware of the dynamical researches of his Dutch contemporary is remote. The 'De Vi Centrifuga' appeared publicly only in 1703 in Huygens' *Opera Posthuma*, though bare enunciations of its 'Theoremata' (especially 'III. Si duo mobilia æqualia in circumferentijs æqualibus ferantur, celeritate inæquali, sed utraque motu æquabili, ...erit vis centrifuga velocioris ad vim tardioris in ratione duplicata celeritatum') were appended, without prior introduction or any explanation, on pages 159–61 of his 1673 *Horologium Oscillatorium* (see note (4) above).

(25) If, where T is the time of periodic orbit in a circle of radius r at a uniform speed v ($\propto r/T$), we suppose generally that $T^2 \propto r^n$, then the central force inducing this motion is $v^2/r \propto r/T^2 \propto r^{1-n}$.

(26) This reference to the satellites of Saturn was copied by Humphrey Newton into the revised version (see note (2)), but soon afterwards, having by way of Edward Paget asked Flamsteed's opinion of 'y[e] supernumeray satellits of ♄' and been given on 27 December 1684 the dampening news that 'I can not find the 2 new ones [announced by Cassini in 1681] with a 24 foot glasse' (*Correspondence of Isaac Newton*, **2** (Cambridge, 1960): 405), Newton deleted the phrase 'et Saturnum' (and Saturn): in Book 3 'De Mundi Systemate' of the first (1687) edition of his *Principia*, likewise, Newton makes no reference to any 'planetæ circumsaturnii', but the satellites of Saturn were restored to grace—after due verification of their existence was given—in Phænomenon II of the second edition ($_2$1713: 359–60). In his further letter to Flamsteed on 30 December 1684 Newton queried whether the mean radius of orbit of Saturn itself, as 'defined' by Kepler in his *Tabulæ Rudolphinæ* (Ulm, 1627), 'is...too little for y[e] sesquialterate proportion' (*Correspondence*, **2**: 407) and failed to receive in Flamsteed's reply on 5 January following (*ibid.*: 408–9) a confident rejection of his worry that it might suffer a perturbed

Corollary 5. If the squares of the periodic times are as the cubes of the radii, the centripetal forces are reciprocally as the squares of the radii. And conversely so.[25]

Scholium. The case of the fifth corollary holds true in the heavenly bodies: the squares of their periodic times are as the cubes of their distances from the common centre round which they revolve. That it does obtain in the major planets circling round the Sun and also in the minor ones orbiting round Jupiter and Saturn[26] astronomers are agreed.

Theorem 3. If a body P in orbiting round the centre S shall describe any curved line APQ, and if the straight line PR touches that curve at any point P and to this tangent from any other point Q of the curve there be drawn QR parallel to the distance SP,[27] *and if QT be let fall perpendicular to this distance SP: I assert that*[28] *the centripetal force is reciprocally as the 'solid'* $SP^2 \times QT^2/QR$, *provided that the ultimate quantity of that solid when the points P and Q come to coincide is always taken.*

For in the indefinitely small configuration $QRPT$ the[29] line-element QR is, given the time, as the centripetal force and, given the force, as[a] the square of the [a]Hypoth. 4.
time, and hence, when neither is given, as the centripetal force and the square of the time jointly; that is, as the centripetal force taken once and the area SQP

'exorbitation' of significant size when in the vicinity of Jupiter. Fifteen years before, in a manuscript annotation on 'pag 173 & 304' of Vincent Wing's *Astronomia Britannica* (London, 1669) inserted in the endpapers of his library copy of it (now Trinity College, Cambridge. NQ. 18.36) he had similarly written: 'An Jovis orbita ad hanc analogiam reduci potest haud scio, id vero suspicor sed hæ ejus tabulæ non satis bene conveniunt cum observationibus'. None the less, in his published Hypothesis V of Book 3 of his *Principia* ($_1$1687: 403) he matched his present confidence in the general validity of Kepler's third planetary 'law'—first propounded in the latter's *Harmonices Mundi Libri V* (Linz, 1619)—by asserting without reservation that '*Planetarum quinque primariorum, &...Terræ circa Solem tempora periodica esse in ratione sesquialtera mediocrium distantiarum à Sole.* Hæc à *Keplero* inventa ratio in confesso est apud omnes [Astronomos]'.

(27) In his manuscript figure, whether intentionally or no, Newton has in fact drawn QR to be more nearly in line with SQ. It is tempting to think that he still wishes to make QR closely approximate the true deviation arc, which will (see note (19)) be exactly parallel to SP only at its end-point R. The distinction will, of course, have no significance in the sequel, where only the length of QR—and not its infinitesimal slope to SP—matters.

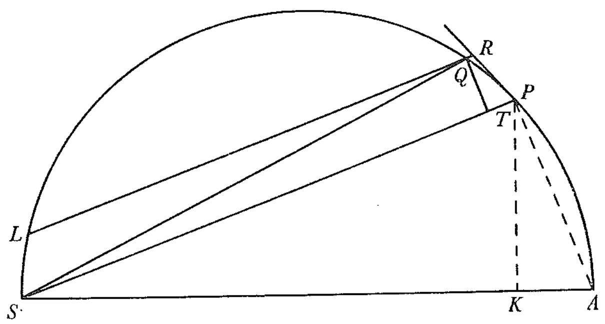
(28) Newton here first began to write 'punctis [?P et Q coeuntibus]' (with the points [?P and Q coming to coincide]). His intention that this 'ultimate' (limit) value of the following ratio must be taken is somewhat cumbrously rephrased in the sequel.

(29) At this place in the redrafted version (see note (2)) Newton further sought to stress that the deviation is vanishingly small by inserting the adjective 'nascens' (nascent). His following 'demonstration' that, where the deviation QR is due to the continuous action of some central force, f say, directed instantaneously to the point S as the body in some infinitesimal time, say dt, traverses the infinitely small orbital arc $\widehat{PQ}$, then $QR \propto f.dt^2$ must inevitably seem superficial to modern eyes attuned to the subtleties glossed over in Newton's present assumptions; compare notes (12) and (19).

area SQP tempori proportionalis (seu duplum ejus $SP \times QT$) bis. Applicetur hujus proportionalitatis pars utraq; ad lineolam QR et fiet unitas ut vis centripeta et $\frac{SP^q \times QT^q}{QR}$ conjunctim, hoc est vis centripeta reciprocè ut $\frac{SP^q \times QT^q}{QR}$.(30) Q.E.D.

Corol. Hinc si detur figura quævis et in ea punctum ad quod vis centripeta dirigitur, inveniri potest lex vis centripetæ quæ corpus in figuræ illius perimetro gyrare faciet. Nimirum computandum est solidum $\frac{SP^q \times QT^q}{QR}$ huic vi reciprocè proportionale. Ejus rei dabimus exempla in problematîs sequentibus.

Prob. 1. Gyrat corpus in circumferentia circuli [:] *requiritur lex vis centripetæ*(31) *tendentis ad punctū aliquod in circumferentia.*



Esto circuli circumferentia $SQPA$, centrum vis centripetæ(31) S, corpus in circumferentia latum P, locus proximus in quem movebitur Q. Ad SA diametrum et SP demitte perpendicula PK QT et per Q ipsi SP parallelam age LR occurrentem circulo in L et tangenti PR in R.(32) Erit RP^q (hoc est QRL) ad QT^q ut SA^q ad SP^q. Ergo $\frac{QRL \times SP^q}{SA^q} = QT^q$. Ducantur hæ æqualia in $\frac{SP^q}{QR}$ et punctis P et Q coeuntibus scribatur SP pro RL. Sic fiet $\frac{SP^{qc}}{SA^q} = \frac{QT^q \times SP^q}{QR}$. Ergo

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(30) More precisely, since (in the terms of the previous note) $QR = \frac{1}{2}f.dt^2$ accurately (compare note (19)), on setting $SP \times QT = c.dt$ there ensues $f = 2(c^2/SP^2).\lim_{Q \to P}(QR/QT^2)$. The full possibilities of this classical Newtonian measure of central force do not well appear in this geometrical form. If we take $S(0, 0)$ to be the origin of a system of polar coordinates in which P is the general point (r, ϕ) and its radius vector $SP = r \equiv r_\phi$ is regarded as a function of the polar angle ϕ, whose infinitesimal increment is $\widehat{PSQ} = o$, then to the infinitely near point $Q(r_{\phi+o}, \phi+o)$ corresponds the radius vector $SQ = r_{\phi+o} = r+o.dr/d\phi+\frac{1}{2}o^2.d^2r/d\phi^2+\ldots$, while the perpendicular $QT = r_{\phi+o}.\sin o = o.r+o^2.dr/d\phi+\ldots$ cuts off

$$ST = r_{\phi+o}.\cos o = r+o.dr/d\phi+\tfrac{1}{2}o^2.(d^2r/d\phi^2-r)+\ldots,$$

and therefore (since PR is tangent at P to the arc $\widehat{PQ}$)

$$QR = SP-ST+QT.(dr/r\,d\phi) = \tfrac{1}{2}o^2.(r-d^2r/d\phi^2+(2/r).(dr/d\phi)^2)+\ldots,$$

so that $\lim_{o \to \text{zero}}(QR/QT^2) = \frac{1}{2}r^{-2}(r-d^2r/d\phi^2+(2/r).(dr/d\phi)^2 = \frac{1}{2}(r^{-1}+d^2(r^{-1})/d\phi^2)$, and so the central force $f \equiv f(r)$ is $c^2r^{-2}(r^{-1}+d^2(r^{-1})/d\phi^2)$. In geometrical equivalent, Newton in his

proportional to the time (or its double, $SP \times QT$) taken twice. Divide each side of this proportionality by the line-element QR and there will come to be 1 as the centripetal force and $SP^2 \times QT^2/QR$ jointly, that is, the centripetal force will be reciprocally as $SP^2 \times QT^2/QR$.(30) As was to be proved.

Corollary. Hence if any figure be given and in it a point to which the centripetal force is directed, there can be ascertained the law of centripetal force which shall make a body orbit in the perimeter of that figure: specifically, you must compute (the quantity of) the 'solid' $SP^2 \times QT^2/QR$ reciprocally proportional to this force. Of this procedure we shall give illustrations in following problems.

Problem 1. A body orbits in the circumference of a circle: the law of centripetal force(31) *tending to some point in its circumference is required.*

Let $SQPA$ be the circle's circumference, S the centre of centripetal force,(31) P the body borne along in the circumference, Q a closely proximate position into which it shall move. To the diameter SA and to SP let fall the perpendiculars PK, QT and through Q parallel to SP draw LR meeting the circle in L and the tangent PR in R.(32) There will be RP^2 (that is, $QR \times LR$) to QT^2 as SA^2 to SP^2, and therefore $QR \times LR \times SP^2/SA^2 = QT^2$. Multiply these equals into SP^2/QR and, with the points P and Q coalescing, let SP be written in place of LR. In this

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following Problems 1–3 will effectively compute the value of this function from the given polar defining equations

$$r^{-1} = R^{-1}\sec\phi, \quad r^{-1} = R^{-1}\sqrt{[1+(e^2/(1-e^2))\sin^2\phi]} \quad \text{and} \quad r^{-1} = R^{-1}(1+e\cos\phi)/(1-e^2)$$

to deduce respectively $f(r) \propto r^{-5}$, $f(r) \propto r$ and $f(r) \propto r^{-2}$. Though Newton himself never made such an inverse application of his present measure, there is in principle no bar to our deducing—by two integrations of the ensuing equation $f(r) = c^2r^{-2}(r^{-1}+d^2(r^{-1})/d\phi^2)$—the polar equation of the general orbit traversible in any given central-force field $f(r)$. In what was to prove historically an *experimentum crucis* of Newton's general dynamical method at the hands of Johann Bernoulli during 1710–19 (see D. T. Whiteside, 'The Mathematical Principles underlying Newton's *Principia Mathematica*', *Journal for the History of Astronomy*, **1**, 1970: 116–38, especially 125–6), it is all but immediate that the conic $r^{-1} = A+B\cos\phi$, resolving the equation $r^{-1}+d^2(r^{-1})/d\phi^2 = k/c^2$, is the most general orbit traversible in the inverse-square force-field $f(r) = k/r^2$. In Newton's own geometrical formulation, unfortunately, such a consequence is far from clearly obvious, and he himself felt forced (as we shall see in commentary upon the scholium to Propositions X–XII of §2 following) to use a less direct approach in demonstrating this inverse of his present Problem 2.

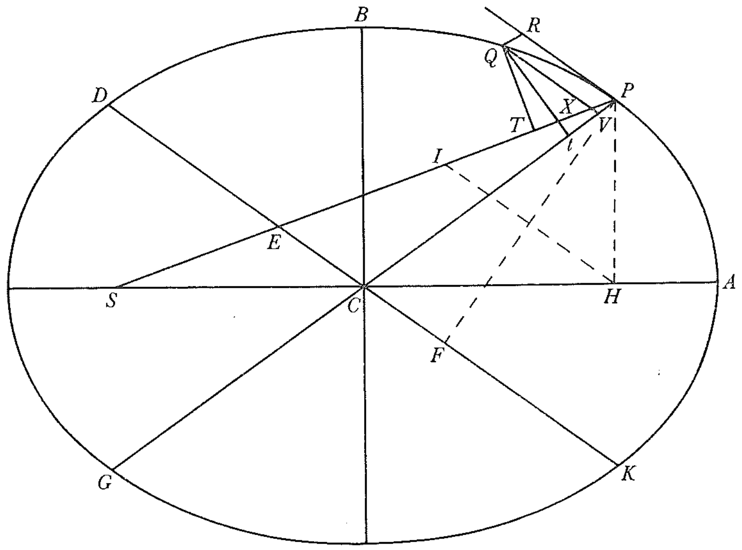
(31) Newton here—and *mutatis mutandis* widely in sequel—initially wrote 'gravitatis'. This broadening beyond the immediate model of solar and terrestrial gravitation is of considerable significance in the developing sequence of his dynamical thought. Compare also note (93).

(32) The logical passage to the next sentence was later amplified by Newton's insertion—for Humphrey Newton to copy into the revised draft—of 'et coeant TQ, PR in Z. Ob similitudinem triangulorum ZQR, ZTP, SPA [erit...]' (and let TQ, PR meet in Z. Because the triangles ZQR, ZTP, SPA are similar [there will be...]). In his accompanying figure, correspondingly, he extended TQ and PR to meet in Z.

vis centripeta[33] reciproce est ut $\frac{SP^{qc}}{SA^q}$, id est (ob datum SA^q) ut quadrato-cubus distantiæ SP.[34] Quod erat inveniendum.

Schol.[35] Cæterum in hoc casu et similibus concipiendum est quod postquam corpus pervenit ad centrum S, id non amplius redibit in orbem sed abibit in tangente.[36] In spirali quæ secat radios omnes in dato angulo[37] vis centripeta tendens ad Spiralis principium est in ratione triplicata distantiæ reciprocè, sed in principio illo recta nulla positione determinata spiralem tangit.

Prob 2. Gyrat corpus in Ellipsi veterum: requiritur lex vis centripetæ tendentis ad centrum Ellipseos.



Sunto CA, CB semiaxes Ellipseos, GP, DK diametri conjugatæ, PF, Qt perpendicula ad diametros[,] QV ordinatim applicata ad diametrum GP et $QVPR$ parallelogrammum.[38] His constructis erit (ex Conicis[39]) PVG ad QV^q ut PC^q ad CD^q et QV^q ad Qt^q ut PC^q ad PF^q[,] et conjunctis rationibus PVG ad Qt^q ut PC^q ad $\frac{CD^q \times PF^q}{PC^q}$. Scribe QR pro PV et[40] $BC \times CA$ pro $CD \times PF$, nec non (punctis P et Q coeuntibus) $2PC$ pro VG[,] et ductis extremis et medijs in se mutuò fiet

(33) Newton again (compare note (31)) initially wrote 'gravitas' (gravity). We will not pinpoint further instances of this change.

(34) In effect, Newton computes $\lim_{Q\to P} (QR/QT^2) = R^2/r^3$ as $r^{-3}(ds/d\phi)^2$, where (in analytical equivalent) the polar equation $r = 2R\cos\phi$ defining $P(r, \phi)$ determines it to be on the circle whose diameter joins $S(0, 0)$ and $A(2R, 0)$. The same result follows equally from computing this limit in the equivalent form $\frac{1}{2}(r^{-1}+d^2(r^{-1})/d\phi^2)$; compare note (30).

(35) This head was probably omitted—by carelessness on its copyist's part?—from the putative fair copy subsequently sent to London (see note (2)) since it is absent in both the Royal Society and Halley transcripts.

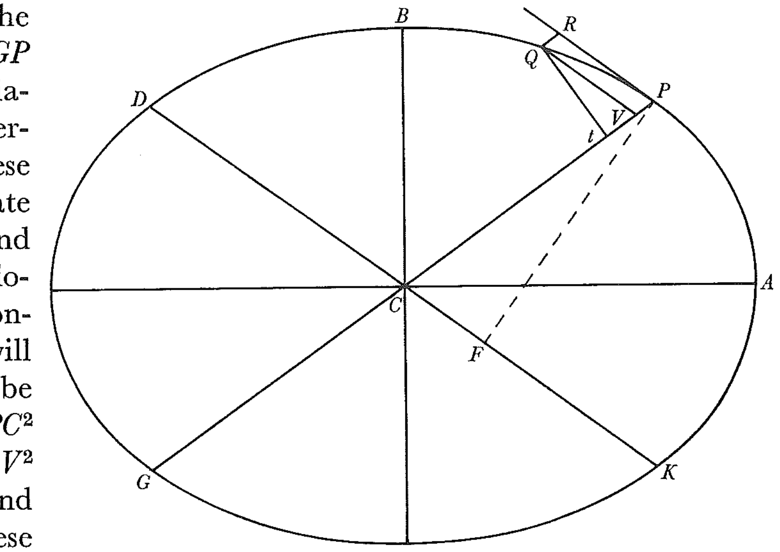
(36) If, of course, we may uniquely define that direction: Newton at once proceeds to give a counter-instance.

(37) The logarithmic (equiangular) spiral, that is. In this curve (already familiar to him as the stereographic projection of the spherical loxodrome which meets all meridians at a constant angle; see IV: 126, note (24)) the infinitesimal elements will, for equal vanishingly

way there will come $SP^5/SA^2 = QT^2 \times SP^2/QR$. Therefore the centripetal force[33] is reciprocally as SP^5/SA^2, that is, (because SA^2 is given) as the fifth power of the distance SP.[34] As was to be found.

Scholium.[35] In this case, however, and similar instances you must conceive that after the body reaches the centre S it will no more return into its orbit but depart along the tangent.[36] In a spiral which cuts all its radii at a given angle[37] the centripetal force tending to the spiral's pole is reciprocally in the tripled ratio of the distance, but at that pole there is no straight line fixed in position which touches the spiral.

Problem 2. A body orbits in a classical ellipse: there is required the law of centripetal force tending to the ellipse's centre.



Let CA, CB be the ellipse's semi-axes, GP and DK conjugate diameters, PF and Qt perpendiculars to these diameters, QV ordinate to the diameter GP, and $QVPR$ a parallelogram.[38] With this construction there will (from the *Conics*[39]) be $PV \times VG$ to QV^2 as PC^2 to CD^2 and again QV^2 to Qt^2 as PC^2 to PF^2, and on combining these proportions $PV \times VG$ to Qt^2 as PC^2 to $CD^2 \times PF^2/PC^2$. Write QR in place of PV and[40] $BC \times CA$ in place of $CD \times PF$, and in addition (with the points P and Q coalescing) $2PC$ in place of VG, and when extremes and middles are multiplied into each other

small increments $\widehat{PSQ}$ of its polar angle, retain the same proportion both to themselves and to the radius vector SP; whence at once $\lim_{Q\to P} (QR/QT^2) \propto 1/SP$, and so the central force to the pole S which induces such a spiral orbit varies as $1/SP^3$. This example is given the status of a separate Propositio VIII in the amplified text reproduced in §2 following.

(38) However, in Newton's accompanying figure (which is economically but somewhat confusingly fashioned to illustrate both this and the following problem) QR is slightly distorted in direction to be parallel to some mean between CVP and SXP. In our English version, here and below, we split the manuscript figure into the two simpler diagrams which it combines, there accurately drawing QR parallel to CP or SP accordingly as the text directs.

(39) Understand Apollonius, *Conics* I, 11–19 or equivalent propositions in any of its more modern reformulations. The reference was inserted as an afterthought.

(40) 'Per Lem: [2]' (by Lemma 2), as Newton was to add in his revise.

$\frac{Qt^q \times PC^q}{QR} = \frac{2BC^q \times CA^q}{PC}$. Est ergo vis centripeta reciprocè ut $\frac{2BC^q \times CA^q}{PC}$, id est (ob datum $2BC^q \times CA^q$) ut $\frac{1}{PC}$, hoc est directè, ut distantia PC.[41] Q.E.I.

Prob. 3. Gyrat corpus in ellipsi: requiritur lex vis centripetæ tendentis ad umbilicum[42] *Ellipseos.*

Esto Ellipseos superioris umbilicus S. Agatur SP secans Ellipseos diametrum DK in E.[43] Patet EP æqualem esse semi-axi majori AC eò, quod actâ ab altero Ellipseos umbilico H linea HI ipsi EC parallela, ob æquales CS, CH æquentur ES, EI, adeo ut EP semisumma sit ipsarum PS, PI id est[44] ipsarum PS, PH quæ conjunctim axem totum $2AC$ adæquant.[45] Ad SP demittatur perpendicularis QT. Et Ellipseos latere recto principali $\left(\text{seu } \frac{2BC^q}{AC}\right)$ dicto L, erit $L \times QR$ ad $L \times PV$ ut QR ad PV id est ut PE (seu AC) ad PC. et $L \times PV$ ad GVP ut L ad GV. et GVP ad QV^q ut CP^q ad CD^q. et QV^q ad QX^q puta ut M ad N.[46] et QX^q ad QT^q ut EP^q ad PF^q id est ut CA^q ad PF^q sive ut CD^q ad CB^q. et conjunctis his omnibus rationibus, $L \times QR$ ad QT^q ut AC ad $PC + L$ ad $GV + CP^q$ ad $CD^q + M$ ad N[47] $+ CD^q$ ad CB^q, id est ut $AC \times L$ (seu $2BC^q$) ad $PC \times GV + CP^q$ ad CB^q

(41) Newton straightforwardly evaluates his geometrical measure $CP^{-2}. \lim_{Q \to P} (QR/Qt^2)$ of the force directed to C by compounding $QR \times VG : QV^2 = PC^2 : CD^2$ and $QV^2 : Qt^2 = PC^2 : PF^2$ to yield $\lim_{V \to P} (PC^2/VG \times CD^2 \times PF^2) = \frac{1}{2}PC/BC^2 \times CA^2 \propto PC$.

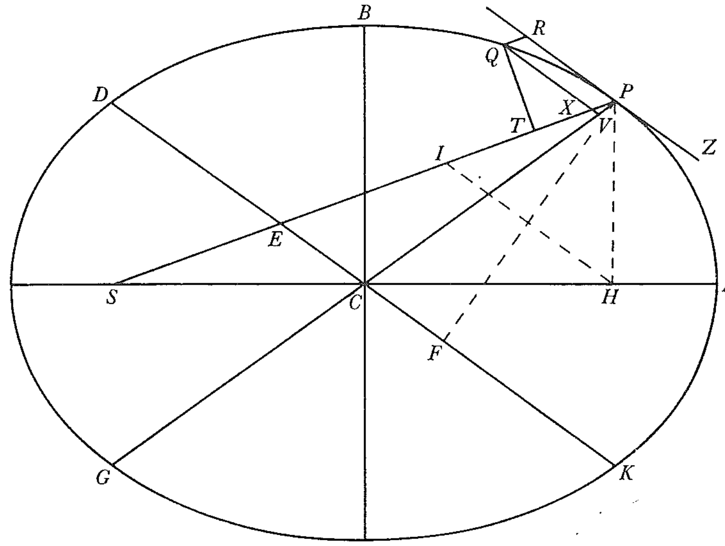
(42) Literally 'navel'. This anthropomorphic designation had been invoked twenty years before by Nicolaus Mercator in the preface of his *Hypothesis Astronomica Nova, et Consensus ejus cum Observationibus* (London, 1664) to denote—as one component of an elaborate 'humanist' image of the planetary ellipse, according to which the line of apsides is divided in the proportions of a male figure set with head at aphelion and feet at perihelion—the 'belly-button' centre of a Keplerian 'vicarious' equant circle dividing the distance between the solar focus (at the 'knees') and upper focus (at the 'breast') in divine section. With the rise to popularity of a simpler Boulliauist hypothesis of elliptical motion (see §2: note (163) below) the 'umbilic' soon came to be identified—for instance, by Claude Milliet Dechales in his *Mundus Mathematicus* (Paris, 1674)—rather with the upper focus itself, and then by extension to be applied to any focus of a conic in general: Isaac Barrow so denotes the focus of a parabola in his *Lectiones XVIII... Opticorum Phænomenωn...* (London, 1669): 28: §XII.1, for example. Newton's anatomically freakish present innovation of permitting his planetary conic to have two 'navels' S and H was afterwards continued by him in his published *Principia*. Earlier, in his youth (see I: 32) he had been content to employ the now standard term 'focus'—introduced by Kepler in Caput IV, §4 of his *Ad Vitellionem Paralipomena...* (Frankfurt, 1604) and afterwards popularised by Mydorge and Descartes—and to this nomenclature he was, as we shall see in the next volume, to return in his private papers in the 1690's.

(43) In his revise (see note (2)) Newton later added in amplification 'et lineam QV in X et compleatur parallelogrammum $QXPR$' (and the line QV in X, and complete the parallelogram $QXPR$).

(44) Newton here subsequently inserted the parenthesis 'ob parallelas HI, PR & angulos æquales IPR, HPZ' (because of the parallels HI, PR and the equal angles $\widehat{IPR}$, $\widehat{HPZ}$), correspondingly extending the tangent RP in his preceding figure to Z.

there will come to be $Qt^2 \times PC^2/QR = 2BC^2 \times CA^2/PC$. The centripetal force is therefore reciprocally as $2BC^2 \times CA^2/PC$, that is, (because $2BC^2 \times CA^2$ is given) as $1/PC$, or in other words directly as the distance PC.[41] As was to be found.

Problem 3. A body orbits in an ellipse: there is required the law of centripetal force tending to a focus[42] *of the ellipse.*



Let S be a focus of the preceding ellipse. Draw SP cutting the ellipse's diameter DK in E.[43] It is evident that EP is equal to the semi-major-axis AC, seeing that, when from the ellipse's other focus H the line HI is drawn parallel to CE, because CS and CH are equal so are ES and EI, and hence EP is the half-sum of PS and PI, that is,[44] of PS and PH which are jointly equal to the total axis $2AC$.[45] To SP let fall the perpendicular QT. Then, on calling the ellipse's principal *latus rectum* (viz. $2BC^2/AC$) L, there will be $L \times QR$ to $L \times PV$ as QR to PV, that is, as PE (or AC) to PC; and $L \times PV$ to $GV \times VP$ as L to GV; and $GV \times VP$ to QV^2 as CP^2 to CD^2; and QV^2 to QX^2 as, say, M to N;[46] and QX^2 to QT^2 as EP^2 to PF^2, that is, as CA^2 to PF^2 or as CD^2 to CB^2: and, when all these proportions are combined, $L \times QR$ is to QT^2 as

$$(AC \text{ to } PC) \times (L \text{ to } GV) \times (CP^2 \text{ to } CD^2) \times (M \text{ to } N)^{(47)} \times (CD^2 \text{ to } CB^2),$$

(45) This 'evident' property of the ellipse—and also, as Newton will soon show (in Proposition XI of §2 following), of the hyperbola—is clearly regarded by Newton as his present discovery, nor indeed have we found it listed in any preceding work on conics: of course, crucial though it is in the context of the present argument, it may (if known) have not been regarded by geometers at large as basic enough to be accorded separate status as a theorem.

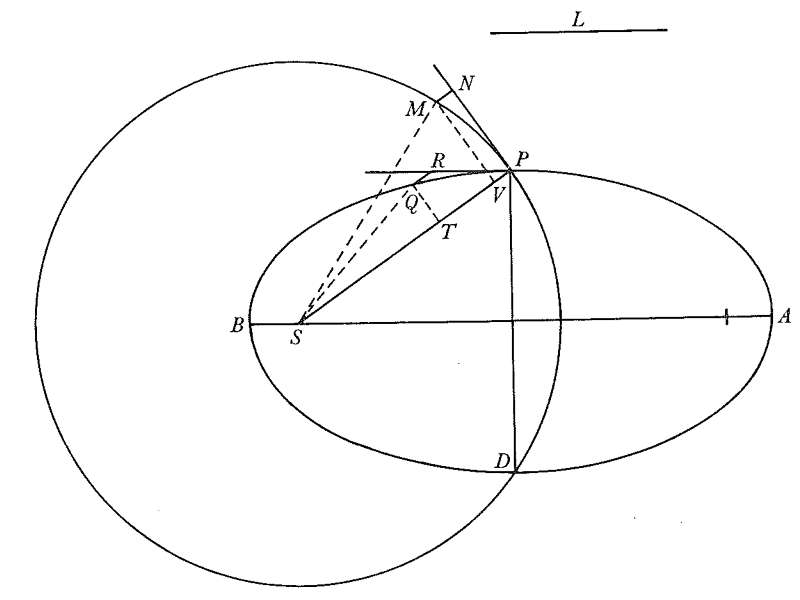
(46) This separate denomination of the ratio QV^2 to QX^2 seems entirely unnecessary, particularly since in the sequel it will, in the limit as Q coincides with P, become unity. Newton realised as much when he amended his subsequent revise (see note (2)), here altering the sentence to read 'et QV^q ad QX^q punctis Q et P coeuntibus fit ratio æqualitatis' (and QV^2 to QX^2 comes, as the points Q and P coincide, to be a ratio of equality), continuing in sequel 'et QX^q seu QV^q est ad QT^q...' (and QX^2, or QV^2, is to QT^2...).

(47) These three '$+M$ ad N' ($\times (M$ to $N)$) were deleted by Newton in the revise in line with the emendation recorded in the previous note. The use of '+' to denote 'addition' (composition) of geometrical ratios is copied from Isaac Barrow, who introduced this some-

$+ M$ ad N,[47] sive ut $2PC$ ad $GV + M$ ad N.[47] Sed punctis Q et P coeuntibus rationes $2PC$ ad GV et M ad N fiunt æqualitatis: Ergo et ex his composita ratio $L \times QR$ ad QT^q.[48] Ducatur pars utraqȝ in $\frac{SP^q}{QR}$ et fiet $L \times SP^q = \frac{SP^q \times QT^q}{QR}$. Ergo vis centripeta reciprocè est ut $L \times SP^q$ id est in ratione duplicata distantiæ.[49] Q.E.I.

[50]*Schol.* Gyrant ergo Planetæ majores in ellipsibus habentibus umbilicum in centro solis, et radijs ad Solem ductis describunt areas temporibus proportionales, omninò ut supposuit Keplerus.[51] Et harum Ellipseon latera recta sunt $\frac{QT^q}{QR}$,[52] punctis P et Q spatio quàm minimo et quasi infinitè parvo distantibus.

Theorem. 4. Posito quod vis centripeta sit reciprocè proportionalis quadrato distantiæ a centro,[53] *quadrata temporum periodicorum in Ellipsibus sunt ut cubi transversorum axium.*



Sunto Ellipseos axis transversus AB,[54] axis alter PD, latus rectum L,[55] umbilicus alteruter S. Centro S intervallo SP describatur circulus PMD. Et eodem tempore describant corpora duo gyrantia arcum Ellipticum PQ et circulum PM, vi centripeta ad umbilicum S tendente. Ellipsin et circulum

what confusing notation in his *Lectiones XXIII... In quibus Opticorum Phænomenωn Genuinæ Rationes investigantur, ac exponuntur* (London, 1669); see especially his introductory list of symbols (signature $a2^v$), where '$A.B+C.D$' is deemed to signify the 'Rationes A ad B, & C ad D compositæ'.

(48) In his revise Newton likewise recast this sentence to read '...punctis Q et P coeuntibus æquantur $2PC$ & GV: Ergo et $L \times QR$ & QT^q æquantur' (...with the points Q and P coalescing, $2PC$ and GV are equal; therefore $L \times QR$ and QT^2 also are equal).

(49) In sum, Newton here evaluates his measure $SP^{-2}.\lim_{Q\to P}(QR/QT^2)$ of the central force to S by compounding (QR or) $PX: PV =$ (PE or) $CA: PC$, $PV \times VG: QV^2 = PC^2: CD^2$ and $QX^2: QT^2 =$ (PE^2 or) $CA^2: PF^2$ to produce

$$SP^{-2}.\lim_{Q,V,X\to P}(CA^3 \times PC \times QV^2/VG \times CD^2 \times PF^2 \times QX^2) = SP^{-2}/L \propto SP^{-2},$$

where the (principal) *latus rectum* $L = 2BC^2/CA$.

that is, as

$$(AC \times L(\text{or } 2BC^2) \text{ to } PC \times GV) \times (CP^2 \text{ to } CB^2) \times (M \text{ to } N),^{(47)}$$

or as ($2PC$ to GV) $\times$ (M to N).[47] But, with the points Q and P coalescing, the ratios $2PC$ to GV and M to N become ones of equality, and so also therefore does the ratio $L \times QR$ to QT^2.[48] Multiply each member by SP^2/QR and there will come to be $L \times SP^2 = SP^2 \times QT^2/QR$. Therefore the centripetal force is reciprocally as $L \times SP^2$, that is, (reciprocally) in the doubled ratio of the distance.[49] As was to be found.

[50]*Scholium.* The major planets orbit, therefore, in ellipses having a focus at the centre of the Sun, and with their *radii* (*vectores*) drawn to the Sun describe areas proportional to the times, exactly as Kepler supposed.[51] And the *latera recta* of these ellipses are QT^2/QR,[52] where the distance between the points P and Q is the least possible and, as it were, infinitely small.

Theorem 4. Supposing that the centripetal force be reciprocally proportional to the square of the distance from the centre,[53] *the squares of the periodic times in ellipses are as the cubes of their transverse axes.*

Let AB be an ellipse's transverse axis,[54] PD its other axis, L its *latus rectum*,[55] S one or other of its foci. With centre S and radius SP describe the circle PMD. Then in the same time let two orbiting bodies describe (respectively) the ellipse-arc $\widehat{PQ}$ and the circle-arc $\widehat{PM}$, with the centripetal force tending to the focus S.

(50) Newton first began to enter a '*Cor.* Punctis P et Q coeu[ntibus ratio $L \times QR$ ad QT fit æqualitatis?]' (*Corollary.* With the points P and Q coming to coalesce [? the ratio of $L \times QR$ to QT^2 becomes one of equality]).

(51) The ellipticity of Mars' orbit was established by Kepler in his *Astronomia Nova* ΑΙΤΙΟΛΟΓΗΤΟΣ, *seu Physica Cœlestis, tradita commentariis De Motibus Stellæ Martis* (Prague, 1609) and of that of the other solar planets—with some remaining degree of doubt as to its exactness—in his later *Epitome Astronomiæ Copernicanæ* (Linz, 1618–21); compare C. A. Wilson's well-documented recent analysis of the former in 'Kepler's Derivation of the Elliptical Path', *Isis*, **59**, 1968: 5–25. On Kepler's formulation of the planetary areal law in his *Astronomia Nova* see note (19) above.

(52) Newton first began to write in sequel 'existentibus figuris $QTPR$ [? quam minimis]' (where the figures $QTPR$ are minimally small).

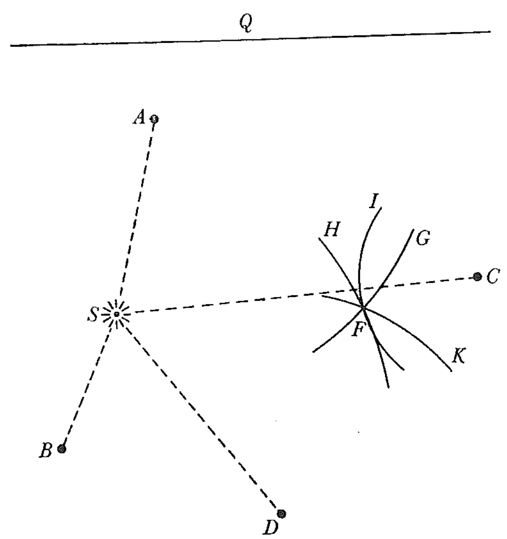
(53) Understand the centre of force (at the ellipses's focus S).

(54) Newton first began: 'Sunto Ellipseos umbilici S, H, centrum C, axis transversus PA, tangens ad verticem PR' (Let the ellipse's foci be S and H, its centre C, transverse axis PA, vertex tangent PR). Correspondingly, in his figure he originally denoted the vertex B by P, and marked the position of the second focus (as here shown) but without naming it; centred on S, furthermore, he drew two vanishingly small focal sectors SPQ and (in general position) SEG such that the force-deviations, RQ and FG respectively, from the tangents PR and EF to the orbit were equal. It is clear that the simplification embodied in rotating $SEFG$ round to coincide with $SPNM$ in the revised figure and resiting $SPRQ$ in its present position occurred to him only as he began to pen the present demonstration.

(55) That is, $L = PD^2/AB = PD^2/2SP$.

tangant PR, PN in puncto P. Ipsi PS agantur parallelæ QR, MN tangentibus occurrentes in R et N. Sint autem figuræ PQR, PMN indefinitè parvæ sic ut (per Schol. Prob. 3) fiat $L\times QR = QT^q$ et[56] $2SP\times MN = MV^q$. Ob communem a centro S distantiam SP et inde æquales vires centripetas sunt MN et QR æquales. Ergo QT^q ad MV^q est ut L ad $2SP$, et QT ad MV ut medium proportionale inter L et $2SP$ seu PD ad $2SP$. Hoc est area SPQ ad aream SPM ut area tota Ellipseos ad aream totam circuli.[57] Sed partes arearum singulis momentis genitæ sunt ut areæ SPQ et SPM atq adeo ut areæ totæ[,] et proinde per numerum momentorum multiplicatæ simul evadent totis æquales. Revolutiones igitur eodem tempore in ellipsibus perficiuntur ac in circulis quorum diametri sunt axibus transversis Ellipseon æquales. Sed (per Cor. 5 Theor 2) quadrata temporum periodicorum in circulis sunt ut cubi diametrorum. Ergo et in Ellipsibus. Q.E.D.[58]

Schol. Hinc in Systemate cœlesti[59] ex temporibus periodicis Planetarum innotescunt proportiones transversorum axium Orbitarum. Axem unum[60] licebit assumere. Inde dabuntur cæteri. Datis autem axibus determinabuntur Orbitæ in hunc modum. Sit S locus Solis seu Ellipseos umbilicus unus[,] A, B, C, D loca Planetæ observatione inventa et Q axis transversus[61] Ellipseos. Centro A radio $Q-AS$ describatur circulus FG et erit ellipseos umbilicus alter in hujus circumferentia. Centris B, C, D, &c intervallis $Q-BS$, $Q-CS$, $Q-DS$ describantur itidem alij quotcunq circuli & erit umbilicus ille alter in omnium circumferentijs atq adeo in omniū intersectione communi F. Si intersectiones omnes non coincidunt, sumendum erit punctum medium[62] pro umbilico.



Praxis hujus commoditas est quod ad unam conclusionem eliciendam adhiberi possint et inter se expeditè comparari observationes quamplurimæ. Planetæ

(56) In the case of the circle it is immediate that the *latus rectum* is equal in length to the diameter.

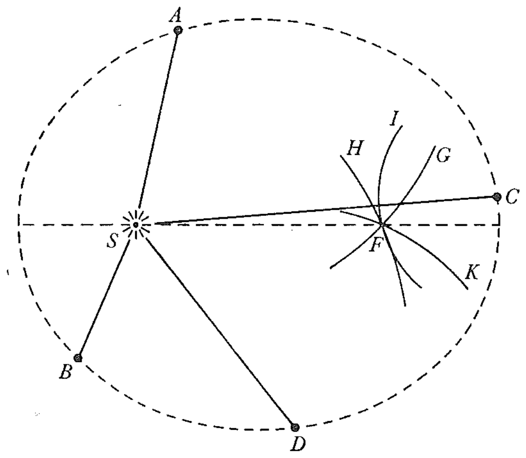
(57) That is, $\frac{1}{4}\pi . AB\times PD : \pi . SP^2 = \frac{1}{2}PD : SP$, since $AB = 2SP$.

(58) In revision (see Propositions XIII/XIV of §2 following) Newton will prefer to prove this third Keplerian planetary law directly from the result $\lim\limits_{Q\to P}(QR/QT^2) = 1/L$ without intervening appeal to the particular case of concentric-circle orbits.

(59) Understand the Keplerian *Systema Copernicanum* in which the planets traverse exact ellipses round the sun set at a common focus.

Let PR, PN be tangent to the ellipse and circle at the point P; and parallel to PS draw QR, MN meeting those tangents in R and N. Now let the figures PQR, PMN be indefinitely small, so that (by the Scholium to Problem 3) there comes to be $L\times QR = QT^2$ and [56]$2SP\times MN = MV^2$. Because of their common distance SP from the centre S and therefore equal centripetal forces producing them, MN and QR are equal. In consequence QT^2 is to MV^2 as L to $2SP$, and so QT to MV as the mean proportional between L and $2SP$, that is, PD to $2SP$; accordingly, the area (SPQ) is to the area (SPM) as the total area of the ellipse to the total area of the circle.[57] But the parts of area generated in individual moments are as the areas (SPQ) and (SPM), and hence as the total areas, and consequently when multiplied by the number of these moments they will simultaneously end up equal to the total areas. Revolutions in ellipses, therefore, are completed in the same time as those in circles whose diameters are equal to the transverse axes of the ellipses. But (by Corollary 5 of Theorem 2) the squares of the periodic times in circles are as the cubes of their diameters. And so also in ellipses, therefore. As was to be proved.[58]

Scholium. Hereby in the heavenly system[59] from the periodic times of the planets are ascertained the proportions of the transverse axes of their orbits. It will be permissible to assume one axis:[60] from that the rest will be given. Once their axes are given, however, the orbits will be determined in this manner. Let S be the position of the Sun—one focus of the ellipse, that is—, A, B, C, D positions of the planet found from observation, and Q the transverse axis[61] of the ellipse. With centre A and radius $Q-AS$ describe the circle FG and the ellipse's other focus will be in its circumference. Correspondingly, with centres $B, C, D, \ldots$ and intervals $Q-BS, Q-CS, Q-DS, \ldots$ describe any number of other circles, and that other focus will be in all their circumferences and hence at the common intersection of all of them. If all their intersections do not coincide, you will need to take a mean point[62] for the focus. The advantage of this technique is that a large number of observations, no matter how many,

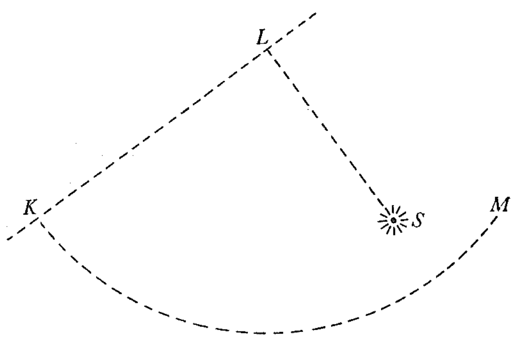


(60) In practice this will, of course, usually be the diameter of the Earth's orbit (the 'Telluris orbis magnus' as Newton names it below).

(61) More precisely, 'longitudo axis transversi' (the length of the transverse axis).

(62) Assuming that, according to some criterion, numerical 'weights' can be set on the relative accuracy with which the initial planetary positions $A, B, C, D, \ldots$ are computed, and

autem loca singula A, B, C, D &c ex binis observationibus, cognito Telluris orbe magno invenire docuit Halleus.(63) Si orbis ille magnus nondum satis exactè determinatus habetur, ex eo propè cognito, determinabitur orbita Planetæ alicujus puta Martis propius: Deinde ex orbita Planetæ(64) per eandem methodum determinabitur orbita telluris adhuc propius: Tum ex orbita Telluris determinabitur orbita Planetæ multò exactiùs quam priùs: Et sic per vices donec circulorum intersectiones in umbilico orbitæ utriusqꝫ exactè satis conveniunt.



Hac methodo determinare licet orbitas Telluris, Martis, Jovis et Saturni, orbitas autem Veneris et Mercurij sic. Observationibus in maxima Planetarum a Sole digressione factis, habentur orbitarum tangentes. Ad ejusmodi tangentem KL demittatur a Sole perpendiculum SL centroqꝫ L et intervallo dimidij axis Ellipseos describatur circulus KM. Erit centrum Ellipseos in hujus circumferentia,(65) adeoqꝫ descriptis hujusmodi pluribus circulis reperietur in omnium intersectione. Tum cognitis orbitarum dimensionibus, longitudines(66)

thereby on the trustworthiness of the trial positions F (say F_i, $i = 1, 2, 3, \ldots$) constructed from these taken in pairs, it will be natural to choose the mean point F as the 'centre of gravity' of the severally constructed points F_i; compare the concluding scholium (page 22) of Roger Cotes's 'Æstimatio Errorum in Mixta Mathesi' (published posthumously in his *Opera Miscellanea* [= *Harmonia Mensurarum* (Cambridge, 1722): 1–121]: 1–22). The simplest instance, in which each of the positions $A, B, C, D, \ldots$ are taken to be computed with equal accuracy, would then define the mean point F as satisfying $\sum_i (F'-F'_i) = 0$, where F', F'_i are the distances of F, F_i from some arbitrary straight line—effectively the modern Gaussian least-squares test.

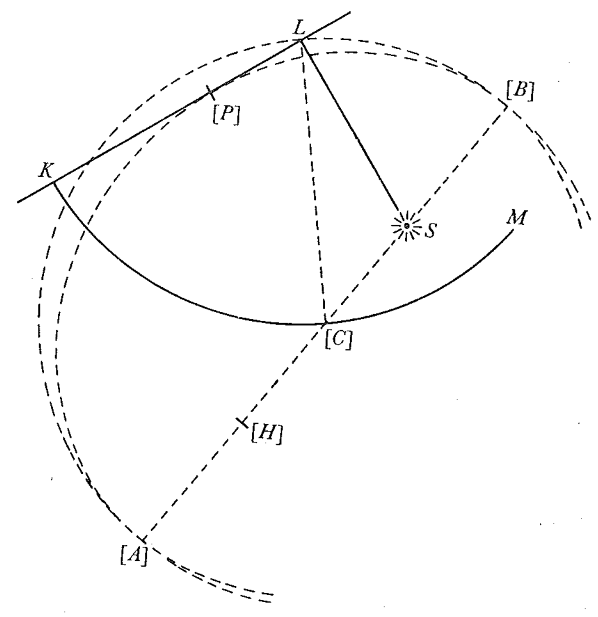
(63) In his 'Methodus directa & Geometrica, cujus ope investigantur Aphelia, Eccentricitates, Proportionesque orbium Planetarum primariorum...', *Philosophical Transactions of the Royal Society*, **11**, No. 128 [for 25 September 1676]: 683–6. (Halley's original English version, 'A direct Geometrical Process to find the Aphelion, Eccentricities, and Proportions of the Orbs of the Primary Planets...', enclosed with his letter to Oldenburg on the previous 11 July, is reproduced in S. P. Rigaud, *Correspondence of Scientific Men of the Seventeenth Century*, **1** (Oxford, 1841): 237–41.) Newton is seemingly unaware that the technique here cited, of fixing the position and relative sizes of planetary *radii vectores* by means of solar oppositions, is Halley's straightforward (if unacknowledged) borrowing from Kepler's *Astronomia Nova* (note (51))—a work which Newton himself almost certainly never read. The method is, of course applicable only to determining the orbits of upper planets, since Mercury and Venus can never be in opposition to the Sun (as viewed from the Earth).

(64) Newton first wrote 'Martis' (Mars), carrying over his preceding instance.

(65) For, if S' is the mirror-image of the orbital focus S in the tangent KL, the line $S'H$ drawn from S' to the second focus H will meet KL in the point P of tangency, and also (since $SL = LS'$ and $SC = CH$) be parallel to LC, where C is the required centre; at once

$$LC = \tfrac{1}{2}S'H = \tfrac{1}{2}(SP+PH) = \tfrac{1}{2}AB,$$

may be employed to elicit a single conclusion and speedily be compared one with another. How, however, the individual positions $A, B, C, D, \ldots$ of a planet may be found from pairs of observations once the 'great' (annual) orbit of the Earth is known, Halley has explained.(63) If that great orbit is not yet considered to be determined with sufficient exactness, the orbit of some planet—say Mars—will be determined more closely from its close delimitation; then from the orbit of the planet(64) the orbit of the Earth will be determined still more closely by the same method; and thereafter from the orbit of the Earth the orbit of the planet will be determined much more exactly than before; and so on in turn until the intersections of the circles concur sufficiently exactly in the focus of each planet.



By this method we are free to determine the orbits of the Earth, Mars, Jupiter and Saturn: to ascertain the orbits of Venus and Mercury, however, do as follows. From observations made at the maximum digression of these planets from the Sun tangents to their orbits are obtained. To a tangent KL of this type let fall the perpendicular SL from the Sun, and then with centre L and radius half the ellipse's axis describe the circle KM: the centre of the ellipse will be in its circumference,(65) and hence when several circles of this sort are described it will be found at their joint intersection. Thereafter, once the dimensions of their orbits are known, the lengths(66)

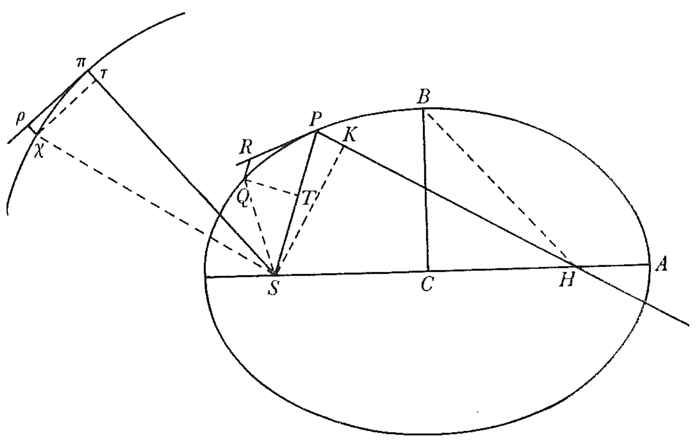
where AB is the orbit's axis. As Newton may well here assume his reader to know, the equivalent proposition that the foot of the perpendicular from the focus of an ellipse to any tangent lies on its circumcircle is Apollonius, *Conics*, III, 49.

(66) Understand their absolute sizes (as distinct from their relative proportions, now—by the two previous techniques—assumed to be known). This use of (twin) observations of the transits of Mercury and especially Venus across the solar disc to determine, by parallax, the distance of the Sun from the Earth (and therefrom the absolute dimensions of the orbits of the solar planets) had been first publicly suggested by James Gregory in a brief scholium to Propositio 87 of the *Appendix, Subtilissimorum Astronomiæ Problematôn resolutionem exhibens* concluding his *Optica Promota* (London, 1663), where (page 130) he wrote that 'Hoc Problema pulcherrimum habet usum, sed forsan laboriosum, in observationibus Veneris, vel Mercurii particulam Solis obscurantis: ex talibus enim solis parallaxis investigari poterit.' Halley, who had witnessed

horum Planetarum exactiùs ex transitu per discum Solis determinabuntur.[67]

Prob. 4. Posito quod vis centripeta sit reciprocè proportionalis quadrato distantiæ a centro, et cognita vis illius quantitate, requiritur Ellipsis[68] *quam corpus describet de loco dato cum data celeritate secundum datam rectam emissum.*

Vis centripeta tendens ad punctum S ea sit quæ corpus π in circulo $\pi\chi$ centro S intervallo quovis $S\pi$ descripto gyrare faciat. De loco P secundum lineam PR emittatur corpus P[69] et mox inde cogente vi centripeta deflectat in Ellipsin PQ. Hanc igitur recta PR tanget in P. Tangat itidem recta $\pi\rho$ circulum in π sitꝗ PR ad $\pi\rho$ ut prima celeritas corporis emissi P ad uniformem celeritatem corporis π. Ipsis SP et $S\pi$ parallelæ agantur RQ et $\rho\chi$[,] hæc circulo in χ illa Ellipsi in Q occurrens, et a Q et χ ad SP et $S\pi$ demittantur perpendicula QT et $\pi\tau$. Est RQ ad $\rho\chi$ ut vis centripeta in P ad vim centripetam in π id est ut $S\pi^{\text{quad.}}$ ad $SP^{\text{quad.}}$, adeoꝗ datur illa ratio. Datur etiam ratio QT ad $\chi\tau$. De hac ratione duplicata auferatur ratio data QR ad $\chi\rho$[70] et manebit data ratio $\frac{QT^q}{QR}$ ad $\frac{\chi\tau^q}{\chi\rho}$, id est (per Schol. Prob. 3) ratio lateris recti Ellipseos ad diametrum circuli: Datur igitur latus rectum Ellipseos. Sit istud L. Datur præterea Ellipseos umbilicus S. Anguli RPS complementū ad duos rectos fiat angulus RPH et dabitur positione linea PH in qua umbilicus alter H locatur. Demisso ad PH perpendiculo SK et erecto semiaxe minore BC est



$$SP^q - 2KPH + PH^q = SH^{q\,(71)} = 4BH^q - 4BC^q = \overline{SP+PH}^{\text{quad.}} - L \times \overline{SP+PH}$$
$$= SP^q + 2SPH + PH^q - L \times \overline{SP+PH}.$$

a transit of Mercury at St. Helena in October 1677, and may well have directed Newton's attention to the possibilities during his visit to Cambridge in August 1684, was later (in papers published in the *Philosophical Transactions* in 1691 and 1716; see A. Armitage, *Edmond Halley* (London, 1966): 103–4) successfully to overcome the formidable practical difficulties involved in pinpointing the moment of occultation, synchronising the clocks of widely distant observers, and making adequate allowance for the diurnal motion of the Earth and inferior planet. Even so, it proved impossible to test the worth of his procedure till the time (1761 and 1769) of the next pair of Venus transits.

of these planets will more exactly be determined from their passage over the Sun's disc.[67]

Problem 4. Supposing that the centripetal force be reciprocally proportional to the square of the distance from its centre, and with the quantity of that force known, there is required the ellipse[68] *which a body shall describe when released from a given position with a given speed following a given straight line.*

Let the centripetal force tending to the point S be that which shall make the body π orbit in the circle $\pi\chi$ described with centre S and any radius $S\pi$. Let the body P be released from the position P following the line PR[69] and directly thereafter under the compulsion of the centripetal force be deflected into the ellipse $\widehat{PQ}$. This, therefore, the straight line PR will touch at P. Let the straight line $\pi\rho$ correspondingly touch the circle at π, and let PR be to $\pi\rho$ as the initial speed of the body P on release to the uniform speed of the body π. Parallel to SP and $S\pi$ draw RQ and $\rho\chi$, the latter meeting the circle in χ, the former the ellipse in Q, and from Q and χ to SP and $S\pi$ let fall the perpendiculars QT and $\chi\tau$. Now RQ is to $\rho\chi$ as the centripetal force at P to the centripetal force at π, that is, as $S\pi^2$ to SP^2, and hence that ratio is given. The ratio of QT to $\chi\tau$ is also given. From this latter ratio doubled take away the given ratio of QR to $\chi\rho$[70] and there will remain given the ratio of QT^2/QR to $\chi\tau^2/\chi\rho$, that is, (by the Scholium to Problem 3) the ratio of the ellipse's *latus rectum* to the circle's diameter; accordingly, the ellipse's *latus rectum* is given. Let that be L. There is given, furthermore, the ellipse's focus S. Make the angle $\widehat{RPH}$ the supplement of the angle $\widehat{RPS}$ and there will be given in position the line PH in which the other focus H is located. On letting fall the perpendicular SK to PH and erecting the semi-minor-axis BC, there is

$$SP^2 - 2KP \times PH + PH^2 = SH^{2\,(71)} = 4BH^2 - 4BC^2 = (SP+PH)^2 - L \times (SP+PH)$$
$$= SP^2 + 2SP \times PH + PH^2 - L \times (SP+PH).$$

(67) In his revise (see note (2)) Newton here added in sequel a further paragraph, touching on the system of solar planets as a collective whole and remarking on the 'superhuman' difficulty of reducing their mutual gravitational perturbations to a 'suitable calculus'. We reproduce it for its intrinsic interest in Appendix 1 following.

(68) Assuming, of course, that the given speed is low enough for this species of conic to be the orbit, as Newton will make clear below.

(69) A cancelled first version of the next two sentences reads in immediate sequel at this point 'ea celeritate quæ sit ad celeritatem uniformem corporis π ut recta quævis PR ad rectam quamvis $\pi\rho$' (with that speed which shall be to the uniform speed of the body π as any straight line PR to any straight line $\pi\rho$).

(70) That is, divide the given ratio $QT^2: \chi\tau^2$ by the given ratio $QR: \chi\rho$. Newton originally continued 'et dabitur' (and there will be given...).

(71) In his later revise (see note (2)) Newton subsequently inserted '$= 4CH^q$' ($= 4CH^2$) in minimal clarification of the next equality.

Addantur utrobiq; $2KPH+L\times\overline{SP+PH}-SP^q-PH^q$ et fiet

$$L\times\overline{SP+PH}=2SPH+2KPH,$$

seu $SP+PH$ ad PH ut $2SP+2KP$ ad L. Unde datur umbilicus alter H. Datis autem umbilicis una cum axe transverso $SP+PH$, datur Ellipsis. Q.E.I.

Hæc ita se habent ubi figura Ellipsis est. Fieri enim potest ut corpus moveat(72) in Parabola vel Hyperbola. Nimirum si tanta est corporis celeritas ut sit latus rectum L æquale $2SP+2KP$, Figura erit Parabola umbilicum habens in puncto S et diametros omnes parallelas lineæ PH. Sin corpus majori adhuc celeritate emittitur movebitur id in Hyperbola habente umbilicum unum in puncto S alterum in puncto H sumpto ad contrarias partes puncti P et axem transversum æqualem differentiæ linearū PS et PH.(73)

Schol. Jam verò beneficio hujus Problematis soluti [Com]etarum(74) orbitas definire concessum est, et inde revolutionum tempora, et ex orbitarum magni-

(72) This was afterwards augmented to the more natural 'moveatur' in Newton's revise.

(73) For, as Newton will prove explicitly in Propositions XI and XII of §2 following, $\lim_{Q\to P}(QT^2/QR)$ may likewise be shown to be the length of the *latus rectum* L when the orbit is a hyperbola or parabola of focus S, and hence the preceding demonstration holds unchanged for the general conic. The very concreteness of Newton's geometrical argument tends to conceal certain of its general implications. If, in modern analytical equivalent, we suppose that the body P sets off in the direction PR—at an angle $\widehat{SPR}=\alpha$, say—with speed v, and is thereupon 'instantaneously' diverted towards the centre S by an inverse-square force of magnitude g at P, and if v be the speed of the body π which rotates in a circle in the same force-field, then in the limit as the arcs $\widehat{PQ}$, $\widehat{\pi\chi}$ vanish to zero there will follow by Newton's line of reasoning (since $PR:\pi\rho=v:v$ and $v^2/S\pi=g.S\pi^{-2}/SP^{-2}$)

$$L/2S\pi=(QT^2/QR)/(\pi\rho^2/\chi\rho)=(PR^2.\sin^2\alpha/\pi\rho^2)\times(\chi\rho/QR)$$
$$=(v^2.\sin^2\alpha/v^2)\times(S\pi^{-2}/SP^{-2})=v^2.\sin^2\alpha/g.S\pi,$$

whence $L=2(v^2/g).\sin^2\alpha$. (This circuitous appeal to an auxiliary circle orbit is not, we may remark, at all necessary, since, where dt is the time of passage from P to Q, at once

$$L=\lim_{Q\to P}(QT^2/QR)=\lim_{R\to P}(PR^2.\sin^2\alpha/\tfrac{1}{2}g.dt^2)$$

with $\lim_{R\to P}(PR/dt)=\lim_{Q\to P}(\widehat{PQ}/dt)=v$.) Further, on taking $SP=R$, there ensues

$$PK=-R\cos 2\alpha$$

and so $SP+PK=2R\sin^2\alpha$, whence $(SP+PH)/PH=2(SP+KP)/L=2/(v^2/gR)$ and therefore $SP+PH=2R/(2-v^2/gR)$; accordingly, the conic's eccentricity is

$$\surd[1-L/(SP+PH)]=\surd[1-(v^2/gR)\ (2-v^2/gR)\sin^2\alpha].$$

The unexpected consequence, considerably veiled in Newton's geometrical guise, that the length $SP+PH$ of the transverse axis of the resulting conic orbit depends only on the speed of projection v and the size g of the central force at P was later to be implicitly invoked by him in criticism of a 'Platonic' hypothesis of Galileo's that, as he noted it in a letter to Richard Bentley on 17 January 1692/3, 'y^e motion of y^e planets is such as if they had all of them been created by God in some region very remote from our Systeme & let fall from thence towards y^e Sun, & so soon as they arrived at their several orbs their motion of falling turned aside into a

Add $2KP\times PH+L\times(SP+PH)-SP^2-PH^2$ to each side and there will come $L\times(SP+PH)=2SP\times PH+2KP\times PH$, that is, $SP+PH$ to PH as $2SP+2KP$ to L. Whence the other focus H is given. Given the foci, however, along with the transverse axis $SP+PH$, the ellipse is given. As was to be found.

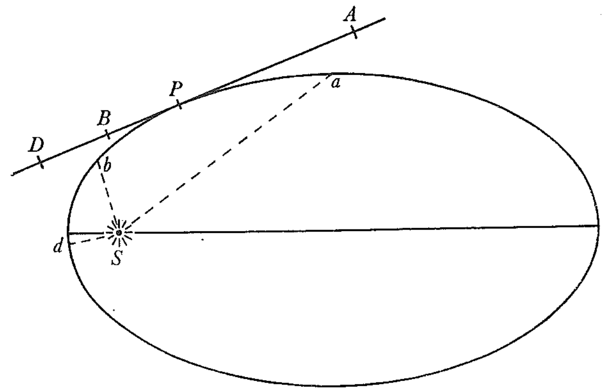
This argument holds when the figure is an ellipse. It can, of course, happen that the body moves in a parabola or hyperbola. Specifically, if the speed of the body is so great that the *latus rectum* L is equal to $2SP+2KP$, the figure will be a parabola having its focus at the point S and all its diameters parallel to the line PH. But if the body is released at a still greater speed, it will move in a hyperbola having one focus at the point S, the second at the point H taken on the opposite side of the point P, and its transverse axis equal to the difference of the lines PS and PH.(73)

Scholium. A bonus, indeed, of this problem, once it is solved, is that we are now allowed to define the orbits of comets,(74) and thereby their periods of revolution,

transverse one' (*Correspondence of Isaac Newton*, **3** (Cambridge, 1961): 240); rather, as he wrote again a month later on 25 February, 'there is no common place from whence all the planets being let fall & descending w^th uniform & equal gravities (as Galileo supposes) would at their arrival to their several Orbs acquire their several velocities w^th w^ch they now revolve in them. If we suppose y^e gravity of all the Planets towards the Sun to be of such a quantity as it really is [*sc.* varying as the inverse-square of their distance from it] & that the motions of the Planets [in their effectively concentric-circle orbits] are turned upwards, every Planet will ascend to twice its height from y^e Sun....And then by falling down again from y^e places to w^ch they ascended they will arrive again at their several orbs w^th the same velocities they had at first & w^th w^ch they now revolve' (*ibid.*: 255). (Compare I. B. Cohen's exhaustive discussion of this 'Galileo–Plato' problem in his 'Galileo, Newton and the Divine order of the solar system' [in (ed. E. McMullin) *Galileo: Man of Science* (New York, 1967): 207–31]. Newton himself went on to remark that, if after a fall from infinity in an inverse-square field 'the gravitating power of y^e Sun' doubled at the moment each planet was diverted into circle orbit, all would be well: equally, he might have supposed that by 'divine' judgement the solar field was inverse-cube at the time the planetary system was created!) The still more fundamental corollary that none but conic orbits are traversible in an inverse-square central-force field—since for every initial speed v and angle of projection α a unique conic trajectory (an ellipse, parabola or hyperbola according as $v<\surd[2gR]$, $v=\surd[2gR]$ or $v>\surd[2gR]$, namely) may correspondingly be defined, so exhausting all possibilities of motion from the given point P, distant R from the force-centre, under a central 'gravity' of intensity g—is here (not unreasonably?) taken by Newton to be self-evident. Twenty five years later, after strong criticism from Johann Bernoulli for a like 'deficiency' in thus failing to underline the obvious in the essentially unaltered equivalent Proposition XVII of Book 1 of his published *Principia* ($_1$1687: 58–9), he decided to insert in its second edition—in the closely similar context of Corollary 1 to the preceding Propositions XI–XIII (*Principia*, $_2$1713: 53)—a sentence justifying the uniqueness of inverse-square conic motion by just such an exhaustion of the possibilities of motion.

(74) In the manuscript Newton initially wrote 'Planetarum' (of planets), a careless slip copied not only into the putative fair copy—since it occurs in Halley's transcript (see note (2))—but also into the revise before Newton caught it, though he then took pains to correct both versions.

tudine, excentricitate, Aphelijs,[75] inclinationibus ad planum Eclipticæ et nodis inter se collatis cognoscere an idem Cometa ad nos sæpius redeat.[76] Nimirum ex quatuor observationibus locorum Cometæ, juxta Hypothesin quod Cometa movetur in linea recta, determinanda est ejus via rectilinea.[77] Sit ea *APBD*, sintq; *A*, *P*, *B*, *D* loca cometæ in via illa temporibus observationum, et *S* locus Solis. Ea celeritate qua Cometa uniformiter percurrit rectam *AD* finge ipsum emitti de locorum suorum aliquo *P* et vi centripeta mox correptum deflectere a recto tramite et abire in Ellipsi *Pbda*. Hæc Ellipsis determinanda est ut in superiore Problemate. In ea sunto *a*, *P*, *b*, *d* loca Cometæ temporibus observationum. Cognoscantur horum locorum e terra longitudines et latitudines. Quanto majores vel minores sunt his longitudines et latitudines observatæ tantò majores vel minores observatis sumantur longitudines et latitudines novæ.[78] Ex his novis inveniatur denuò via rectilinea cometæ et inde via Elliptica ut priùs. Et loca quatuor nova in via Elliptica prioribus erroribus aucta vel diminuta jam congruent cum observationibus exactè satis.[79] Aut si fortè errores etiamnum sensibiles manserint potest opus



(75) Another carelessness: Newton manifestly intends 'Perihelijs' (perihelia). No solar comet known in his day is visible in the region of its aphelion from the Earth.

(76) This periodicity Halley was afterwards in his rare (but frequently reprinted) folio pamphlet, *Astronomiæ Cometicæ Synopsis* (Oxford, 1705), to verify in the celebrated instance of 'his' 1682 comet, not only deducing from his detailed computation of their elements of orbit that it was identical with those which had appeared before in 1531 and 1607—and hence, from the rough equality (75–76 years) of the intervening time-periods, probably also with still earlier ones more vaguely recorded—but accurately predicting its reappearance in late 1758 (after his death).

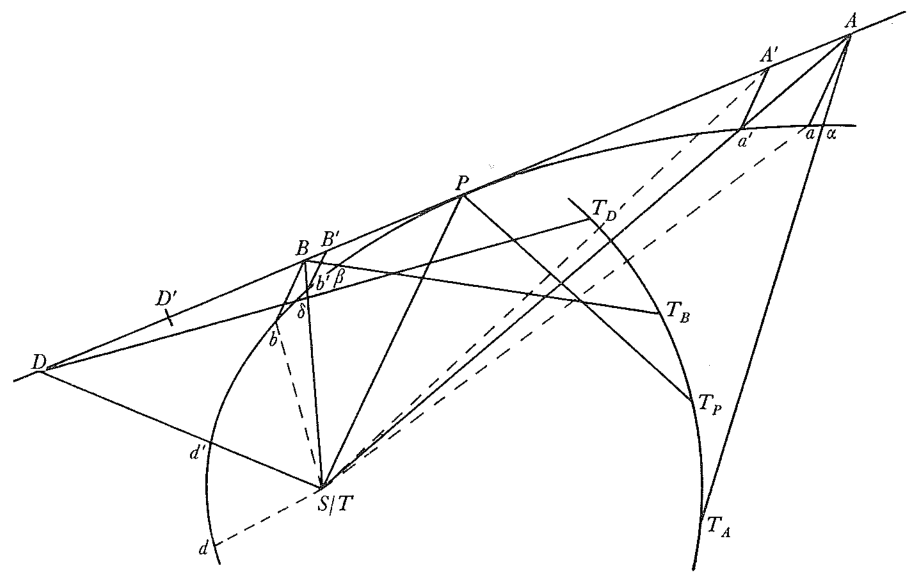
(77) Doubtless by the Wrennian method in 'Prob. 52' of his contemporary Lucasian lectures on algebra (see v: 298–302) or—where the Earth may be considered as motionless to sufficient accuracy—by the simpler 'Prob: 16' preceding (v: 210–12). We have also reproduced in the previous volume (v: 524–9) two versions of an abortive attempt by Newton in the autumn of 1685 to apply the former technique to locating in rough position the out-going arc of the 1680–1 comet (or, more accurately, its orthogonal projection upon the ecliptic).

(78) In his revise (see note (2)) Newton later added the clarifying phrase 'id adeo ut correctiones respondeant erroribus' (this so as to make the corrections correspond to the errors).

(79) This makeshift construction—quickly to be superseded by more viable methods (see 2, §2 below) if indeed it was ever at any time put into practice—seems more optimistic of a chance success than solidly reasoned. If T_A, T_P, T_B, T_D are the points in the Earth's solar

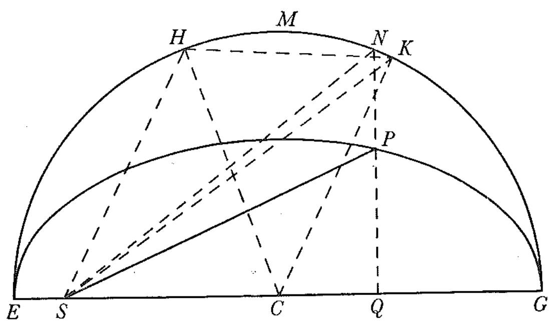
and then to ascertain from a comparison of their orbital magnitude, eccentricities, aphelia,[75] inclinations to the ecliptic plane and their nodes whether the same comet returns with some frequency to us.[76] To be specific, from four observations of a comet's positions we need, under the hypothesis that a comet moves in a straight line, to determine its rectilinear path.[77] Let it be *APBD*, with *A*, *P*, *B*, *D* the positions of the comet in that path at the times of observation and *S* the position of the Sun. At the speed with which it uniformly traverses the straight line *AD* imagine that the comet is released from some one of its places *P* and, snatched up immediately by the centripetal force, is deflected from its straight-line course, going off in the ellipse *Pbda*. This ellipse is to be determined as in the above problem. In it let *a*, *P*, *b*, *d* be the positions of the comet at the times of observation, and ascertain the longitudes and latitudes of these places from the Earth. As much as the observed longitudes and latitudes are greater than these take new longitudes and latitudes greater or less than the observed ones.[78] From these new ones let the comet's rectilinear path be found afresh, and therefrom the elliptical path as before. The four new positions in the elliptical path, increased or diminished by the previous errors, will now agree exactly enough with their observations.[79] Or, should perhaps the errors even

orbit (effectively a minimally eccentric circle round *S*) from which the primitive sightings $T_A A$, $T_P P$, $T_B B$ and $T_D D$ of the orbiting comet at α, P, β and δ respectively are made, Newton's hypothesis that the points A', B' and D' from and to which the comet would uniformly travel in the tangent at *P* with the speed which it there has (in the same time as it in fact, drawn towards the Sun *S*, orbits from and to α, β and δ) approximately satisfy $a\widehat{T_A}A = A\widehat{T_A}A'$, $b\widehat{T_B}B = B\widehat{T_B}B'$ and $d\widehat{T_D}D = D\widehat{T_D}D'$ is palpably false in general. Where the tangential points A, B, D; A', B', D' rigorously correspond to contemporaneous points a, b, d; a', b', d' in the



totum repeti. Et nè computa Astronomos molestè habeant suffecerit hæc omnia per praxin Geometricam[80] determinare.

Sed areas aSP, PSb, bSd temporibus[81] proportionales assignare difficile est. Super Ellipseos axe majore EG describatur semicirculus EHG. Sumatur angulus ECH tempori proportionalis. Agatur SH eiꝗ parallela CK circulo occurrens in K. Jungatur HK et circuli segmento HKM (per tabulam segmentorum vel secus) æquale fiat triangulum SKN. Ad EG demitte perpendiculum NQ, et in eo cape PQ ad NQ ut Ellipseos axis minor ad axem majorem et erit punctum P in Ellipsi[,] atꝗ acta recta SP abscindetur area Ellipseos EPS tempori[82] proportionalis. Namꝗ area $HSNM$ triangulo SNK aucta et huic æquali segmento HKM diminuta fit triangulo HSK id est triangulo HSC æquale. Hæc æqualia adde areæ ESH, fient areæ[83] æquales $EHNS$ et EHC. Cùm igitur Sector EHC tempori proportionalis sit et area EPS areæ $EHNS$, erit etiam area EPS tempori proportionalis.[84]



cometary orbit, then Aa, Bb, Dd; $A'a'$, $B'b'$, $D'd'$ are the deviations due to solar gravity, and therefore (since the total cometary arc $a\widehat{Pb}d$ to be determined is, by implication, small) are all very nearly parallel to the solar vector SP; whence to a close approximation $a\widehat{S}A = A\widehat{S}A'$, $b\widehat{S}B = B\widehat{S}B'$ and $d\widehat{S}D = D\widehat{S}D'$. It follows that Newton's basic premiss for constructing his revised longitudes (and, by simple trigonometry, the corresponding latitudes) holds only when the Earth's orbit $T_A T_P T_B T_D$ effectively shrinks to a point T in the immediate region of the Sun S; here, however, we do not need the full Wrennian method to construct the uniform tangential path $A'PB'D'$ but only the simpler 'Prob: 16' of Newton's algebraic *lectiones* (see note (76)), while the points a', b', d' of orbit are derived immediately as the meets of the parallels $A'a'$, $B'b'$, $D'd'$ to (SP or) TP with the respective sightings TA, TB, TD. Newton came to realise as much, for in the following autumn of 1685 he made at least one determined effort, without conspicuous success, to apply this simplification of his present cometary method to constructing the section of the out-going orbit of the 1680–1 comet visible between 21 December and 25 February; the essence of the worksheet (ULC. Add. 3965.11: 163ʳ) on which he then penned his computations is reproduced in Appendix 2 below.

(80) Newton subsequently amended this to read 'per descriptionem linearum' (by the description of lines). It is unlikely that contemporary astronomers, rigorously trained (in the main) in the practice of numerical and trigonometrical techniques, would have preferred the relative inaccuracy of such an equivalent geometrical construction, however easy to effect.

(81) Understand in which the preceding ellipse-arcs $\widehat{aP}$, $\widehat{Pb}$, $\widehat{Pd}$ are described.

(82) Of orbit, that is, in the ellipse-arc $\widehat{EP}$, from E to P. Notice that Newton measures this 'mean anomaly' in modern style from perihelion (and not, as was the usual contemporary practice, from the aphelion G).

now remain sensible, the whole process can be repeated. And, in case the computations prove troublesome to astronomers, it will be enough to determine all these things by a geometrical procedure.[80]

But to assign areas aSP, PSb, bSd proportional to the times[81] is difficult. On the major axis EG of an ellipse describe the semicircle EHG. Take the angle $E\widehat{C}H$ proportional to the time. Draw SH and parallel to it CK, meeting the circle in K; then join HK and make the triangle SKN equal to the circle's segment HKM (by means of a table of segments or otherwise). To EG let fall the perpendicular NQ and in it take PQ to NQ as the ellipse's minor axis is to its major axis, and the point P will then be in the ellipse, while the straight line SP will, when drawn, cut off an area (EPS) of the ellipse which is proportional to the time.[82] For the area ($HSNM$), augmented by the triangle SNK and diminished by the segment (HKM) equal to this, becomes equal to the triangle HSK, that is, to the triangle HSC. When these equals are added to the area (ESH) they will form equal areas ($EHNS$) and (EHC). Since, therefore, the sector (EHC) is proportional to the time, and the area (EPS) to the area ($EHNS$), the area (EPS) also will then be proportional to the time.[84]

(83) This was trivially changed in Newton's revise (note (2)) to read equivalently '...addita areæ ESH, facient areas'.

(84) Since Newton equates the segment (HKM) to the area of the triangle SKN—and not the minimally larger focal sector (SKN)—his construction will not be rigorously exact, though we may readily show how finely it approximates to the truth. If, in simplest analytical equivalent, we suppose the ellipse to have semi-axes $EC = CG = 1$ and eccentricity $SC = e$, then define the position P of the orbiting body in the arc $\widehat{EPG}$ by the eccentric angle $E\widehat{C}N = \theta$ (not shown in Newton's figure) and the time of orbit over $\widehat{EP}$ by the angle $E\widehat{C}H = T$, where (by the areal law)

$$T:\pi = (ESP):(ESGP) = (ESN)\ [\text{or } \tfrac{1}{2}(\theta - e\sin\theta)] : (ESGN)\ [\text{or } \tfrac{1}{2}\pi],$$

it will be clear that Newton's construction approximately resolves the ensuing equation $\theta - e\sin\theta = T$ (given), and hence the general 'Astronomicum Problema' propounded by Kepler in 1609 in Chapter 60 of his *Astronomia Nova* (see IV: 668, note (38)) which is its geometrical model: namely, he there successively adduces to that end

$$H\widehat{C}K (= S\widehat{H}C) = \alpha = \tan^{-1}[e\sin T/(1 - e\cos T)]$$
$$= e\sin T + \tfrac{1}{2}e^2\sin 2T + \tfrac{1}{3}e^3\sin 3T + \tfrac{1}{4}e^4\sin 4T + \ldots,$$

segment $(HKM) = \frac{1}{2}(\alpha - \sin\alpha)$ where

$$\alpha - \sin\alpha = \beta = \tfrac{1}{6}\alpha^3 + \ldots = \tfrac{1}{24}e^3(3\sin T - \sin 3T) + \tfrac{1}{16}e^4(2\sin 2T - \sin 4T)\ldots,$$

and sector $\quad (CKN) \approx (SKN)/(1 + (SC/CK).\cos K\widehat{C}G) \approx \frac{1}{2}\beta/(1 - e\cos\theta)$

to derive in equivalent terms the solution, correct to $O(e^5)$,

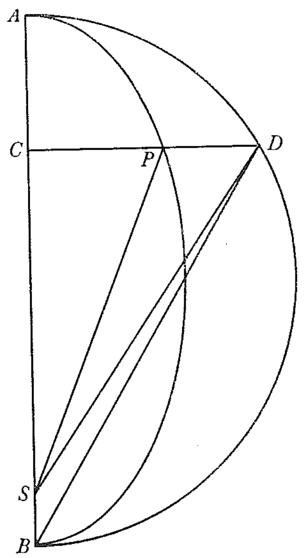
$$\theta \approx T + \alpha - \beta/(1 - e\cos T) = T + e\sin T + \tfrac{1}{2}e^2\sin 2T + \tfrac{1}{8}e^3(3\sin 3T - \sin T)$$
$$+ \tfrac{1}{6}e^4(2\sin 4T - \sin 2T) + \ldots.$$

As Newton may well have known, the simpler approximation

$$\theta \approx E\widehat{C}K = T + \sin\alpha = T + e\sin T/\sqrt{[1 - 2e\cos T + e^2]}$$
$$= T + e\sin T + \tfrac{1}{2}e^2\sin 2T + \tfrac{1}{8}e^3(3\sin 3T - \sin T) + \ldots,$$

Prob. 5. Posito quod vis centripeta sit reciprocè proportionalis quadrato distantiæ a centro,(85) *spatia definire quæ corpus rectà cadendo datis temporibus describit.*

Si corpus non cadit perpendiculariter describet id Ellipsin(86) puta *APB* cujus umbilicus inferior puta *S* congruet cum centro terræ.(87) Id ex jam demonstratis constat. Super Ellipseos axe majore *AB* describatur semicirculus *ADB* et per corpus decidens transeat recta *DPC* perpendicularis ad axem, actisq; *DS*, *PS*, erit area *ASD* areæ *ASP* atq; adeò tempori proportionalis. Manente axe *AB* minuatur perpetuò latitudo Ellipseos, et semper manebit area *ASD* tempori proportionalis. Minuatur latitudo illa in infinitum et Orbita *APB* jam coincidente cum axe *AB* et umbilico *S* cum axis termino *B*(88) descendet corpus in recta *AC* et area *ABD* evadet tempori proportionalis.(89) Definietur itaq; spatium *AC* quod corpus de loco *A* perpendiculariter cadendo tempore dato describit si modò tempori proportionalis capiatur area *ABD* et a puncto *D* ad rectam *AB* demittatur perpendicularis *DC*. Q.E.F.



Schol.(90) Priore Problemate(91) definiuntur motus projectilium(92) in aere nostro,

correct to $O(e^4)$, had long before been derived by Bonaventura Cavalieri in an effectively identical manner (by drawing *CK* parallel to *SH*) in his *Directorium Generale Uranometricum, In quo Trigonometriæ Logarithmicæ Fundamenta, ac Regulæ demonstrantur, Astronomicæque Supputationes ad solam fere Vulgarem Additionem reducuntur* (Bologna, 1632): 152, and repeated—with full credit given to Cavalieri—in G. B. Riccioli's widely studied *Almagestum Novum, Astronomiam Veterem Novamque complectens* (Bologna, 1651): 535. Just three years before Newton contrived his present improvement—unpublished in his lifetime—Christiaan Huygens had made the Cavalierian approximation the basic regulator of the varying planetary speeds (in eccentric circle orbits) in the 'automatic' planetarium whose model he completed in 1682; see his *Œuvres complètes*, **21** (The Hague, 1944): 143–8.

(85) Newton has deleted the qualification 'terræ' (of the earth) in line with his systematic alteration throughout the present text of (solar or terrestrial) 'gravitas' into a general, unspecified 'vis centripeta' (compare note (31)). Correspondingly, the unrestricted 'corpus' *P* whose motion is here determined was initially described throughout as a 'grave' (gravitating body).

(86) Since Newton's argument requires only a vanishingly small initial speed of projection transverse to the path *AS* of rectilinear fall, there is no deficiency in his thus restricting the ensuing conic orbit.

(87) This unwanted survivor of an earlier, more restricted dynamical viewpoint—still uncancelled in both the Royal Society and Halley transcripts of the putative fair copy (see note (2))—was afterwards deleted by Newton (compare §2: note (166)) in line with his preceding enunciation (see note (85)). In making similar omission of a terrestial location for *S* we stress in our English version that it is a general 'centrum virium'.

(88) Since the eccentricity of the ellipse *APB* approaches indefinitely close to unity.

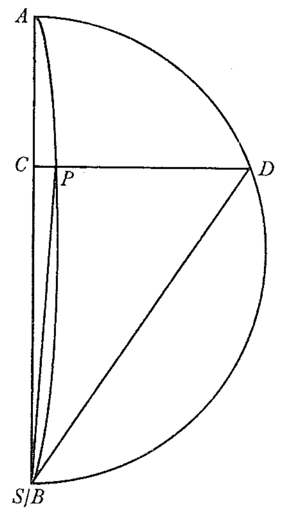
(89) Newton's procedure of maintaining the length of the orbital axis *AB* unchanged while

Problem 5. Supposing that the centripetal force be reciprocally proportional to the square of the distance from the centre,(85) *to define the distances which a body falling straight down describes in given times.*

If the body does not fall perpendicularly it will describe an ellipse,(86) suppose $\widehat{APB}$, whose lower focus, say *S*, will coincide with the centre [of force].(87) This is settled from what we have already demonstrated. On the ellipse's major axis *AB* describe the semicircle *ADB* and let the straight line *DPC* pass through the dropping body perpendicular to the axis, and when *DS*, *PS* are drawn the area (*ASD*) will be proportional to the area (*ASP*) and so also to the time. With the axis *AB* remaining fixed, perpetually diminish the width of the ellipse and the area (*ASD*) will ever remain proportional to the time. Diminish that width indefinitely and, with the orbit $\widehat{APB}$ now coming to coincide with the axis *AB* and the focus *S* with the end-point *B* of the axis,(88) the body will descend in the straight line *AC* and the area (*ABD*) will turn out to be proportional to the time.(89) In consequence, the distance *AC* described by a body falling perpendicularly from the position *A* in a given time will be defined if only the area (*ABD*) be taken proportional to the time and then from the point *D* the perpendicular *DC* be let fall to the straight line *AB*. As was to be done.

Scholium.(90) By the previous problem (91) the motions of projectiles(92) in our

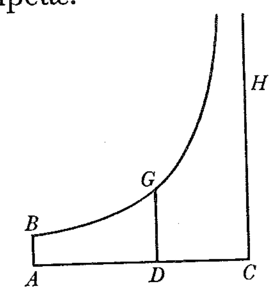
continuously enlarging the initial distance *AS* of the body from the centre of force till it coincides with *AB*—and so, correspondingly, varying the gravitational constant in the force-field, k/SP^2 round *S*—seems needlessly complicated and its subtlety is far from adequately explored. The natural approach, avoiding such a complex detour, is to keep the distance *AS* unchanged but (by allowing the initial velocity of projection at *A*, normal to *AS*, to become vanishingly small) continuously to diminish the orbital diameter *AB* till *B* coincides with *S*; the 'last' ratio of the elliptical sector (*ASP*) and the infinitely narrow semi-ellipse (*ABP*) is then straightforwardly that of the segment (*ASD*) and the full semicircle, and Newton's desired result that the time of fall along the orbital arc $\widehat{AP}$, that is (in the limit) *AC*, is measured by the circle segment (*ASD*) follows immediately. Unless the falling body, when it attains the force-centre *S*, is (by its impact with it or some other means) conceived to have its direction of motion there instantaneously reversed, there will be a discontinuity in this argument from the vanishingly small approximating ellipse which Newton fails to appreciate: namely, the body will continue past *S* at a slowing speed analogously proportional to the inverse-square of its distance from it till it comes momentarily to a halt at (say) *A'* where $SA' = AS$, thereafter falling back towards *S* and past it to halt momentarily at *A* and then going on to repeat this cycle indefinitely.



(90) In his revise (see note (2)) Newton extensively amplified this scholium, asserting that any postulated interplanetary *æther* cannot offer appreciable resistance to orbiting bodies (and so will not be a set-back to the application of the preceding Problems 3 and 4) and referring to his correspondence with Robert Hooke (in the winter of 1679–80) on the apparent deviation

hacce motus gravium perpendiculariter cadentium ex Hypothesi quod gravitas reciprocè proportionalis sit quadrato distantiæ a centro terræ quodꝗ medium aeris nihil resistat. Nam gravitas est species una vis centripetæ.[93]

Prob. 6. Corporis sola vi insita per medium similare resistens delati motum definire.



Asymptotis rectangulis ADC, CH describatur Hyperbola secans perpendicula AB, DG in B, G. Exponatur tum corporis celeritas tum resistentia medij ipso motus initio per lineam[94] AC[,] elapso tempore aliquo per lineam[95] DC[,] et tempus exponi potest per aream $ABGD$ atꝗ spatium eo tempore descriptum per lineam AD. Nam celeritati proportionalis est resistentia medij et resistentiæ proportionale est decrementum celeritatis,[96] hoc est, si tempus in partes æquales

—to the south as well as east in the northern hemisphere, he would argue—of a body let fall from a height onto the rotating Earth. For its considerable intrinsic interest we reproduce its major portion in Appendix 1 below.

(91) Problem 4, that is.

(92) Newton first wrote 'gravium', carelessly anticipating his next clause. It is, of course, his wish to distinguish here between bodies impelled 'artificially' at some point and those which fall 'naturally' under terrestrial (inverse-square) gravitation alone.

(93) We have already pointed (see note (31)) to the significance of Newton's specification of 'gravity' (solar or terrestrial) as but one instance of the general central force (*vis centripeta*) whose accelerative effects he begins—for the first time—to discuss abstractly in his present treatise. In the manuscript he has, for no clear reason, cancelled an immediately preceding sentence 'Sequentibus resistentia medij similaris primùm absꝗ gravitate dein cum gravitate consideratur' (In the following [problems] the resistance of a homogeneous medium is considered, first in divorce from gravity and then in company with it). This smooth transition to the final Problems 6 and 7 (and their scholium), so harshly broken, was repaired in Newton's revise by the insertion of an appropriate subhead underscoring the change of theme to 'the motion of bodies in resisting media'. Compare Appendix 1 below.

The extant portion of Halley's transcript of the fair copy terminates at this point. The remaining pages were sent by him to John Wallis two years after he penned them, and are now lost. In his covering letter to Wallis on 11 December 1686 he recalls that 'Mr Isaac Newton about 2 years since gave me the inclosed propositions, touching the opposition of the Medium to a direct impressed Motion, and to falling bodies, upon supposition that the opposition is as the Velocity; which tis possible is not true: however I thought any thing of his might not be unacceptable to you, and I begg your opinion thereupon, if it might not be (especially the 7th problem) somewhat better illustrated' (from the original in Trinity College, Cambridge. R.4.45.111B, first published in E. F. MacPike, *Correspondence and Papers of Edmond Halley* (London, 1932): 74–5). As newly appointed Clerk to the Royal Society, responsible among other things for restoring the health of its ailing *Transactions*, Halley was primarily concerned not to divulge the content of Newton's researches into resisted motion (the text of which, let it be said, had already (see note (2)) been transcribed into the Society's Register Book, and so was accessible to any Fellow) but to use the communicated propositions as bait to persuade out of Wallis his own 'conclusions concerning the opposition of the Medium to projects moving through it; ...not doubting but that your extraordinary talent in matters of this nature, will be able to clear up this subject which hitherto seems to have been only mencōned among

atmosphere are defined, and by the present one those of heavy bodies falling perpendicularly, in accord with the hypothesis that gravity is reciprocally proportional to the square of the distance from the earth's centre and that the medium of the air resists not at all. For gravity is one species of centripetal force.[93]

Problem 6. To define the motion of a body borne by its innate force alone through a homogeneous resisting medium.

With rectangular asymptotes ADC, CH describe a hyperbola cutting the perpendiculars AB, DG in B and G. Express both the body's speed and the resistance of the medium by the[94] line AC at the very start of motion and by the[95] line DC after some lapse of time: the time can then be expressed by the area $ABGD$ and the distance described in that time by the line AD. For the resistance of the medium is proportional to the speed and the decrement of the speed[96] is proportional to the resistance; that is, if the time be divided into equal

Mathematicians, never yet fully discussed' (*ibid.*). (Like every one else in England at the time, Newton included, he was unaware that Huygens had fully solved the problem of motion under simple gravity when the 'opposition' is instantaneously proportional to the flight speed more than seventeen years earlier; see note (113) below.) After some initial hesitancy (compare A. R. Hall, *Ballistics in the Seventeenth Century* (Cambridge, 1952): 130–1) Wallis was soon afterwards coaxed into composing a short, woollily argued 'Discourse concerning the Measure of the Airs resistance to Bodies moved in it' (*Philosophical Transactions*, **16**, No. 186 [for January–March 1687]: 269–80) in which the horizontal and vertical components of the projectile path defined by $dv/dt = -rv+g.dy/ds$ are accurately rendered as the limit ($\Delta t \to 0$, where t is the base variable) of the respective difference-equations

$$v_{x+1}-v_x = -rv_x \quad \text{and} \quad v_{y+1}-v_y = -rv_y+g, \quad x, y = 0, 1, 2, 3, \ldots,$$

and the former's 'integral' $t \propto \log(v_0/v_x)$ is correctly deduced in the geometrical model of a hyperbolic area, but the corresponding solution ($v_x \to v_y - g/r$) is missed, and accordingly no rigorous justification is given for his assertion (*ibid.*: 27) that 'the line of Projects...resembles a Parabola deformed', an insight perhaps derived largely from his private view of Newton's concluding scholium. That Halley had not sought Newton's prior approval for his action is clear from his letter to Newton on 24 February 1686/7, where, in referring to 'Dr Wallis his papers', he remarked that Wallis 'had the hint from an account I gave him of what you had demonstrated' (*Correspondence of Isaac Newton*, **2**, 1960: 469). A fortnight or so before Newton had made independent enquiries, through Edward Paget, of Wallis' 'things about projectiles pretty like those of mine in ye papers Mr Paget first shewed you' and had been reassured that he would be 'consulted whether I intend to print mine', as he wrote to Halley on 13 February (*ibid.*: 464). Luckily for Halley (and Wallis), Newton had the manuscript of the second book of his *Principia* with, in its first section, improved versions of his present Problems 6 and 7 and terminal scholium (see Appendix 3 following) ready to go to press, and a potentially nasty squabble was averted by its arrival in London on 28 March (*ibid.*: 473).

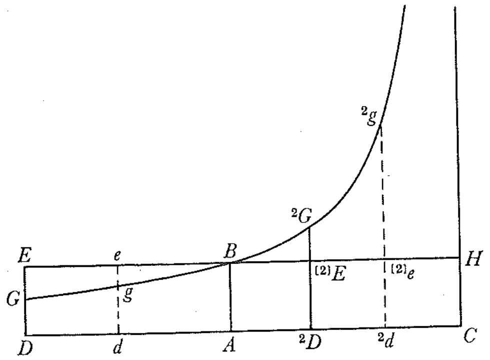
(94) The clarification 'quamvis datam' (any given) is deleted. In revise (see note (2)) Newton inserted the equivalent 'datæ longitudinis' (of given length) after '...AC'.

(95) Later qualified as 'indefinitam' (indefinite) by Newton in his revise.

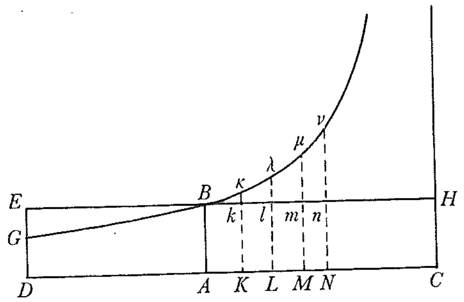
(96) Understand, as Newton at once effectively specifies, over a correspondingly infinitesimal increment of time.

dividatur, celeritates ipsarum initijs sunt differentijs suis proportionales.
[a]Lem [1] Decrescit ergo celeritas in [a]proportione Geometrica dum tempus crescit in Arithmetica. Sed tale est decrementum lineæ DC et incrementum areæ $ABGD$,(97) ut notum est. Ergo tempus per aream et celeritas per lineam illam rectè exponitur. Q.E.D. Porrò celeritati atq adeo decremento celeritatis proportionale est incrementum spatij descripti[,] sed et decremento lineæ DC proportionale est incrementum lineæ AD. Ergo incrementum spatij per incrementum lineæ AD, atq adeo spatium ipsum per lineam illam rectè exponitur. Q.E.D.(98)

Prob. 7. Posita uniformi vi centripeta, motum corporis in medio similari(99) *rectà ascendentis ac descendentis definire.*(100)



Corpore ascendente exponatur vis centripeta per datum quodvis rectangulum BC et resistentia medij initio ascensus per rectangulum BD sumptum ad contrarias partes. Asymptotis rectangulis AC, CH, per punctum B describatur Hyperbola secans perpendicula DE, de in G, g et corpus ascendendo tempore $DGgd$ describet spatium $EGge$, tempore $DGBA$ spatium ascensus totius EGB, tempore AB^2G^2D spatium descensus $B^{[2]}E^2G$ atq tempore ${}^2D^2G^2g^2d$ spatium descensus ${}^2G^{[2]}E^{[2]}e^2g$: et celeritas corporis resistentiæ medij proportionalis, erit in horum temporum periodis $ABED$, $ABed$, nulla,(101) $AB^{[2]}E^2D$, $AB^{[2]}e^2d$; atq maxima celeritas quam corpus descendendo potest acquirere erit BC.



Resolvatur enim rectangulum(102) AH in rectangula innumera Ak, Kl, Lm, Mn &c quæ sint ut incrementa celeritatum æqualibus totidem temporibus facta[,] et erunt Ak, Al, Am, An &c ut celeritates totæ atq
[a]Hyp [1] adeo [a]ut resistentiæ medij in fine(103) singulorum temporum æqualium. Fiat AC ad AK, vel $ABHC$ ad $ABkK$ ut vis cen-

(97) Changed by Newton in his revise to read 'Sed proportione priore decrescit linea DC et posteriore crescit area $ABGD$' (But the line DC decreases in the former proportion and the area $ABGD$ increases in the latter one).

(98) If (in more easily assimilable modern analytical equivalent) we suppose that the body, starting off from A along ADC with an initial speed V_x, is in time t slowed by the resistance acting over $AD = x$ down to the speed v_x at D, then, (with Newton) appropriately absorbing the constant of proportionality into t, we may set the resisting force $\dot{v}_x$ $(= dv_x/dt)$ to be equal to the speed $v_x = \dot{x}$ $(= dx/dt)$; accordingly, by Lemma 1 (compare note (13) above) the fluxional equation $\dot{v}_x = -v_x$ yields as its 'integral' $\log(v_x/V_x) = -t$, while its equivalent

parts, the speeds at the beginnings of these are proportional to their own differences. The speed therefore decreases in a geometrical proportion[a] while the time increases in an arithmetical one. But this is the manner of decrease of the line DC and of increase of the area $ABGD$,(97) as is known. Therefore the time is correctly expressed by the area and the speed by that line. As was to be proved. Furthermore, the increment of the space described is proportional to the speed and hence to the decrement of the speed, but so also is the increment of the line AD proportional to the decrement of the line DC. Therefore the increment of the space is correctly expressed by the increment of the line AD, and hence the space itself by that line. As was to be proved.(98) [a]Lemma 1

Problem 7. Supposing a uniform centripetal force, to define(100) *the motion of a body ascending and descending straight up and down in a homogeneous*(99) *medium.*

Where the body ascends, let the centripetal force be represented by any arbitrary rectangle $(A)B(H)C$ and the resistance of the medium at the start of ascent by the rectangle $(A)B(E)D$ taken the opposite way. With rectangular asymptotes AC, CH through the point B describe a hyperbola cutting the perpendiculars DE, de in G and g: the body in ascent in the time $(DGgd)$ will then describe the distance $(EGge)$ and in the time $(DGBA)$ a distance of total ascent (EGB), while in the time (AB^2G^2D) it will cover the distance (B^2E^2G) in descent and in the (further) time $({}^2D^2G^2g^2d)$ a descent $({}^2G^2E^2e^2g)$; also the body's speed, proportional to the resistance of the medium, will at these points in time be $(ABED)$, $(ABed)$, nothing,(101) (AB^2E^2D) and (AB^2e^2d), while the greatest speed which the body can acquire in its descent will be BC.

For resolve the rectangle $A(B)H(C)$ into innumerable rectangles Ak, Kk, Lm, Mn, ..., which shall be as the increments of speed brought about in a corresponding number of divisions of time, and Ak, Al, Am, An, ..., will then be as the whole speeds and hence[a] as the resistances of the medium at the end(103) of each of the corresponding equal times. Make AC to AK, or $ABHC$ to $ABkK$, as the [a]Hypoth. 1.

$\dot{v}_x = -\dot{x}$ produces straightforwardly $v_x - V_x = -x$. In the terms of Newton's hyperbolic model, therefore, it follows, on putting $AC = V_x$, that

$$DC\ (= V_x - x) = v_x \quad \text{and} \quad (ABGD) \propto \log\,(AC/DC) = \log\,(V_x/v_x) = t.$$

His proof demonstrates somewhat cumbrously that, where dt is a vanishingly small increment of the base variable t, then $d(DC) = -d(AD)$ and $d(ABGD) = GD \times d(AD) \propto -d(DC)/DC$, so that (as Newton puts it) if the area $(ABGD)$ increases arithmetically—and so its increments $d(ABGD)$ are constant—then $d(DC) \propto DC$ and therefore the increments of DC vary geometrically with their distance DC from C.

(99) Understand 'resistente' (resisting) as before.

(100) Newton first wrote 'exponere' (to represent).

(101) This replaces 'nihil' (nil).

(102) The equivalent 'parallelogrammum' was first written.

(103) In revise Newton improved his phrasing slightly to read ' et erunt nihil, Ak, Al, Am, An &c...in principio' (and nil, Ak, Al, Am, An, ... will then be...at the beginning).

tripeta ad resistentiam in fine temporis primi[104] et erunt *ABHC*, *KkHC*, *LlHC*, [*Mm*]*HC* &c ut vires absolutæ quibus corpus urgetur atq; adeo ut
[b]Lem. [1] incrementa celeritatum, id est ut rectangula *Ak*, *Kl*, *Lm*, *Mn* &c & [b]proinde in progressione geometrica. Quare si rectæ *Kk*, *Ll*, *Mm*, *Nn* [&c] productæ occurrant Hyperbolæ in κ, λ, μ, ν &c[105] erunt areæ $AB\kappa K$, $K\kappa\lambda L$, $L\lambda\mu M$, $M\mu\nu N$ &c æquales, adeoq; tum temporibus æqualibus tum viribus centripetis semper æqualibus analogæ. Subducantur rectangula *Ak*, *Kl*, *Lm*, *Mn* &c viribus absolutis analoga et relinquentur areæ $Bk\kappa$, $k\kappa\lambda l$, $l\lambda\mu m$, $m\mu\nu n$ &c resistentijs medii in fine singulorum temporum, hoc est celeritatibus atq; adeo descriptis spatijs analogæ.[106] Sumantur analogarum summæ et erunt areæ $Bk\kappa$, $Bl\lambda$, $Bm\mu$, $Bn\nu$ &c spatijs totis descriptis analogæ, nec non areæ $AB\kappa K$, $AB\lambda L$, $AB\mu M$, $AB\nu N$ &c temporibus.[107] Corpus igitur inter descendendum tempore quovis $AB\lambda L$ describit spatium $Bl\lambda$ et tempore $L\lambda[\nu]N$ spatium $\lambda ln\nu$. Q.E.D. Et similis est demonstratio motus expositi in ascensu. Q.E.D.[108]

Schol. Beneficio duorum novissimorum problematum innotescunt motus projectilium in aëre nostro, ex hypothesi quod aer iste similaris sit quodq; gravitas uniformiter & secundum lineas parallelas agat. Nam si motus omnis obliquus corporis projecti distinguatur in duos, unum ascensus vel descensus, alterum progressus horizontalis: motus posterior determinabitur per Problema sextum, prior per septimum ut fit in hoc diagrammate.[109]

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(104) This, correspondingly, was later changed to 'in principio temporis secundi' (at the beginning of the second time-division), while the sequel was considerably amplified to read 'deq; vi centripeta subducantur resistentiæ et manebunt *ABHC*, ...quibus corpus in principio singulorum temporum urgetur...' (from the centripetal force take away the resistances and *ABHC*, ...will then remain...by which the body is urged at the beginning of the individual times).

(105) Newton subsequently inserted a clarifying parenthesis 'ob proportionales *AK* ad *KL* ut *KC* ad *LC* hoc est ut $L\lambda$ ad $K\kappa$' (because of the proportionals *AK* to *KL* as *KC* to *LC*, that is, as $L\lambda$ to $K\kappa$).

(106) This sentence was afterwards substantially augmented to read: 'Est autem area $AB\kappa K$ ad aream $Bk\kappa$ ut $K\kappa$ ad $\frac{1}{2}k\kappa$ seu *AC* ad $\frac{1}{2}AK$ hoc est ut vis centripeta ad resistentiam in medio temporis primi. Et simili argumento areæ $\kappa KL\lambda$, $\lambda LM\mu$, $\mu MN\nu$ &c sunt ad areas $\kappa kl\lambda$, $\lambda lm\mu$, $\mu mn\nu$ &c ut vires centripetæ ad resistentias in medio temporis secundi tertij quarti &c. Proinde cum areæ æquales $BAK\kappa$, $\kappa KL\lambda$, $\lambda LM\mu$, $\mu MN\nu$ &c sint viribus centripetis analogæ, erunt areæ $Bk\kappa$, $\kappa kl\lambda$, $\lambda lm\mu$, $\mu mn\nu$ &c resistentijs in medio singulorum temporum, hoc est... descriptis spatijs analogæ' (Now the area ($AB\kappa K$) is to the area ($Bk\kappa$) as $K\kappa$ to $\frac{1}{2}k\kappa$ or *AC* to $\frac{1}{2}AK$, that is, as the centripetal force to the resistance at the middle of the first time-interval. And by a similar reasoning the areas ($\kappa KL\lambda$), ($\lambda LM\mu$), ($\mu MN\nu$),...are to the areas ($\kappa kl\lambda$), ($\lambda lm\mu$), ($\mu mn\nu$),...as the centripetal forces to the resistances at the middle of the second, third, forth,...time-intervals. Consequently, since the equal areas ($BAK\kappa$), ($\kappa KL\lambda$), ($\lambda LM\mu$), ($\mu MN\nu$),...are proportional to the centripetal forces, the areas ($Bk\kappa$), ($\kappa kl\lambda$), ($\lambda lm\mu$), ($\mu mn\nu$),...will be proportional to the resistances at the middle of the individual time-intervals, that is,...to the distances described.)

(107) In revise Newton stressed the passage to the limit implied in sequel by inserting 'Et hæ areæ ubi rectangula numero infinita et infinite parva evadunt coincidunt cum Hyperbolicis'

centripetal force to the resistance at the end of the first time-division,[104] and *ABHC*, *KkHC*, *LlHC*, *MmHC*, ... will then be as the absolute forces by which the body is urged and hence as the increments of speed, that is, as the rectangles *Ak*, *Kl*, *Lm*, *Mn*, ... and consequently[b] in geometrical progression. Wherefore, if [b]Lemma 1.
the lines *Kk*, *Ll*, *Mm*, *Nn*, ... when produced meet the hyperbola in κ, λ, μ, ν, ...,[105] the areas ($AB\kappa K$), ($K\kappa\lambda L$), ($L\lambda\mu M$), ($M\mu\nu N$), ... will be equal and hence in proportion both to the equal times and to the ever equal centripetal forces. Take away the rectangles *Ak*, *Kl*, *Lm*, *Mn*, ... proportional to the absolute forces and there will be left areas ($Bk\kappa$), ($k\kappa\lambda l$), ($l\lambda\mu m$), ($m\mu\nu n$), ... proportional to the resistances of the medium at the end of the separate intervals of time, that is, to the speeds and hence to the distances described.[106] Take the sums of these proportionals and the areas ($Bk\kappa$), ($Bl\lambda$), ($Bm\mu$), ($Bn\nu$), ... will be in proportion to the total distances described, and also the areas ($AB\kappa K$), ($AB\lambda L$), ($AB\mu M$), ($AB\nu N$), ... to the times.[107] The body, therefore, during its descent in any time ($AB\lambda L$) describes the distance ($Bl\lambda$) and in the (further) time ($L\lambda\nu N$) the distance ($\lambda ln\nu$). As was to be proved. The demonstration for the motion represented in ascent is similar. As was to be proved.[108]

Scholium. With the aid of the two most recent problems the motions of projectiles in our air are discoverable, on the hypothesis that this air is homogeneous and that gravity acts uniformly and following parallel lines. For if every oblique motion of a projected body be distinguished into two, one of ascent or descent, the other of horizontal advance, the latter motion will be determined by the sixth problem and the former by the seventh, as happens in this diagram.[109]

(And when the rectangles come to be infinite in number and infinitely small, these areas coincide with the [corresponding] hyperbolie ones).

(108) If, in analytical clarification, we suppose that the moving body starts off with speed V_y and, ever subject to a constant decelerating force g, is in time t further slowed by the resistance, acting over a distance y, down to the speed $v_y = dy/dt$, then we may set the equation of motion to be $dv_y/dt = -v_y - g$, that is, $d(v_y+g)/dt = -(v_y+g)$; whence (compare note (98)) at once $\log((v_y+g)/(V_y+g)) = -t$ and $(v_y+g)-(V_y+g) = -y-gt$. In the terms of Newton's hyperbolic model, therefore, it follows on putting $CA = g$, $AD = V_y$, $Ad = v_y$ and (for simplicity) $AB = DE = 1$ that

$$(DGgd) = CA\log(CD/cd) = g\log((V_y+g)/(v_y+g)) = gt$$

and

$$(DEed) = Dd = V_y - v_y = y+gt,$$

so that $(GEeg) = y$. Conversely, where the constant force g accelerates the motion, the corresponding equation of motion $dv_y/dt = -v_y+g$ yields

$$\log((v_y-g)/(V_y-g)) = -t \quad \text{and} \quad (v_y-g)-(V_y-g) = -y+gt:$$

hence in the geometrical model, on taking $CA = g$ and $AB = 1$ as before but now $A^2D = V_y$ and $A^2d = v_y$, there ensues

$$({}^2D^2G^2g^2d) = CA\log(C^2D/C^2d) = gt \quad \text{and} \quad ({}^2D^2E^2e^2d) = {}^2D^2d = -y+gt,$$

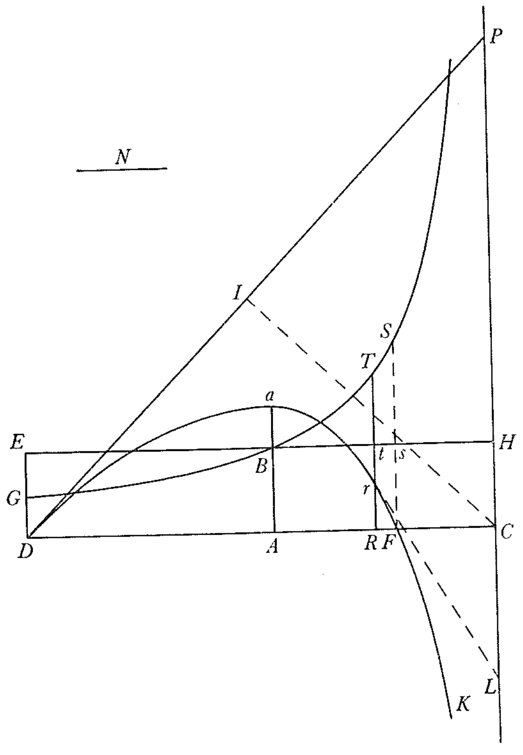
so that $({}^2G^2E^2e^2g) = y$.

(109) Here, if the body shot off at D along DP traverses the arc $\overset{\frown}{Dar} = s$ under simple gravity g (acting vertically downwards) to $r(x, y)$, defined by the coordinates $DR = x$ and $Rr = y$,

Ex loco quovis D ejaculetur corpus secundum lineam quamvis rectam DP, & per longitudinem DP exponatur ejusdem[110] celeritas sub initio motus. A puncto P ad lineam horizontalem DC demittatur perpendiculum PC, ut et ad DP[111] perpendiculum CI, ad quod sit DA ut est resistentia medij ipso motus initio ad vim gravitatis. Erigatur perpendiculum AB cujusvis longitudinis et completis parallelogrammis $DABE$, $CABH$, per punctum B asymptotis DC CP describatur Hyperbola secans DE in G. Capiatur linea N ad EG ut est DC ad CP et ad rectæ DC punctum quodvis R erecto perpendiculo RtT, in eo cape

$$Rr = \frac{DRtE - DRTBG}{N}$$

et projectile tempore $DRTBG$ perveniet ad punctum r, describens curvam lineam $DarFK$ quam punctum r semper tangit, perveniens autem ad maximam altitudinem a in perpendiculo AB, deinde incidens in lineam horizontalem DC ad F ubi areæ $DFsE$, $DFSBG$ æquantur,[112] et postea semper appropinquans Asymptoton PCL. Estq; celeritas ejus in puncto quovis r ut curvæ tangens rL.[113]



there attaining the orbital speed $v = ds/dt$ after time t, then (on presuming, with Newton, that the constant of proportionality of resistance to instantaneous velocity is suitably absorbed) the equation of motion of the body at r is $dv/dt = -v - g . dy/ds$. Newton's present assumption that this may be split into horizontal and vertical components, respectively $dv_x/dt = -v_x$, $v_x = dx/dt$ (see note (98)) and $dv_y/dt = -v_y - g$, $v_y = dy/dt$ (see note (108)), is accurate (for, because $\widehat{DRr}$ is right, $v^2 = v_x^2 + v_y^2$ and therefore

$$dv/dt = v . dv/ds = v_x . dv_x/ds + v_y . dv_y/ds = (dx/ds) . dv_x/dt + (dy/ds) . dv_y/dt)$$

but, for all its plausibility, not a self-evident truth; it is, indeed, impermissible to split the general equation $dv/dt = -v^n - g . dy/ds$ $(n \neq 1)$ into comparable 'components' $dv_x/dt = -v_x^n$ and $dv_y/dt = -v_y^n - g$, as Leibniz in 1688 claimed might be done—'sed prolixitas hic vitanda est'—in the instance $n = 2$ (that of *resistentia respectiva*, defined by $dv/ds = -v$ when $g = 0$) in Article VI of his insufficiently pondered 'Schediasma de Resistentia Medii, & Motu projectorum gravium in medio resistente' (purportedly mapped out 'jam pro magna parte Parisiis duodecim abhinc annis' [*sc.* in 1676] but composed only shortly before its publication in *Acta Eruditorum* (January 1689): 38–47 [= (ed. C. I. Gerhardt) *Leibnizens Mathematische Schriften*, **6** (Halle, 1860): 135–43]). Newton seemingly remained ignorant of the 'Schediasma'

Let the body be hurled from any position D following any straight line DP, and let its[110] speed at the start of motion be expressed by the length DP. From the point P let fall the perpendicular PC to the horizontal DC, and also drop the perpendicular CI to DP;[111] then let DA be to CI as the resistance of the medium at the very start of motion is to the force of gravity. Erect the perpendicular AB of any length and, having completed the parallelograms $DABE$, $CABH$, through the point B and with asymptotes DC, CP describe an hyperbola cutting DE in G. Take the line N to EG as DC is to CP and, having erected the perpendicular RtT at any point R of the line DC, take in it $Rr = ((DRtE) - (DRTBG))/N$ and the projectile will in time $(DRTBG)$ reach the point r, describing the curve $DarFK$ with which the point r is ever in contact—attaining, indeed, its maximum height at a in the perpendicular AB, then falling onto the horizontal DC at F, where the areas $(DFsE)$ and $(DFSBG)$ are equal,[112] and ever afterwards approaching the asymptote PCL; and its speed at any point r will be as the curve's tangent rL.[113]

for a quarter of a century till John Keill freshly brought it to his notice in the heat of the fluxion priority squabble as one more stick with which to belabour Leibniz, but afterwards he was only too ready to criticise Leibniz' 'erroneous' attempt 'to find the Curve described in a Medium where the resistance was in a duplicate ratio of the velocity [by] compos[ing] the horizontal & perpendicular motions of the projectile' (ULC. Add. 3968.31: 457[r], extracted from a draft 'Supplement' to Desmaizeaux' editorial *Preface* introducing his *Recueil* (note (4)), **1**: i–lxxxi in 1720). Above all, in a developed critique of Leibniz' paper which he sent to John Keill in May 1714 he wrote: 'In sexto [Articulo] Propositiones sunt tantum duæ, et utraq; falsa est. Corpus enim, ubi resistentia est in duplicata ratione velocitatis, non fertur motu composito ex motibus duorum Articulorum præcedentium [$dv_x/dt = -v_x^2$ and $dv_y/dt = -v_y^2 + g$ respectively]' (ULC. Res. 1893.8, published in Joseph Edleston, *Correspondence of Sir Isaac Newton and Professor Cotes* (London, 1850): 309–10). Newton himself straightforwardly discussed these 'component' Leibnizian rectilinear motions in Propositions V–VII and VIII/IX respectively of Book 2 of his published *Principia* (see Appendix 2: note (17) below), but, in seeking in his following Proposition X to reduce the general equation $dv/dt = -\rho + g. dy/ds$ of motion under constant gravity g and resistance ρ, he was forced to concoct *ad hoc* a cumbrous, circuitous method which in a particular instance, as Johann Bernoulli brought to his attention in autumn 1712 (though he himself could not accurately trace the error—an inexact infinitesimal approximation—in Newton's general argument), led ineluctably to a numerically false result. (Compare D. T. Whiteside, 'The Mathematical Principles underlying Newton's *Principia Mathematica*' (note (30)): 128–9.) Extracts from the sequence of manuscripts in which Newton came to appreciate his mistake and to give several variant proofs of the correct result—only one of which was set, in a last minute stop-gap, in the new edition of his *Principia* ($_2$1713: 232–40)—will be reproduced in the eighth volume.

(110) Initially 'projecti' (the projectile's) was written.

(111) Accurately, this should (see note (113) below) be '...ad parallelam ipsi DP ductam per A' (to the parallel to DP drawn through A), or some equivalent which reduces the length of Newton's perpendicular CI in the ratio AC/DC.

(112) Perhaps because it merely states the obvious, this clause 'deinde...æquantur' (then ...are equal) was cancelled by Newton in his redraft.

(113) Where (as in note (109)) the body, shot off at D under simple downwards gravity g and perpetually slowed by a resistance equal to its orbital speed, traverses $\widehat{Dar}$ in time t,

Si proportio resistentiæ aeris ad vim gravitatis nondum innotescit: cognoscantur (ex observatione aliqua) anguli ADP, AFr(114) in quibus curva $DarFK$ secat lineam horizontalem DC. Super DF constituatur rectangulum $DFsE$ altitudinis cujusvis, ac describatur Hyperbola rectangula ea lege ut ejus una Asymptotos sit DF, ut areæ $DFsE$, $DFSBG$ æquentur et ut sS sit ad EG sicut tangens anguli AFr(114) ad tangentem anguli ADP. Ab hujus Hyperbolæ centro C ad rectam DP(115) demitte perpendiculum CI ut et a puncto B ubi ea secat rectam Es, ad rectam DC perpendiculum BA, et habebitur proportio quæsita DA ad CI, quæ est resistentiæ medij ipso motus initio ad gravitatem projectilis. Quæ omnia ex prædemonstratis facilè eruuntur. Sunt et alij modi inveniendi resistentiam aeris quos lubens prætereo. Postquam autem inventa est hæc resistentia in uno casu, capienda est ea in alijs quibusvis ut corporis celeritas et superficies sphærica conjunctim, (nam projectile sphæricum esse passim suppono;) vis autem gravitatis innotescit ex pondere. Sic habebitur semper pro-

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attaining at point r (defined by $DR = x$, $Rr = y$) the speed $ds/dt = v$ of respective horizontal and vertical components $dx/dt = v_x$ and $dy/dt = v_y$, we may integrate the corresponding equations of motion $dv_x/dt = -v_x$ and $dv_y/dt = -v_y - g$ to produce $V_x - v_x = x$, $\log(V_x/v_x) = t$ (note (98)) and $V_y - v_y = y + gt$, $\log((V_y+g)/(v_y+g)) = t$ (note (108)), where V_x, V_y are the initial values (at D) of v_x, v_y. At once $V_x/v_x = V_x/(V_x - x) = (V_y+g)/(v_y+g)$ and so

$$x/V_x = (V_y - v_y)/V_y + g),$$

whence $y = ((V_y+g)/V_x)x - g\log(V_x/(V_x - x))$ is the defining Cartesian equation of $r(x, y)$. Clearly, the projectile will reach its maximum height at $a(x, y)$ when $v_y = 0$ and hence $DA = V_xV_y/(V_y+g)$, and attain a maximum horizontal range $DC = X$ at $v_X = V_x - X = 0$ and so $DC = V_x$; whence $CP = V_y$, $DP = V$, $AC = g.V_x/(V_y+g)$ and $RC = V_x - x = v_x$, so that the tangent $rL = v_x.ds/dx = v$. Further, since $AB \times AC/EG = AC \times DC/DA = (V_x/V_y)g$, on taking (with Newton) $V_x/V_y = N/EG$, there results

$$(DRtE) = AB \times DR = (N.g/AC)\,DR = N.((V_y+g)/V_x)x$$

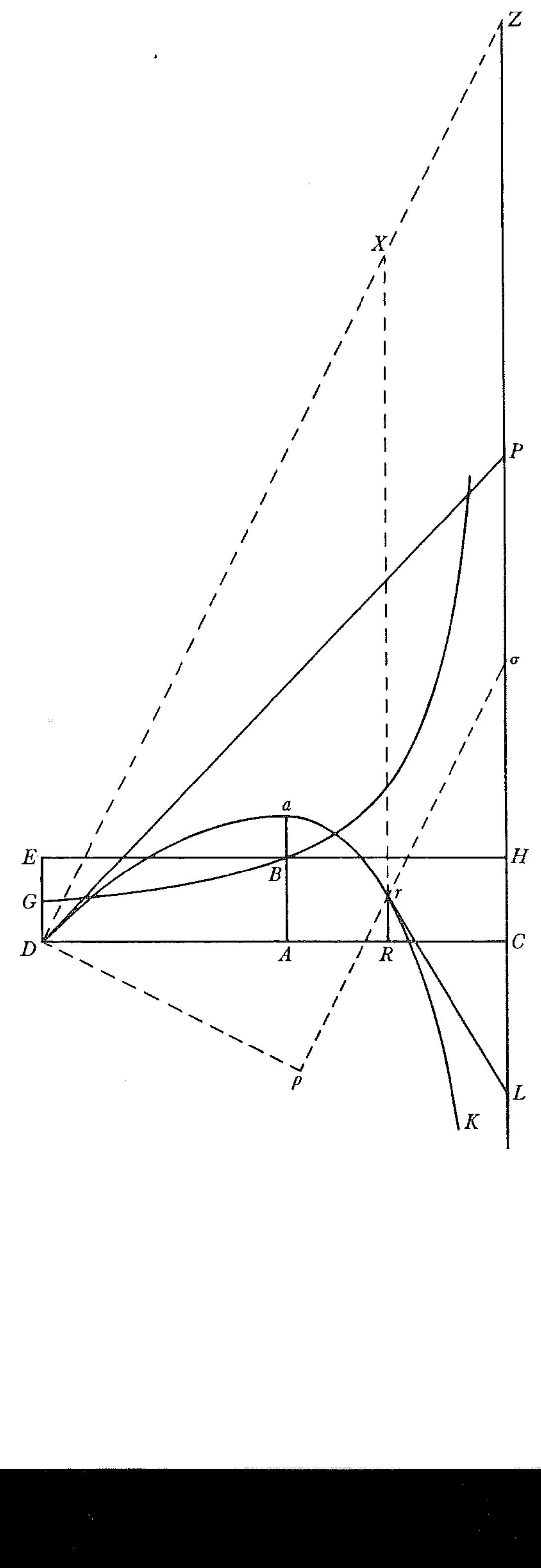
and

$$(DRTBG) = AB \times AC.\log(DC/RC) = N.gt \propto t,$$

so that $Rr = ((DRtE) - (DRTBG))/N$. Lastly, the 'resistentia medij ipso motus initio' V is to the 'vis gravitatis' g as $V_xV_y/g.CI$, that is,—on correcting a trivial Newtonian slip—as DA/CI', where $CI' = CI.g/(V_y+g) = CI.(AC/DC)$.

Newton was to set Problems 6 and 7 and the preceding portion of the present scholium with minimal revision as the opening Section I of Book 2 of his *Principia* ($_1$1687: 236–45), there presenting in Proposition IV a geometrical demonstration of what he here merely asserts. For purposes of comparison with our modern analytical justification we reproduce the text of this geometrical revision in Appendix 3. 1/3 following. The novelty of his present solution to the problem of motion under resistance varying as the speed and a constant uni-directional diverting force should not be over-stressed: though Newton was still himself unaware of it in 1687, Christiaan Huygens had here anticipated him by nearly two decades. (Huygens made public announcement of his own near-identical researches only three years afterwards in the concluding pages of his reworked *Discours de la Cause de la Pesanteur* (Leyden, 1690): 168–80 [=*Œuvres complètes*, **21** (The Hague, 1944): 478–93], there remarking that 'J'ay vu avec plaisir ce que Mr. Newton écrit touchant les chûtes & les jets des corps pesants dans l'air, ou dans quelqu' aut[r]e milieu qui resiste au mouvement; m'estant appliqué autrefois à la mesme

If the ratio of the resistance of the air to the force of gravity is not yet ascertained, let there (from some observation) be learnt the angles $\widehat{ADP}$ and $\widehat{AFr}$(114) in which the curve $DarFK$ intersects the horizontal DC. On DF form the rectangle $DFsE$ of any height, then describe a rectangular hyperbola with the restrictions that DF be one of its asymptotes, that the areas $(DFsE)$, $(DFSBG)$ be equal, and that sS be to EG as the tangent of the angle $\widehat{AFr}$(114) to the tangent of $\widehat{ADP}$. From the centre C of this hyperbola let fall the perpendicular CI to the line DP,(115) and also from the point B where it cuts the line Es drop the perpendicular BA to the line DC, and the required ratio DA to CI—that of the resistance of the medium at the very start of motion to the gravity of the projectile—will be had. All these results are easily derived from what has previously been demonstrated. There are other methods, too, of finding out the resistance of the air, but these I readily pass over. After this resistance has been determined in one case, however, it needs to be taken in any others jointly as the body's speed and its spherical surface (for I suppose throughout that the projectile is spherical); while the force of gravity is ascertainable from its



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recherche....J'examinay premierement ces mouvemens, en supposant que les forces de la Resistance sont comme les Vitesses des corps, ce qui alors me paroissoit fort vraisemblable' (*ibid.*: 168–9). His original manuscript article 'De proportione gravium cadentium habita ratione resistentiæ aeris vel aquæ', precisely dated in his usual way by a '*εὕρηκα* 28 Oct. 1668' [N.S.], is printed in his *Œuvres complètes*, **19**, 1937: 102–18.) In the second edition of his *Principia* ($_2$1713: 215–19) Newton amplified his earlier discussion of the logarithmic projectile-path $DarK$, inserting two new Corollaries 1 and 2 which we likewise reproduce in Appendix 3.2. In essence, if CP is extended to Z so that $PZ = g$ and Rr drawn to meet DZ in X, then $XR = (V_x/(V_y+g))\,x$ and so $rX = gt$. It follows at once that the body r moves uniformly away from DZ as it traverses $\widehat{DarK}$—in other words, if $D\rho$ is the perpendicular from D to the line $\rho r\sigma$ drawn through r parallel to DZ, then portio resistentiæ ad gravitatem seu lineæ DA ad lineam CI. Hac proportione et angulo ADP determinatur specie figura $DarFKLP$: et capiendo longitudinem DP proportionalem celeritati projectilis in loco D determinatur eadem magnitudine sic ut altitudo Aa maximæ altitudini projectilis et longitudo DF longitudini horizontali inter ascensum et casum projectilis semper sit proportionalis, atq; adeò ex longitudine DF in agro semel mensurata semper determinet tum longitudinem illam DF tum alias omnes dimensiones figuræ $DarFK$ quam projectile describit in agro. Sed in colligendis hisce dimensionibus usurpandi sunt logarithmi pro area Hyperbolica $DRTBG$.(116)

Eadem ratione determinantur etiam motus corporum gravitate vel levitate & vi quacunq; simul et semel impressa moventium in aqua.

APPENDIX 1. THE AUGMENTED TRACT 'DE MOTU CORPORUM' (DECEMBER 1684?).(1)

Excerpts from the corrected amanuensis copy(2) in the University Library, Cambridge

DE MOTU SPHÆRICORUM CORPORUM IN FLUIDIS.(3)

Def. 1. Vim centripetam appello qua corpus attrahitur vel impellitur versus punctum aliquod quod ut centrum spectatur.

Def. 2. Et vim corporis seu corpori insitam qua id conatur perseverare in motu suo secundum lineam rectam.

$D\rho\,(= (V_\infty/V)g.t) \propto t$. The basic subtangential property of the *logarithmica DarK*, defined by $Xr = (-g.\log(RC/DC)$ or$) -g.\log(ZX/ZD)$ with respect to the oblique Cartesian coordinate-lengths ZX and Xr, gives straight-forwardly $L\sigma = ZX.\,d(Xr)/d(ZX) = g = PZ$, constant, with $rL = v$ its oblique projection; whence, as a corollary, the terminal speed of the projectile as it nears the asymptote ZPC is g. (See also J. A. Lohne, 'The Increasing Corruption of Newton's Diagrams' [*History of Science*, **6**, 1967: 69–89]: 76–80.)

(114) Newton assumes, for simplicity of reference, that the small arc $\widehat{Fr}$ is effectively a straight line.

(115) Again (compare note (111)) this should be '... ad parallelam rectæ DP per A transeuntem' (to the parallel to the line DP passing through A), or some equivalent diminishing CI in the ratio AC/DC.

(116) A somewhat needless reminder to anyone who has read the preceding paragraphs with understanding, surely?

(1) As we have earlier indicated (see §1: note (2) preceding), this immediate revise is basically Humphrey Newton's secretarial transcript of the corrected state of Newton's primary autograph, amplified by an augmented set of introductory 'Definitions' (now also including five 'Laws') and 'Lemmas' and by new *scholia* to Theorem 4 and Problem 5, and further altered and lightly amended by Newton's own hand. By and large these latter changes convert

weight. In this way the ratio of the resistance to gravity—that is of the line DA to the line CI—will always be had. From this ratio and the angle $\widehat{ADP}$ the configuration $DarFKLP$ is determined in species; and by taking the length DP proportional to the speed of the projectile at the position D it is determined in magnitude: accordingly the altitude Aa is ever proportional to the maximum altitude of the projectile and the length DF to the horizontal length covered by the projectile during its rise and fall, and hence, once the length DF is measured in the field, it will always determine not merely that length DF but also all other dimensions of the figure $DarFK$ which the projectile describes in the field. But in obtaining these dimensions logarithms must be employed in place of the hyperbolic area ($DRTBG$).(116)

By the same procedure are determined also the motions of bodies moving in water under gravity or levity and any arbitrary force impressed once and instantaneously.

Def. 3. Et resistentiam quæ est medij regulariter impedientis.

Def. 4. Exponentes quantitatum sunt aliæ quævis quantitates proportionales expositis.(4)

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the present text into corresponding portions of a still more developed tract 'De motu Corporum' (§2 following) which Humphrey Newton was likewise entrusted to pen out. Though its principal innovations are already published in more than one place (see next note) and not of narrow mathematical interest, we have thought fit to outline the broad features of this present interim revise because of the unique glimpse it affords of Newton's rapidly changing and developing ideas on general celestial and terrestrial motion in the early winter of 1684–5. This date of composition, while not explicitly supported by contemporary documentation, is narrowly delimited on the one hand by the preparation about early November 1684 of the putative fair copy of the original 'De motu Corporum' (see §1: note (2)), and on the other by the elaboration in the first months of 1685 of its several revises, beginning with the remoulding of its preliminaries (§2, Appendix 1) and then continuing with its large-scale recasting (§§2/3).

(2) ULC. Add. 3965.7: 40^r–54^r, first published in full—but interblended with the analytical table of contents set by Halley at the head of his transcript of the primary version (see §1: note (2))—by A. R. and M. B. Hall in their *Unpublished Scientific Papers of Isaac Newton* (Cambridge, 1962): 243–5/247–67(with English translation on 267–70/271–92 following). The principal innovations in its text were earlier recorded by W. W. R. Ball in his *Essay on Newton's Principia* (London, 1893): 51–6, and are also given in J. W. Herivel's *Background to Newton's Principia* (Oxford, 1966): 294–9 (with English renderings on 299–303).

(3) On this change in title see §1: note (3) above.

(4) A somewhat trivial innovation in the primary text. In practice, of course, Newton will employ an 'exponent' which, by suitably absorbing a constant factor of proportionality, will result in a simplest possible representing mathematical (fluxional) equation.

(5)*Lex 1.* Sola vi insita corpus uniformiter(6) in linea recta semper pergere si nil impediat.

Lex 2. Mutationem status movendi vel quiescendi proportionalem esse vi impressæ et fieri secundum lineam rectam qua vis illa imprimitur.

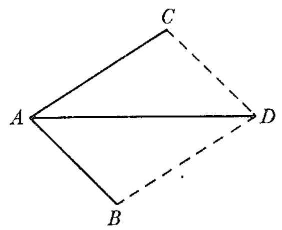
Lex 3. Corporum dato spatio inclusorum eosdem esse motus inter se sive spatium illud quiescat sive moveat id perpetuò et uniformiter in directum absqꝫ motu circulari.

Lex 4. Mutuis corporum actionibus commune centrum gravitatis non mutare statum suum motus vel quietis. Constat ex Lege 3.

Lex 5. Resistentiam medij esse ut medij illius densitas et corporis moti sphærica superficies & velocitas conjunctim.

(7)*Lemma 1.* Corpus viribus conjunctis diagonalem parallelogrammi eodem tempore describere quo latera separatis.

Si corpus dato tempore vi sola *M* ferretur ab *A* ad *B* et vi sola *N* ab *A* ad *C*, compleatur parallelogrammum *ABDC* et vi utraqꝫ feretur id eodem tempore ab *A* ad *D*. Nam quoniam vis *M* agit secundum lineam *AC* ipsi *BD* parallelam, hæc vis per Legem 2 nihil mutabit celeritatem accedendi ad lineam illam *BD* vi altera impressam. Accedet igitur corpus eodem tempore ad lineam *BD* sive vis *AC* imprimatur sive non, atqꝫ adeò in fine illius temporis reperietur alicubi in linea illa *BD*. Eodem argumento in fine temporis ejusdem reperietur alicubi in linea *CD*, et proinde in utriusqꝫ lineæ concursu *D* reperiri necesse est.



Lemma 2. Spatium quod corpus urgente quacunqꝫ vi centripeta ipso motus initio describit, esse in duplicata ratione temporis.

Exponantur tempora per lineas *AB*, *AD* datis *Ab Ad* proportionales, et urgente vi centripeta æquali exponentur spatia descripta per areas rectilineas *ABF*, *ADH* perpendiculis *BF*, *DH* et rectâ quavis *AFH* terminatas ut exposuit Galilæus. Urgente autem vi centripeta inæquabili(8) exponantur spatia descripta

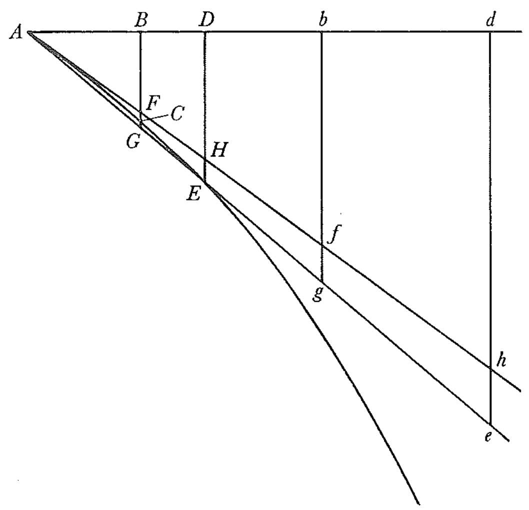
(5) In the following recasting, 'Laws' 1 and 5 replace the earlier equivalent 'Hypotheses' 2 and 1 respectively, and Law 2 explicitly justifies the basic Newtonian 'Axiom 2' of motion (as it will be renamed in §2) which is fundamental in the demonstration of Theorems 1–3 but was earlier merely assumed; Laws 3 and 4 (needed in the newly introduced final paragraph of the scholium to Theorem 4 below) have, of course, no earlier correlatives.

(6) Initially 'motu uniformi' before being changed back by Newton to his original adverb.

(7) Of these four following 'Lemmas', 1 and 2 restate—now with full demonstration (and explicit appeal to *Lex* 2 preceding)—Newton's earlier 'Hypotheses' 3 and 4 (compare §1: notes (10) and (12)), while 3 and 4 merely reiterate his earlier Lemmas 1 and 2.

(8) Lightly changed by Newton from the initially copied equivalent 'Sit autem vis centripeta inæquabilis et perinde'.

per areas *ABC*, *ADE* curva quavis *ACE* quam recta *AFH* tangit in *A*, comprehensas. Age rectam *AE* parallelis *BF*, *bf*, *dh* occurrentem in *G*, *g*, *e*, et ipsis *bf dh* occurrat *AFH* producta in *f* et *h*. Quoniam area *ABC* major est area *ABF* minor area *ADEG* erit area *ABC* ad aream *ADEG* major quam area *ABF* ad aream *ADEG* minor quam area *ABG* ad aream *ADH*[,] hoc est major quam area *Abf* ad aream *Ade* minor quam area *Abg* ad aream *Adh*. Diminuantur jam lineæ *AB*, *AD* in ratione sua data usqꝫ dum puncta *ABD* coeunt et linea *Ae* conveniet cum tangente *Ah*; adeoqꝫ ultimæ rationes *Abf* ad *Ade* et *Abg* ad *Adh* evadent eædem cum ratione *Abf* ad *Adh*. Sed hæc ratio est dupla rationis *Ab* ad *Ad* seu *AB* ad *AD*[,] ergo ratio *ABC* ad *ADEC* ultimis illis intermedia jam fit dupla rationis *AB* ad *AD* id est ratio ultima evanescentium spatiorum seu prima nascentium dupla est rationis temporum.



Lemma 3. Quantitates differentijs suis proportionales sunt continuè proportionales. Ponatur *A* ad *A*−*B*, ut *B* ad *B*−*C* & *C* ad *C*−*D* &c et dividendo fiet *A* ad *B* ut *B* ad *C* et *C* ad *D* &c.

Lemma 4. Parallelogramma omnia circa datam Ellipsin descripta, esse inter se æqualia. Constat ex Conicis.

De motu corporum in mediis non resistentibus.(9)

Theorema 1. Gyrantia omnia...describere. Dividatur tempus . . . . . . et constabit propositio.

Theorem. 2. Corporibus in...circulorum. Corpora *B*, *b* . . . . . . constat Propositio.

Cor. 1. . . . . *Cor. 2.* . . . . *Cor. 3.* . . . . *Cor. 4.* . . . . *Cor. 5.* . . . .

Schol. Casus Corollarij quinti...circa Jovem jam statuunt Astronomi.

Theor. 3. Si corpus P...puncta P et Q. Namqꝫ in figura.... Q.E.D.

Corol. Hinc si...in problematîs sequentibus.

Prob. 1. Gyrat corpus...in circumferentia. Esto circuli . . . . . . inveniendum.

Schol. Cæterum in hoc casu...tangit.

Prob. 2. Gyrat corpus...centrum Ellipseos. Sunto *CA*, *CB* .... Q.E.I.

(9) This inserted subhead counterbalances the parallel one '...IN MEDIJS RESISTENTIBUS' which fills (compare §1: note (93)) a more necessary rôle below.

Prob. 3. Gyrat corpus...umbilicum Ellipseos. Esto Ellipsis.... Q.E.I.

Schol. Gyrant ergo Planetæ majores...sunt quantitates $\frac{QT^q}{QR}$, quæ ultimò fit ubi coeunt puncta P et Q.

Theor. 4. Posito quod vis...axium. Sunto Ellipseos...Ellipsibus. Q.E.D.

Schol. Hinc in Systemate cœlesti...conveniant. Hac methodo determinare licet...determinabuntur.

Cæterum totum cœli Planetarij spatium vel quiescit (ut vulgò creditur) vel uniformiter movetur in directum et perinde Planetarum commune centrum gravitatis (per Legem 4) vel quiescit vel una movetur. Utroq; in casu motus Planetarum inter se (per Legē 3) eodem modo se habent, et eorum commune centrum gravitatis respectu spatij totius quiescit, atq; adeo pro centro immobili Systematis totius Planetarij haberi debet. Inde verō Systema Copernicæum probatur a priori. Nam si in quovis Planetarum situ computetur commune centrum gravitatis[,] hoc vel incidet in corpus Solis vel ei semper proximum erit. Eo Solis a centro gravitatis errore fit ut vis centripeta non semper tendat ad centrum illud immobile et inde ut planetæ nec moveantur in Ellipsibus exactè neq; bis revolvant in eadem orbita. Tot sunt orbitæ Planetæ cujusq; quot revolutiones, ut fit in motu Lunæ, et pendet orbita unaquæq; ab omnium Planetarum motibus conjunctis, ut taceam eorum omnium actiones in se invicem. Tot autem motuum causas simul considerare et legibus exactis calculum commodum admittentibus motus ipsos definire superat ni fallor vim omnem humani ingenij. Omitte minutias illas et orbita simplex et inter omnes errores mediocris erit Ellipsis de qua jam egi. Si quis hanc Ellipsin ex tribus observationibus per computum trigonometricum (ut solet) determinare tentaverit, hic minus caute rem aggressus fuerit. Participabunt observationes illæ de minutijs motuum irregularium hic negligendis adeoq; Ellipsim de justa sua magnitudine et positione (quæ inter omnes errores mediocris esse debet) aliquantulum deflectere facient, atq; tot dabunt Ellipses ab invicem discrepantes quot adhibentur observationes trinæ. Conjungendæ sunt igitur et una operatione inter se conferendæ observationes quamplurimæ, quæ se mutuò contemperent et Ellipsin positione et magnitudine mediocrem exhibeant.(10)

(10) Curtis Wilson has given a lengthy analysis of this paragraph in his 'From Kepler's Laws, So-called, to Universal Gravitation: Empirical Factors' (*Archive for History of Exact Sciences*, **6**, 1970: 89–170): 161–2. We concur in his criticism that the concept of a 'Planetarum commune centrum gravitatis' (instantaneous centre of interacting planetary force) to which Newton here appeals is ill-defined and too readily supposed to lie within or closely near to the Sun, and can likewise trace no earlier Newtonian statement regarding 'eorum omnium actiones in se invicem' or dismissal of such mutual planetary interactions as merely yielding minimal periodic (and non-cumulative) divergences from the 'true' mean Keplerian, exactly elliptical orbits. (Kepler himself had been less sure, in his 1627 *Tabulæ Rudolphinæ*, that such 'physic[æ] minim[æ] intensiones et remissiones extra ordinem' were effectively negligible

Prob. 4. Posito quod...emissum. Vis centripeta tendens...Ellipsis. Q.E.I. Hæc ita se habent...*PS* et *PH*.

Schol. Jam verò...determinare. Sed areas...proportionalis.

Prob. 5. Posito quod vis...describit. Si corpus...perpendicularis *DC*. Q.E.F.

Schol: Hactenus motum corporum in medijs non resistentibus exposui; id adeo ut motus corporum cœlestium in æthere determinarem. Ætheris enim puri resistentiam quantum sentio vel nulla est vel perquam exigua....(11) Interfluit æther liberrimè nec tamen resistit sensibiliter. Cometas infra orbitam Saturni descendere jam sentiunt Astronomi saniores quotquot distantias eorum ex orbis magni parallaxi præterpropter colligere norunt: hi igitur celeritate immensa in omnes cœli nostri partes indifferenter feruntur, nec tamen vel crinem seu vaporem capiti circundatum resistentia ætheris impeditum et abreptum amittunt. Planetæ verò jam per annos millenos in motu suo perseverarunt, tantum abest ut impedimentum sentiant.

Demonstratis igitur legibus reguntur motus in cœlis. Sed et in aere nostro, si resistentia ejus non consideratur, innotescunt motus projectilium per Prob. 4. et motus gravium perpendiculariter cadentium per Prob. 5, posito nimirum quod gravitas sit reciprocè proportionalis quadrato distantiæ a centro terræ. Nam virium centripetarum species una est gravitas; et computanti mihi prodijt vis centripeta qua luna nostra detinetur in motu suo menstruo circa terram, ad vim gravitatis hic in superficie terræ, reciprocè ut quadrata distantiarum a centro terræ quamproximè.(12) Ex horologij oscillatorij motu tardiore in cacu-

over a period or the product of inaccurate observations or inadequately computed planetary elements, but such an 'extreme' view was later heavily criticised by Jeremiah Horrocks; see Curtis Wilson, 'Kepler's Derivation of the Elliptical Path' (*Isis*, **59**, 1968: 5–25): 24.)

(11) We omit several sentences digressing to consider the relative resistance of 'air' (the terrestrial atmosphere), quicksilver (mercury) and water.

(12) The first unimpeachable reference by Newton to a reasonably successful testing of the moon's orbit as traversible—solar and other deviations apart—in an inverse-square terrestrial force-field. Notice that he is still unwilling to identify the lunar *vis centripeta* with terrestrial *gravitas* (which is but one species of centripetal force) but merely states that their deviating effects are 'very nearly' the same. It is well known that Henry Pemberton (in the preface to his *A View of Sir Isaac Newton's Philosophy* (London, 1728): [a1r/a1v]), Abraham de Moivre (in a private 'Memorandum' he gave to John Conduitt in November 1727 [compare ULC. Add. 4007: 706r–707r]) and William Whiston (in his *Memoirs* (London, 1749): 36–8) agree in asserting that Newton had much earlier—Pemberton says 'when he retired from Cambridge in 1666', but this is considerably suspect—had such an 'Inclination...to try, whether the same Power did not keep the Moon in her Orbit, notwithstanding her projectile Velocity,... which makes Stones and all heavy Bodies with us fall downward, and which we call Gravity', but that when he tested this 'Postulatum' he was 'in some Degree' disappointed to find that 'the Power that restrained the Moon in her Orbit, measured by the versed Sines of that Orbit, appeared not to be quite the same that was to be expected, had it been the Power of Gravity alone, by which the Moon was there influenc'd. Upon this Disappointment, which made [him] suspect that this Power was partly that of Gravity, and partly that of *Cartesius's* Vortices,

mine montis prælalti quàm in valle liquet etiam gravitatem ex aucta nostra a terræ centro distantia diminui, sed qua proportione nondum observatum est.

Cæterum projectilium motus in aere nostro referendi sunt ad immensum et revera immobile cœlorum spatium, non ad spatium mobile quod una cum terra et aere nostro convolvitur, et a rusticis ut immobile spectatur. Invenienda est Ellipsis quam projectile describit in spatio illo verè immobili et inde motus ejus in spatio mobili determinandus. Hoc pacto colligitur grave, quod de ædeficij sublimis vertice demittitur, inter cadendum deflectere aliquantulum a perpendiculo, ut et quanta sit illa deflexio et quam in partem. Et vicissim ex deflexione experimentis comprobata colligitur motus terræ. Cum ipse olim hanc deflexionem Clarissimo Hookio significarem, is experimento ter facto rem ita se habere confirmavit, deflectente semper gravi a perpendiculo versus orientem et austrum ut in latitudine nostra boreali oportuit.(13)

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he threw aside the Paper of his Calculation, and went to other Studies' (Whiston, *Memoirs*: 36–7). (Newton's early belief in the physical existence of solar, lunar and terrestrial Cartesian vortices is otherwise documented; see D. T. Whiteside, 'Before the *Principia*: the Maturing of Newton's Thoughts on Dynamical Astronomy, 1664–1684' (*Journal of the History of Astronomy*, 1, 1970: 5–19): 11–12.) No such 'old imperfect Calculation' can now be traced, but its present successful reworking in late 1684 was doubtless along the lines of that afterwards set down by him in Proposition IV, 'Lunam gravitare in terram, & vi gravitatis retrahi semper à motu rectilineo, & in orbe suo retineri', of Book 3 of his published *Principia* ($_1$1687: 406–7), founded on the Copernican estimate of the semi-diameter of lunar orbit being approximately sixty times the Earth's radius.

(13) On the letters relating to this topic (printed in *The Correspondence of Isaac Newton*, 2, 1960: 297–313) which passed between Hooke and Newton during the six weeks from late November 1679 see A. Koyré, 'An Unpublished Letter of Robert Hooke to Isaac Newton' (*Isis*, 43, 1952: 312–37, reprinted with slight emendations in his *Newtonian Essays* (London, 1965): 221–60) and especially J. A. Lohne, 'Hooke *versus* Newton. An Analysis of the Documents in the Case on Free Fall and Planetary Motion' (*Centaurus*, 7, 1960: 6–52). Newton had originally communicated his 'fansy' that a falling body 'outrunning y^e parts of y^e earth will shoot forward to y^e east side of the perpendicular describing in it's fall a spiral line' (*Correspondence*, 2: 301) on 28 November, but, after Hooke pointed out in reply on 9 December that (because its plane of motion must pass through the Earth's centre) 'the fall...will not be exactly east of the perpendicular but South East and indeed more to the south then the east' (*ibid.*: 306) he came quickly to agree four days later that 'y^e body in o^r latitude will fall more to y^e south then east if y^e height it falls from be any thing great' (*ibid.*: 307). Hooke reported on 6 January 1679/80 that he had by then 'made three tryalls of the Experiment of the falling body in Every of which the Ball fell towards the south east of the perpendicular...the Least being about a quarter of an inch' (*ibid.*: 310), adding on 17 January that 'by two tryalls since made in two several places w^{th}in doors it succeeded also. Soe that I am now perswaded the Experiment is very certaine' (*ibid.*: 312–13). According to Lohne's computations ('Hooke *versus* Newton': 31–3), however, the deflection to the east which Hooke measured is several tens of times too much, while he queries whether—since an equal Coriolis force acts on the plumb-line which fixes the vertical—any deviation at all to the south could possibly have been observed.

De motu Corporum in medijs resistentibus

Prob. 6. Corporis sola vi...definire. Asymtotis rectangulis...exponitur. Q.E.D. Porrò celeritati...exponitur. Q.E.D.

Prob. 7. Posita uniformi vi...definire. Corpore ascendente...erit *BC*. Resolvatur enim...in ascensu. Q.E.D.

Schol. Beneficio...diagrammate. Ex loco quovis...tangens *rL*. Si proportio ...*DRTBG*.

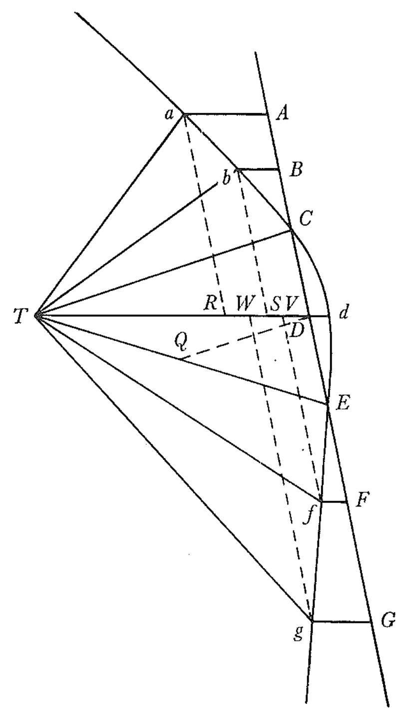
Eadem ratione...moventium in aqua.

APPENDIX 2. COMPUTATION OF AN APPROXIMATION TO THE CURVED PATH OF THE COMET OF 1680–1 BY A MODIFIED RECTILINEAR TECHNIQUE.(1)

[*c.* October 1685?]

From the original worksheet(2) in the University Library, Cambridge

[1] Dec 21. [6^h. 36′. 59″. *a*.]
[Dec] 26. [5^h. 20′. 44″. *b*.]
[Dec] 30. [8^h. 10′. 26″. *C*.]
Jan 5. [6^h. 1′. 38″. *d*.]
[Jan] 13. [7^h. 8′. 55″. *E*.]
[Jan] 30. [8^h. 21′. 53″. *f*.]
Feb 25 [8^h. 41′. 9″. *g*.](3)



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(1) As we have already remarked (see §1: note (79) above) this represents a considerable simplification of the method loosely outlined in the preceding 'De motu Corporum' (opening paragraph of the scholium to Problem 4) for approximately tracing the projection upon the

DTa $63^{\mathrm{gr}}\lfloor 6922$.	DA $359^{\mathrm{h}}\lfloor 410833$.	DTb $40\lfloor 41777$.	DB $240\lfloor 68166$.
DTC $21\lfloor 201888$.	DC $141\lfloor 85333$.	DTE $17\lfloor 17333$.	DE $193\lfloor 12138$.
DTf $34\lfloor 507217$.	DF $602^{\mathrm{h}}\lfloor 3375$.	DTg $47\lfloor 50333$.	DG $1226\lfloor 6586$.

s. Q	$38\lfloor 37472$.	[Ejus Log.]	9. 79295306.		
TD	10000000		10.		
s. TDQ	$21\lfloor 20139$		9. 55828509		
TQ			9. 76533203.		
s. DTQ	$17\lfloor 17333$		9. 47020925		
DQ			9. 67725619.		
$CD)\ CE$(4)			0. 3731724		
$DE)\ CE$(4)			0. 2391815		
TE	13756391.		10. 1385044.		
TC	8249606.		9. 9164377.		
$TC+TD$	18249606.	4.2612534.	$TE+TD$	23756391.	4. 3757804.
$TC-TD$	1750394.	3. 2431358.	$TE-TD$	3756391.	3. 5747707.
[co]tang $\frac{1}{2}CTD$	$79\lfloor 3993$.	10.7277928	cotang $\frac{1}{2}DTE$	$81\lfloor 41333$.	10. 8210296
Tang(5)	$27\lfloor 13414$.	9. 7096752.	Tang(6)	$46\lfloor 32132$.	10. 0200199.

ecliptic of the curving orbit of a comet—here, in exemplification, that of 1680–1. In line with that earlier imposed on Newton's parallel calculations at this period using the unmodified Wrennian rectilinear technique (v: 524–9), we hazard the following date of composition on the basis of his observation to Flamsteed on 19 September 1685 that 'I have not yet computed y$^{\mathrm{e}}$ orbit of a comet but am now going about it...taking that of 1680 into fresh consideration' (*Correspondence of Isaac Newton*, **2**, 1960: 419), and our coupled surmise (see note (3) following) that the timed sightings from which he here works are those, corrected for atmospheric refraction, which were sent by Flamsteed with his reply to Newton's letter a week later.

(2) ULC. Add. 3965.11: 163$^{\mathrm{r}}$. For brevity and clarity we have slightly compressed and trivially reordered Newton's rough calculations.

(3) As in v: 525, note (3) we follow J. A. Ruffner's eminently plausible conjecture that the cometary sightings made on these dates (from which the following differences in cometary longitude, as viewed from the earth, and in corrected 'true' times of observation are straightforwardly computed) were—except for that on 25 February, made by Newton himself at Cambridge—among those listed in a (now lost) 'tablet' included by Flamsteed with his letter to Newton of 26 September 1685, 'in which you will not wonder...to find a difference of some few minutes from y$^{\mathrm{e}}$ former I sent you [on 7 March 1680/1]' (*Correspondence of Isaac Newton*, **2**: 422; compare *ibid.*: 354) and which Newton afterwards published in Book 3 of his *Principia* ($_{1}$1687: 490) along with his own sighting on February 25 at an unadjusted *tempus apparens* of '$8^{\mathrm{h}}.30'$' (*ibid.*: 491).

(4) Understand 'ratio CE ad CD' and 'ratio CD ad DE' respectively. Since Newton's computation of TE (from TQ) and of TC (from DQ) requires him to find only the logarithms of the analogous ratios TE/TQ and TC/QD, he does not bother to list their explicit numerical values alongside.

(5) Read 'Tang $\overline{\tfrac{1}{2}TCD-\tfrac{1}{2}TDC}$', corresponding (in this application of the familiar trigonometrical tangent-rule to resolving the triangle CTD, given its sides TC, TD and their included angle $C\widehat{T}D$) to 'cotang $\frac{1}{2}CTD$' [$=\tan\frac{1}{2}(T\widehat{C}D+T\widehat{D}C)$] in the previous line.

$TCD = 106\lfloor 53344$.
$TDC = 52\lfloor 26516$.

$TED = 35\lfloor 092$
$TDE = 127\lfloor 73465$.
[sive] $TDC = 52\lfloor 26535$ ut supra.(7)

sin DTC	[Ejus Log.]	9. 55828509
sin DCT		9. 98166183
$DC = 141\lfloor 853333$		2. 1518395
$TD = 376^{\mathrm{h}}\lfloor 02464$.		2. 57521624.

s: aTD	$63\lfloor 6922$.	9. 9525145	s: bTD	$40\lfloor 41777$.	9. 8118136
s: TaR	$64\lfloor 04245$.	9. 9538177	s: TbS	$87\lfloor 31688$.	9. 9995236
AD	$359\lfloor 41083$.	2. 5555911	BD	$240\lfloor 68166$.	2. 3814430
$TR=$	$360\lfloor 49041$.	2. 5568937.	$TS=$	$370\lfloor 81138$.	2. 5691530.
$Aa=RD=$	$15^{\mathrm{h}}\lfloor 53423$.	Dec 21.	$Bb[=SD]=$	$5\lfloor 21326$.	Dec. 26.
s: fTD	$34\lfloor 507217$.	9. 7532088	s: gTD	$47\lfloor 50333$.	9. 86765406
s. TfV	$17\lfloor 758133$.	9.4842991	s: TfW	$4\lfloor 76202$.	8. 91916831
DF	$602\lfloor 3375$.	2. 7798399	DG	$1226\lfloor 6586$.	3. 08872366
$TV=$	$324\lfloor 2875$.	2. 5109302.	TW	$138\lfloor 1141$.	2. 14023791.
$Ff[=VD]=$	$51\lfloor 7371$.	Jan 30.	$Gg[=WD]=$	$137\lfloor 91054$.	Feb. 25.(8)

(6) Here, similarly, read

'Tang $\overline{\tfrac{1}{2}TDE-\tfrac{1}{2}TDE}$'

in parallel to 'cotang $\frac{1}{2}DTE$' [$=\tan\frac{1}{2}(T\widehat{D}E+T\widehat{E}D)$] above.

(7) The slight divergence in the last two figures is of course the additive accumulation of small inaccuracies in the seventh decimal places of the logarithmic and trigonometrical tables here employed. It will be clear that, given timed sightings Ta, Tb, TC, Td, TE, Tf, ... of the comet from the earth ('T[erra]', here assumed to be fixed in position), Newton's first step is to determine the direction of motion of the comet at some mean point d: this he approximates as that of the chord CE of the small surrounding (near-parabolic) arc $\widehat{CdE}$, further supposing that (because the comet's speed over this arc is effectively uniform?) the base-line Td will intersect CE at D such that $CD: DE = \mathrm{time}_{C\to D}: \mathrm{time}_{D\to E}$. The construction of the angle $T\widehat{D}C$ at which the line AG, parallel to the tangent at d, is inclined to TD (assigned a conventional length of 10^7 units) is then accomplished by (geometrical) Problem 16 of Newton's contemporary Lucasian lectures on algebra (see v: 210–12): namely, on drawing DQ parallel to CT, the ratios

$$TD: TQ: QD\,(=\sin T\widehat{Q}D: \sin T\widehat{D}Q: \sin Q\widehat{T}D),\quad TE: TQ\,(=CE: CD)$$

and

$$TC: QD\,\,(=CE: DE)$$

are given, and hence the figure $TCDE$ is given in species and its elements straightforwardly calculable. As a gloss, Newton in immediate sequel converts the length of TD to the horary units in which CD and DE are expressed.

(8) Making the final blanket assumption (which we have sought tentatively to justify in §1: note (79) above) that the oblique distances $aR\,(=AD)$, $bS\,(=BD)$, $fV\,(=FD)$, $gW\,(=GD)$ from TDd in the direction of $ACEG$—taken (see note (7)) to be that of the tangent at d, or so

[2] Jan 25. [7^h. 58′. 42″. *f*.]
Feb 5. [7^h. 4′. 41″. *g*.][(3)]

DTf 30⌊77722.[(9)] *DF* 481⌊95111. *DTg* 38⌊17833. *DG* 745⌊050833.

s.*fTD*	30⌊77722.	9. 7090164	s:*gTD*	38⌊168333.	9. 7909708
s.*TfV*	21⌊48813.	9. 5638454	s.*TfW*	14⌊09702.	9. 3861968
DF	481⌊95111.	2. 6830029	*DG*	745⌊050833.	2. 8721859
TV=	345⌊010238.	2. 5378319.	*TW*	293⌊3675.	2. 4674119.
F*f*=	31⌊01440.	Jan 25.	*Gg*=	82⌊65714.	Feb 5.

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[3] Dec 29. [8^h. 3′. 2″. *b*.]
Jan 9. [7^h. 0′. 53″. *f*.][(3)]

[*DTb* 25⌊62361. *DB* 165⌊97666.] *DTf* 9, 90222. *DF* 96⌊9875.

	Dec 29.			Jan 9.	
s:*bTD*	25⌊62361.	9. 6359432	s:*fTD*	9⌊90222.	9. 2354458
s:*TbG*	102⌊11104.	9.9902257	s:*TfV*	42⌊36313.	9. 8285484
DB	165⌊97666.	2. 2200469	*DF*	96⌊9875.	1. 9867157
ST	375⌊25756.	2. 5743294.	*TV*	380⌊030434.	2. 5798183.
Bb	0⌊76708.		[*Ff*]	4⌊005794.[(10)]	

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[4] Jan 10. [6^h. 6′. 10″. *f*.][(3)]

DTf	11,863055.	*DF*	120⌊07555.
s:*fTD*	11⌊863055.		9. 3129664
s*TfV*	40⌊40230.		9. 8116759

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we surmise—are proportional to the corresponding differences in time between the sighting *Td* and those along *Ta*, *Tb*, *Tf*, *Tg* and hence given in ratio to *CD*, *DE* and so to *TD*, Newton by single applications of the sine-rule straightforwardly computes the ratios of the subtenses *TR*, *TS*, *TV*, *TW* and therefrom those of the cometary 'deviations' *aA* (= *RD*), *bB* (= *SD*), *fF* (= *VD*) and *gG* (= *WD*) to *TD* which fix the 'orbital' points *a*, *b*, *f*, *g* in position. Though the 'vertex' *d* is manifestly not constructable in a like manner, in his manuscript figure (here accurately reproduced) Newton roughly locates it by eye at the intersection of a 'smooth' parabolic curve drawn freehand through *a*, *b*, *C*, *E*, *f* and *g*.

(9) This should be (30° 6′ 38″ =) '30^{gr}⌊11055'. The effect of the correction will be slightly to decrease Newton's ensuing value for *Ff*.

(10) Strictly '−4⌊005794' since *TV* is greater than *TD*; whence (as it should) the point *f* will lie to the right of *ADG* in the arc $\widehat{dE}$. In this position *f* the 'comet' will, in Newton's scheme of its orbit, attain its rightmost point *d* very nearly. The final computation following reveals that a day later the 'comet' is only a little more than $2\frac{1}{2}$ horary units to the right of *ADG*.

DF	120⌊07555.	2. 0794545
[*TV*]	378⌊611755.	2. 5781940.
[*Ff*]	2⌊587115.[(11)]	

APPENDIX 3. THE BALLISTIC CURVE (RESISTANCE PROPORTIONAL TO VELOCITY) REWORKED.[(1)]

[spring 1685/*c.* mid-1692]

Extracts from the first and second editions of Newton's *Principia*

[1][(2)] PROP. IV. PROB. II.

Posito quod vis gravitatis in Medio aliquo similari uniformis sit, ac tendat perpendiculariter ad planum Horizontis; definire motum Projectilis, in eodem resistentiam velocitati proportionalem patientis.

E loco quovis *D* egrediatur Projectile secundum lineam quamvis rectam *DP*, & per longitudinem *DP* exponatur ejusdem velocitas sub initio motus. A puncto *P* ad lineam Horizontalem *DC* demittatur perpendiculum *PC*, & secetur *DC* in

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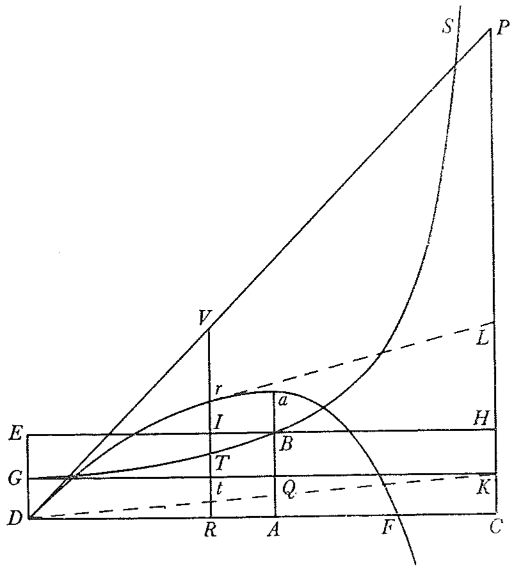
(11) With *TV* again greater than *TD*, this should read '−2⌊587115', whence *f* is (as it should be, once more) in $\widehat{dE}$ to the right of *ADG*. Newton's computations terminate abruptly at this point, and we have no reason to think that he ever again was tempted to apply this badly deficient method to the computation of any real cometary orbit. The would-be 'simplifying' assumption (compare §1: note (79)) that the earth be supposed to be at rest in the immediate vicinity of the sun is, even over a very short interval of time, here a crucial defect. As the rapidly varying cometary longitudes at this period indicate (see Flamsteed's tabulation in *Principia*, $_1$1687: 490), the earth in late December 1680/January 1681 was moving almost directly away from the very nearly rectilinear path of the 1680–1 comet, travelling at a little more than half its mean speed. Newton's present calculation yields a 'cometary' path which, in wide divergence from physical reality, closely approximates a parabola with its vertex near to the point *d*.

(1) The classically composed demonstration here reproduced (in [1]) from Book 2 of Newton's published *Principia* ($_1$1687) essentially mirrors our analytical justification (§1: note (113)) of his unproved equivalent construction of the present projectile orbit in his preceding 'De motu Corporum', and may straightforwardly be recast in its terms. To it we append from the *Principia's* second edition ($_2$1713) two opening corollaries which reduce the geometrical definition of this *logarithmica* to standard form. In [3] we reproduce the five corollaries originally added in 1687 (but here renumbered 3–7 as in the second edition) which minimally elaborate the basic construction and, in the case of the last, somewhat forlornly attempt to determine the ballistic orbit empirically by points 'ex Phænomenis quamproximè'.

(2) *Philosophiæ Naturalis Principia Mathematica* (London, $_1$1687): 241–2. The main emendations made in revise in 1713 are noticed in following footnotes.

A ut sit *DA* ad *AC* ut resistentia Medii ex motu in altitudinem sub initio orta,(3) ad vim gravitatis; vel (quod perinde est) ut sit rectangulum sub *DA* & *DP* ad rectangulum sub *AC* & *CP* ut resistentia tota sub initio motus ad vim Gravitatis. [Asymptotis *DC*, *CP*](4) [d]escribatur Hyperbola quævis *GTBS* secans erecta perpendicula *DG*, *AB* in *G* & *B*; & compleatur parallelogrammum *DGKC*, cujus latus *GK* secet *AB* in *Q*. Capiatur linea *N* in ratione ad *QB* qua *DC* sit ad *CP*; & ad rectæ *DC* punctum quodvis *R* erecto perpendiculo *RT*, quod Hyperbolæ in *T*, & rectis *GK*, *DP* in *t* & *V*(5) occurrat; in eo cape *Vr* æqualem $\frac{tGT}{N}$,(6) & Projectile tempore *DRTG* perveniet ad punctum *r*, describens curvam lineam *DraF*, quam punctum *r* semper tangit; perveniens autem ad maximam altitudinem *a* in perpendiculo *AB*, & postea semper appropinquans ad Asymptoton PLC. Estq; velocitas ejus in puncto quovis *r* ut Curvæ Tangens *rL*. Q.E.[I].



Est enim *N* ad *QB* ut *DC* ad *CP* seu *DR* ad *RV*, adeoq; *RV* æqualis $\frac{DR \times QB}{N}$, & $Rr\left(\text{id est } RV - Vr \text{ seu } \frac{DR \times QB - tGT}{N}\right)$ æqualis $\frac{DR \times AB - RDGT}{N}$. Exponatur jam tempus per aream *RDGT*, & (per Legum Corol. 2)(7) distinguatur motus

(3) That is, the vertical component of the initial 'resistentia medij' (V_y in the notation of §1: notes (108) and (113)). Observe that Newton now implicitly corrects his earlier slip (see §1: note (110)) in assigning the ratio of $V = (DP/CP).V_y$ to g.

(4) This very necessary phrase was later inserted by Newton himself in the second edition.

(5) In preparation for his addition in the next line, Newton in his 1713 edition expanded this to read '...rectis *EH*, *GK*, *DP* in *I*, *t* & *V*'. The intersection of *EH* and *RV* in the accompanying figure was correspondingly marked '*I*' as shown.

(6) In 1713 (compare previous note) Newton added in sequel the minimal clarification 'vel quod perinde est, cape *Rr* æqualem $\frac{GTIE}{N}$'.

(7) See §2: note (20) following. This corollary formally (in a Cartesian coordinate system, as here, where the acceleration in the tangential direction is zero) justifies Newton's splitting of the orbital acceleration $dv/dt = -v-g.dy/ds$ into horizontal and vertical components (constant in direction), but his further assumption in sequel that these are $(dv_x/dt =) -v_x$ and $(dv_y/dt =) -v_y - g$ respectively is not as immediately obvious, perhaps, as he would have it (compare §1: note (109)).

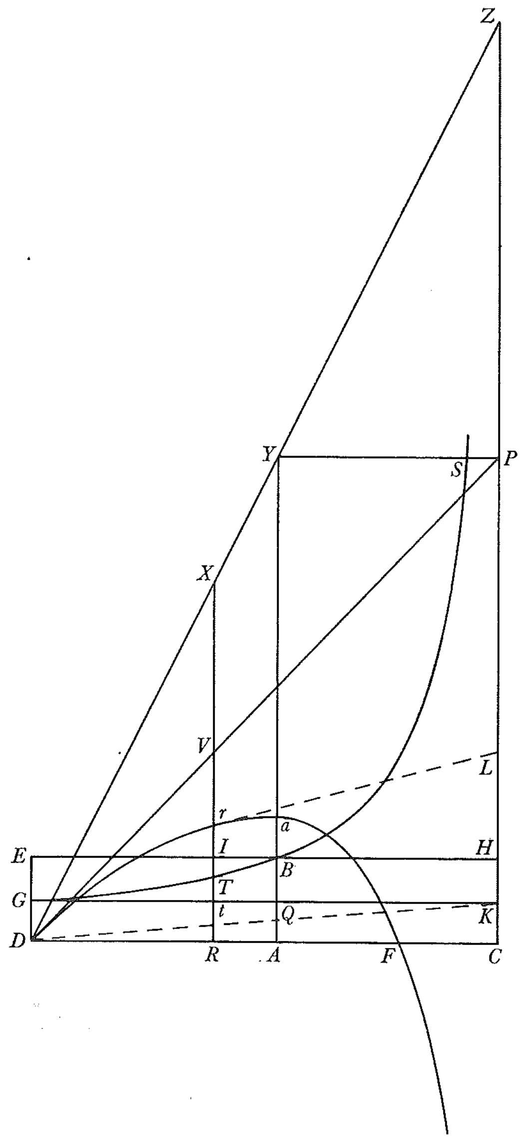
corporis in duos, unum ascensus, alterum ad latus. Et cum resistentia sit ut motus,(8) distinguetur etiam hæc in partes duas partibus motus proportionales & contrarias: ideoq; longitudo a motu ad latus descripta erit (per Prop. II. hujus)(9) ut linea *DR*, altitudo vero (per Prop. III. hujus)(9) ut area

$$DR \times AB - RDGT,$$

hoc est ut linea *Rr*. Ipso autem motus initio area *RDGT* æqualis est rectangulo $DR \times AQ$, ideoq; linea illa *Rr*

$$\left(\text{seu } \frac{DR \times AB - DR \times AQ}{N}\right)$$

tunc est ad *DR* ut $AB - AQ$ (seu *QB*) ad *N*, id est ut *CP* ad *DC*; atq; adeo ut motus in altitudinem ad motum in longitudinem sub initio. Cum igitur *Rr* semper sit ut altitudo, ac *DR* semper ut longitudo, atq; *Rr* ad *DR* sub initio ut altitudo ad longitudinem: necesse est ut *Rr* semper sit ad *DR* ut altitudo ad longitudinem, & propterea ut corpus moveatur in linea *DraF*, quam punctum *r* perpetuo tangit. Q.E.D.



[2](10) *Corol. 1.* Est igitur *Rr* æqualis $\frac{DR \times AB}{N} - \frac{RDGT}{N}$, ideoque si producatur *RT* ad *X* ut sit *RX* æqualis $\frac{DR \times AB}{N}$, (id est, si compleatur parallelogrammum *ACPY*, jungatur *DY*

(8) In his working, of course, Newton suitably absorbs the constant factor of proportionality, thereby (for simplicity) equating the instantaneous resistance upon the body to its *motus* (orbital speed).

(9) These are essentially identical with Problems 6 and 7 respectively of the preceding 'De motu Corporum'; for their analytical equivalents see §1: notes (98) and (108).

(10) These two opening corollaries are additions in the revised edition of the *Principia* ($_2$1713: 217).

secans CP in Z, & producatur RT donec occurrat DY in X;) erit Xr æqualis $\frac{RDGT}{N}$, & propterea tempori proportionalis.(11)

Corol. 2. Unde si capiantur innumeræ CR vel, quod perinde est, innumeræ ZX, in progressione Geometrica; erunt totidem Xr in progressione Arithmetica.(12) Et hinc Curva *DraF* per tabulam Logarithmorum facile delineatur.

[3](13) *Corol.* [3]. Hinc si Vertice D, Diametro DE deorsum producta, & latere recto quod sit ad $2DP$ ut resistentia tota, ipso motus initio, ad vim gravitatis, Parabola construatur: velocitas quacum corpus exire debet de loco D secundum rectam DP, ut in Medio uniformi resistente describat Curvam *DraF*, ea ipsa erit quacum exire debet de eodem loco D, secundum eandem rectam $D[P]$, ut in spatio non resistente describat Parabolam.(14) Nam Latus rectum Parabolæ hujus, ipso motus initio, est $\frac{DV^{\text{quad.}}}{Vr}$ & Vr est $\frac{tGT}{N}$ seu $\frac{DR \times Tt}{2N}$. Recta autem quæ, si duceretur, Hyperbolam GTB tangeret in G, parallela est ipsi DK,(15) ideoq; Tt est $\frac{CK \times DR}{DC}$, & N erat $\frac{QB \times DC}{CP}$. Et propterea Vr est $\frac{DR^q \times CK \times CP}{2CD^q \times Q[B]}$, id est (ob proportionales DR & DC, DV & DP) $\frac{DV^q \times CK \times CP}{2DP^q \times QB}$. & Latus rectum

(11) In the terms of §1: note (113), since $N = AB \times AC/g$, therefore

$$rX = (RDGT)/N = g.\log(DC/RC) = gt \propto t.$$

(12) For in analytical equivalent (see §1: note (113)) the defining 'symptom' of the *logarithmica DraF* is, in standard form, $Xr = -g.\log(ZX/ZD)$.

(13) These five final corollaries (here renumbered as in the second edition) are lightly corrected reproductions of the equivalent Corollaries 1–5 in *Principia* ($_1$1687): 242–5.

(14) When, in the equivalent analytical terms of §1: note (113), the resistance is zero, the component equations of motion of the orbiting body $r(x, y)$ defined by $DR = x$, $Rr = y$ are $dv_x/dt = 0$ and $dv_y/dt = -g$, yielding respectively $v_x - V_x = 0$ and $v_y - V_y = -gt$, whence $x = V_x t$ and so $y = V_y t - \frac{1}{2}gt^2 = (V_y/V_x)x - \frac{1}{2}(g/V_x^2)x^2$. On setting

$$DV = (V/V_x).x = X \quad \text{and} \quad Vr = (V_y/V_x)x - y = Y,$$

the Cartesian defining equation of $r(X, Y)$ proves to be $Y = \frac{1}{2}(g/V^2)\,X^2$, a 'Galileian' parabola of diameter $Aa\,(X = VV_y/g)$ parallel to DE $(X = 0)$ and of *latus rectum* $2V^2/g = 2DP.(V/g)$. Newton's following derivation of this last result ingeniously proceeds from the assumption that the resisted *logarithmica* of the main proposition coincides with this parabola in the immediate vicinity of the firing point D (before the resistance has had opportunity to decelerate the projectile's orbital speed), but has somewhat unsatisfactorily to assume the verticality of the parabola's diameter as its initial step.

(15) For, since $CD \times DG = CA \times AB$, constant, at once $-d(DG)/d(CD) = DG$ (or CK)/CD. Newton assumes, of course, that in the limit as r comes to coincide with D the vanishingly small hyperbolic segment (GTt) approaches a right triangle whose hypotenuse is tangent to the arc $\overset{\frown}{GBS}$ at G.

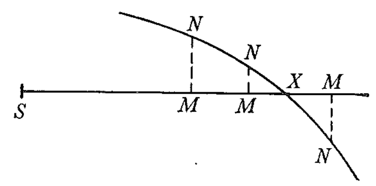
$\frac{DV^{\text{quad.}}}{Vr}$ prodit $\frac{2DP^q \times QB}{CK \times CP}$, id est (ob proportionales QB & CK, DA & AC) $\frac{2DP^q \times DA}{AC \times CP}$, adeoq; ad $2DP$ ut $DP \times PA$ ad $PC \times AC$; hoc est ut resistentia ad gravitatem. Q.E.D.

Corol. [4]. Unde si corpus de loco quovis D, data cum velocitate, secundum rectam quamvis positione datam DP projiciatur, & resistentia Medii ipso motus initio detur, inveniri potest Curva *DraF*, quam corpus idem describet. Nam ex data velocitate datur latus rectum Parabolæ, ut notum est. Et sumendo $2DP$ ad latus illud rectum ut est vis Gravitatis ad vim resistentiæ, datur DP. Dein secando DC in A, ut sit $CP \times AC$ ad $DP \times DA$ in eadem illa ratione Gravitatis ad resistentiam, dabitur punctum A. Et inde datur Curva *DraF*.

Corol. [5]. Et contra, si datur curva *DraF*, dabitur & velocitas corporis & resistentia Medii in locis singulis r. Nam ex data ratione $CP \times AC$ ad $DP \times DA$, datur tum resistentia Medii sub initio motus, tum latus rectum Parabolæ: & inde datur etiam velocitas sub initio motus. Deinde ex longitudine tangentis rL, datur & huic proportionalis velocitas, & velocitati proportionalis resistentia in loco quovis r.

Corol. [6]. Cum autem longitudo $2DP$ sit ad latus rectum Parabolæ ut gravitas ad resistentiam in D; & ex aucta Velocitate augeatur resistentia in eadem ratione, at latus rectum Parabolæ augeatur in ratione illa duplicata: patet longitudinem $2DP$ augeri in ratione illa simplici, adeoq; velocitati semper proportionalem esse, neq; ex angulo CDP mutato augeri vel minui, nisi mutetur quoq; velocitas.

Corol. [7]. Unde liquet methodus determinandi Curvam *DraF* ex Phænom[e]nis quamproxime, & inde colligendi resistentiam & velocitatem quacum corpus projicitur. Projiciantur corpora duo similia & æqualia eadem cum velocitate, de loco D, secundum angulos diversos CDP, $[C]Dp$ (minuscul[æ] liter[æ] loc[o] subintellect[o]) & cognoscantur loca F, f ubi incidunt in horizontale planum DC. Tum assumpta quacunq; longitudine pro DP vel Dp, fingatur quod resistentia in D sit ad gravitatem in ratione qualibet, & exponatur ratio illa per longitudinem quamvis SM. Deinde per computationem, ex longitudine illa assumpta DP, inveniantur longitudines DF, Df, ac de ratione $\frac{Ff}{DF}$ per calculum inventa, auferatur ratio eadem per experimentum inventa, & exponatur differentia per perpendiculum MN. Idem fac iterum ac tertio, assumendo semper novam resistentiæ ad gravitatem rationem SM, & colligendo novam differentiam MN. Ducantur autem differentiæ affirmativæ ad unam partem rectæ SM, & negativæ ad alteram; & per puncta



N, *N*, *N* agatur curva regularis *NNN* secans rectam *SMMM* in *X*,(16) & erit *SX* vera ratio resistentiæ ad gravitatem, quam invenire oportuit. Ex hac ratione colligenda est longitudo *DF* per calculum; & longitudo quæ sit ad assumptam longitudinem *DP* ut modo inventa longitudo *DF* ad longitudinem eandem per experimentum cognitam, erit vera longitudo *DP*. Qua inventa, habetur tum Curva Linea *DraF* quam corpus describit, tum corporis velocitas & resistentia in locis singulis.(17)

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(16) A familiar Newtonian technique of attaining a rough approximation; compare IV: 560, note (113).

(17) As Newton well knew, the hypothesis that (in the earth's atmosphere) a projectile is instantaneously decelerated by resistance in proportion to its speed is exceedingly unrealistic, even for the low muzzle velocities obtaining in his day. It is difficult to believe that in outlining this present elaborate method for determining the elements of his ensuing *logarithmica DraF* 'ex Phænomenis quamproximè' he did not have his tongue stuck firmly—if still a little hopefully?—in this cheek. More than eighteen years before the *Principia* was published Christiaan Huygens had modestly declined to foist so mathematically elegant but physically useless a 'Theorie' upon the learned world, preferring to devote his effort to exploring the experimentally truer supposition that 'la resist[a]nce de l'air, & de l'eau, estoit comme les quarrez des vitesses', accurately divining that in 'ce veritable fondement des Resistances...la chose estoit beaucoup plus difficile, & sur tout en ce qui regarde la ligne courbe que parcourent les corps jettez obliquement' (*Discours de la Cause de la Pesanteur* (Leyden, 1690): 169; compare §1: note (113) above). Much like Huygens before him in 1669 (see his *Œuvres complètes*, **19**, 1937: 144–57; and compare A. R. Hall, *Ballistics in the Seventeenth Century* (Cambridge, 1952): 111–17, especially 116), Newton was able in the following Propositions V–IX of his *Principia* ($_1$1687: 246–60) correctly to resolve the problem of motion under a decelerating force varying instantaneously as the square of the speed in cases where the motion is confined to be in a straight line: namely, if [Propositions V–VII] $d^2x/dt^2 = dv_x/dt = -v_x^2$, then, on assuming the initial conditions $x = t = 0$ and $v_x = V_x$, from $dt/dv_x = -1/v_x^2$ there follows $t = 1/v_x - 1/V_x$, whence $dx/dt = v_x = 1/(t+1/V_x)$ and so $x = \log(V_x t+1)$; while if more generally [Propositions VIII/IX] $d^2y/dt^2 = dv_y/dt = -v_y^2 \pm g$, g constant, then with similar initial conditions $y = t = 0$ and $v_y = V_y$, from $dt/dv_y = 2dy/d(v_y^2) = -(v_y^2 \mp g)$ there ensues

$$t = \log((V_y+g^{\frac{1}{2}})/(V_y-g^{\frac{1}{2}})) - \log((v_y+g^{\frac{1}{2}})/(v_y-g^{\frac{1}{2}}))$$

or alternatively $t = \tan^{-1}(V_y/g^{\frac{1}{2}}) - \tan^{-1}(v_y/g^{\frac{1}{2}})$ according as $-g$ or $+g$ is taken, while in either case $y = \frac{1}{2}\log((V_y^2 \mp g)/(v_y^2 \mp g))$. His following Propositions XI–XIV (*Principia*, $_1$1687: 274–84) comparably—but to no real purpose other than to reveal his mastery of the geometrical quadratures (in terms of logarithmic/inverse-tangent functions) involved—depart from the compounded equations of motion

$$dv_x/dt = -v_x^2 - 2kv_x \quad (\text{or } dv_x/dx = -v_x - 2k) \quad \text{and} \quad dv_y/dt = v_y . dv_y/dy = -v_y^2 - 2kv_y \pm g.$$

Wise, however, to the fallacy of seeking vectorially to combine these resultant 'component' motions—unlike Leibniz when he four years later vainly proffered this 'solution' (see §1: note (109))—Newton was unable to approach the problem of defining the general orbit traversed, under constant 'gravity' directed vertically downwards, in a medium resisting (in the instantaneous direction of motion) as the square of the orbital speed other than by implicitly sub-

suming it, as a particular case not there further explored but which (he told David Gregory in May 1694) was in his 'power', in the intervening Proposition X (*Principia*, $_1$1687: 260–9) where an arbitrary resistance to motion is assumed and then related to the gravity by the equivalent of a third-order differential equation whose solution—here readily effectable parametrically—resolves the problem. (Compare D. T. Whiteside, 'The Mathematical Principles underlying Newton's *Principia Mathematica*' (§1: note (30)): 126–30 and 137, note 5.)
