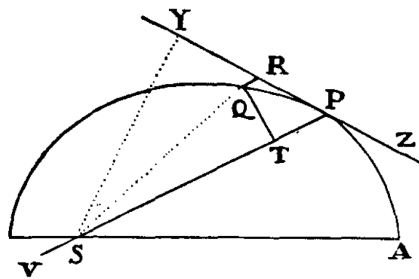


# Module 1

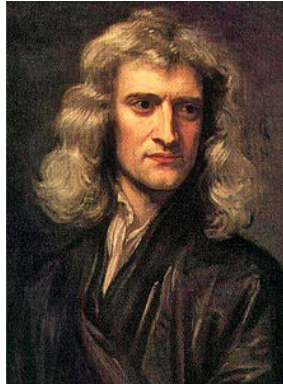
## Newton's mathematical force



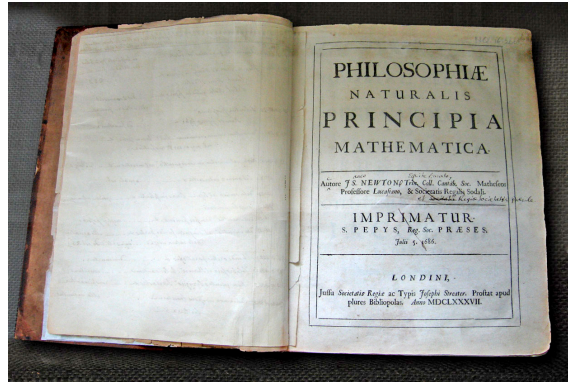
UNIVERSITY OF COPENHAGEN



# Philosophiæ Naturalis *Principia* Mathematica (1687)



## Newton (1687)



Dec 14, 2016



## Isaac Newton masterwork becomes most expensive science book sold

First edition of Principia Mathematica, which was published in 1687 and sets out Newton's laws of motion, raises £3m at auction

"The *Principia* is perhaps the **greatest intellectual stride** that it has ever been granted to any man to make" (Einstein)

"The *Principia* marked the epoch of a great **revolution in physics**. The method followed by its illustrious author Sir Newton ... spread the **light of mathematics on a science** which up to then had remained in the **darkness of conjectures and hypotheses**" (Clairaut)

"The *Principia* is one of the **most influential** works in Western culture, but it is a work **more revered than read**" (Brackenridge)

# Mathematics and Reality?

## Divergence

- Geometry is geometry only through the **abstract simplicity of its object**. Only that makes it certain and demonstrative. The **object of physics is much vaster**. That is what makes it difficult, uncertain and obscure. But this is essential to it: **one is not a better physicist because one is the best of geometers** (Castel, 1743)

## Convergence

- It is not sufficient for a system to satisfy the phenomena only in a vague and general manner, or to **provide plausible explanations of some of them**: the details and the **precise calculations are the touchstone**; only they can tell if one must adopt, reject, or modify an hypothesis (D'Alembert, 1749)

What did mathematics do to physics? (Gingras, 2001)

# Motivation to write the *Principia*

January 1684



Hooke



Wren



Halley

How to derive the laws of planetary motion?

Hooke claims to have derived that an inverse square law leads to an ellipse, but shows no evidence.

August 1684

Months passed and Hooke had yet to produce his evidence. Edmund Halley traveled to Cambridge to find out what Isaac Newton had to say on the matter.

When Halley put the question to Newton, Newton surprised him by saying that he had already made the derivations some time ago; but that he could not find the papers...

November 1684

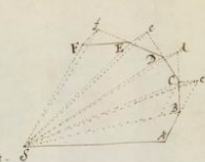
Newton sent Halley a nine-page manuscript titled *De Motu Corporum in Gyrum* (On the Motion of Orbiting Bodies).

Halley is so fascinated by its content and method that he demands Newton to send more of his work to the Royal Society – which leads to the *Principia* (1687)



# De Motu Corporum in Gyrum

Describimus hanc Be aequali ipsi AB  
Dico ut radii AB, BE, et ad centrum  
actis constructi forent aequalis area  
ASB, BSE. Verum ubi corpus venit  
ad B agat vis contripeta impulsu  
unico sed magno, faciatque corpus  
a recta Be deflectere et pergere  
in recta BC. Ipsi BE parallela aga-



hur CE occurrat BC in C et completa secunda temporis parte  
corpus reperitur in C. Iungo SC et triangulum SBC ob pa-  
rallelos SB, CE aequale erit triangulo SBE atque duo etiam tri-  
angulo SAB. Simili argumento si vis contripeta successively agat  
in C, D, E etc. faciens corpus singulis temporis momentis singulis  
describere rectas CD, DE, EF etc. triangulum SED triangulo  
SBC et SED ipsi SCB et SET ipsi BE aequale erit. Aequalibus  
igitur temporibus aequalis area describuntur. Sunt jam hae  
triangula numero infinita et infinita parva, sic, ut singulis  
temporis momentis singula respondent triangula, agentis vi  
contripeta sine intermissione, et constabit propositio.

Theorem. 2. Corporibus in circumferentiis circularium  
uniformiter gyranlibus vires contripetas esse ut arcuum simul  
descriptorum quadrata applicata ad radios circularium.

Corpora B, b in circumferentiis

circularium BB, bd gyranlibus simul  
describant arcus BB, bd. Sola vi  
nisi describerent tangentibus BC,  
be his arcibus aequalis. Vires contri-  
petae sunt quae perpendiculae retrahunt  
corpora de tangentibus ad circum-



ferentias, atque dico haec sunt ad initium ut spatia ipsi perstrata  
erunt, id est, si productis CD, ed ad F et f ut  $\frac{CD^2}{CF}$  ad  $\frac{be^2}{cf}$   
sic ut  $\frac{CD^2}{CF}$  ad  $\frac{be^2}{cf}$ . Quare de spatii BB, bd minutissimi  
vires in infinitum diminuuntur sic ut pro  $\frac{1}{2} CF$ ,  $\frac{1}{2} cf$  scribere  
hinc circularium radios SB, sb. Quo facto constat propositio.

Cor 1. Hinc vires contripetae sunt ut velocitatum quadrata  
applicata ad radios circularium.

Cor 2. Et reciproci ut quadrata temporum periodiceorum ap-  
plicata ad radios.

Cor 3. Unde si quadrata temporum periodiceorum sunt ut  
radii circularium vires contripetae sunt aequales, et vis versa

Cor 4. Si quadrata temporum periodiceorum sunt ut qua-  
drata radiorum vires contripetae sunt reciproci ut radii. Et  
vis versa.

Cor 5. Si quadrata temporum periodiceorum sunt ut cubi  
radiorum vires contripetae sunt reciproci ut quadrata radio-  
rum. Et vis versa.

Schol. Caput Corollaris quinti obtinet in corporibus  
celestibus. Quadrata temporum periodiceorum sunt ut cubi  
distansiarum a communi centro circum quod volutur. Id  
obtinere in Planis maioribus circa solem gyranlibus virge  
minoribus circa Iovem et Saturnum jam statuunt Astronomi.

Theorem. 3. Si corpus P circa centrum S gyraudo, descri-

bat hanc quamvis curvam APQ,  
et si tangat recta PR curvam illam  
in puncto quovis P et ad tangentem  
ab alio quovis puncto Q aga-  
hur QR distantia PR parallela  
ac demittatur QT perpendicularis  
ad distantiam SP: Dico quod vis  
contripeta sit reciproci ut solidum  $\frac{SP^2 \times QT^2}{QR}$ , si modo  
solidi illius ea temporis sumatur quantitas quae ultimo fit  
ubi coeunt puncta P et Q.

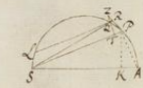


Namque in figura indefinita parva QRPT hincola QR  
dato tempore est ut vis contripeta et data vi ut  $\frac{1}{2}$  quadra-  
tum temporis atque adeo usque dato ut vis contripeta et qua-  
dratum temporis conjunctum, id est ut vis contripeta simul et  
area SPQR temporis proportionalis (vel duplum eius  $SP \times QT$ ) sit.  
Applicatur huius proportionalitatis pars utraque ad distantiam QR et  
fiet unitas ut vis contripeta et  $\frac{SP^2 \times QT^2}{QR}$  conjunctum, hoc est vis  
contripeta reciproci ut  $\frac{SP^2 \times QT^2}{QR}$  Q.E.D.

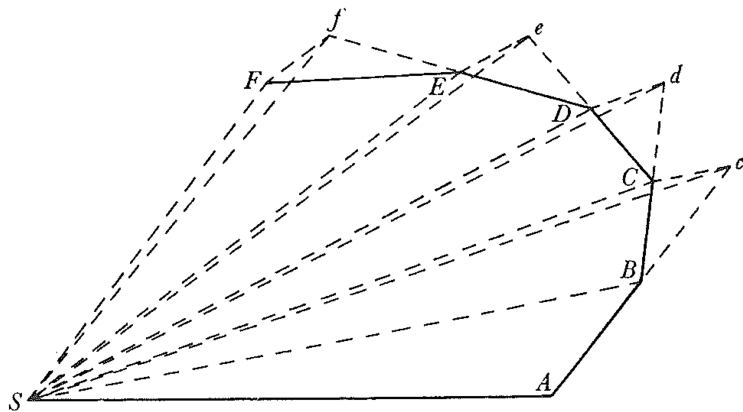
Corol. Hinc si datur figura quavis et in ea punctum ad quod  
vis contripeta dirigatur, inveniri potest haec vis contripeta quae corpus  
in figura illius perimetro gyraudo faciat. Ximiam computandum  
est solidum  $\frac{SP^2 \times QT^2}{QR}$  hanc vi reciproci proportionalis. Eius rei  
dabimus exempla in problematis sequentibus.

Prob. 1. Gyraudo corpus in circumfe-  
rentia circulari, requiritur haec vis contripeta  
tangente ad punctum aliquod in circumferentia.

Esto circuli circumferentia RPA,  
centrum sit contripeta S, corpus in circumfe-  
rentia habet P, locus proximius in quo mo-  
vetur Q. Ad SA diametrum et SP demittit perpendicularis PK, QT

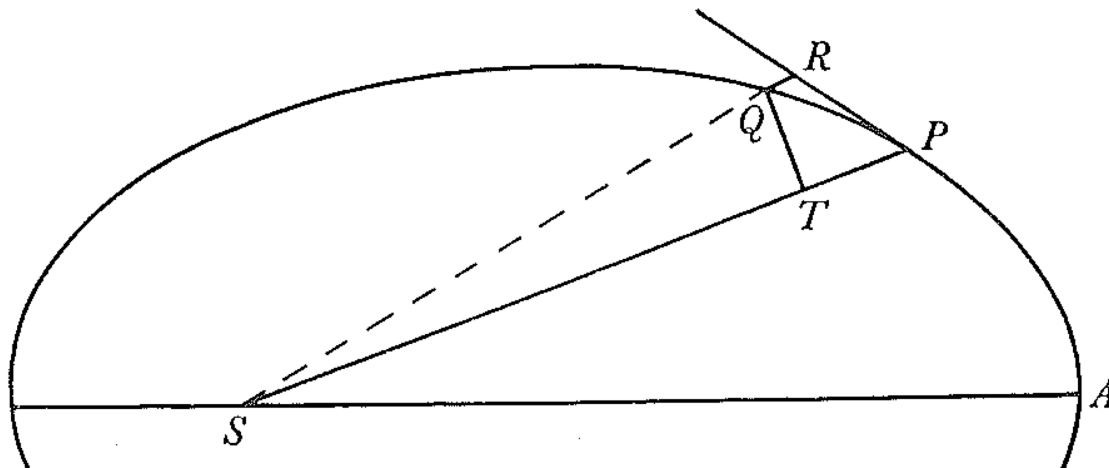


# Theorem 1: Central force → Equal areas



Wikipedia: Newtons proof of Keplers second law.gif

### Theorem 3: Force proportional to $QR/(SP^2 \times QT^2)$



*Corollary.* Hence if any figure be given and in it a point to which the centripetal force is directed, there can be ascertained the law of centripetal force which shall make a body orbit in the perimeter of that figure: specifically, you must compute (the quantity of) the ‘solid’  $SP^2 \times QT^2/QR$  reciprocally proportional to this force. Of this procedure we shall give illustrations in following problems.

# Problem 1: Center in the circumference

$$F \propto QR/(SP^2 \times QT^2)$$

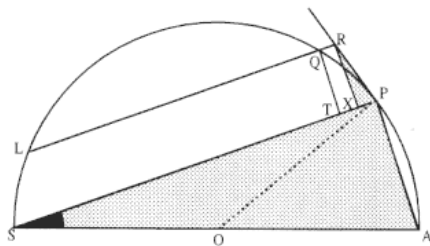


Figure 5.3A  
A revised diagram for Problem 1. The perpendicular  $RX$  and the radius  $OP$  are added.

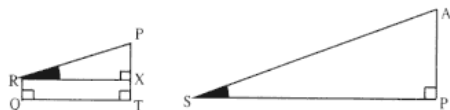


Figure 5.3B  
The triangle  $RPX$  is similar to the triangle  $SAP$ .

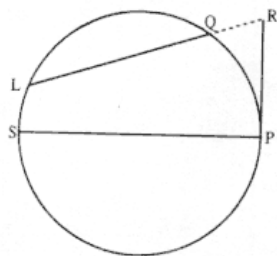
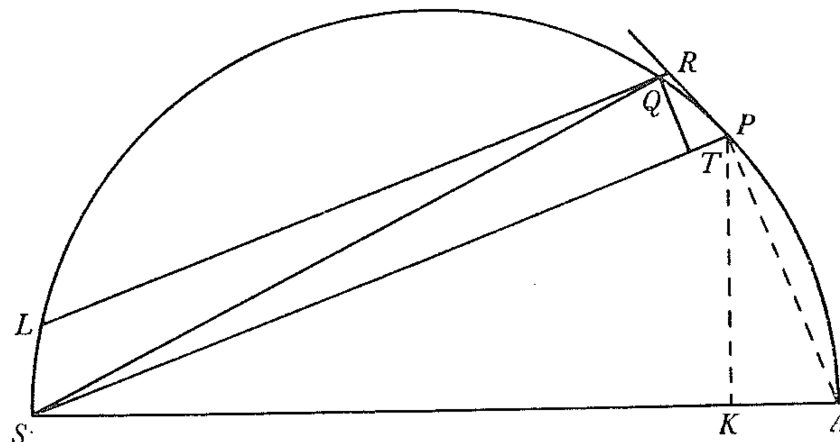


Figure 5.4B  
Thus,  $RL / RP = RP / QR$  or  
 $RP^2 = QR \times RL$  as required in Problem 1



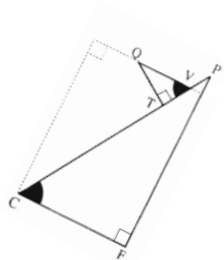
$$1: \triangle SAP \sim \triangle RPX \therefore (SA/SP)^2 = (RP/QT)^2$$

$$2: RP^2 = (QR).(LR)$$

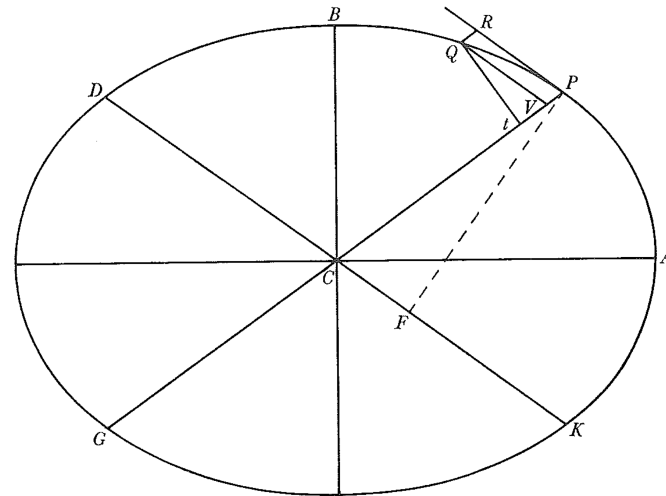
$$3: R \rightarrow P \quad LR \rightarrow SP$$

$$QR/QT^2 = SA^2/SP^3$$

$$F(r) \propto 1/r^5$$

$$F \propto QR/(CP^2 \times QT^2)$$


$$2: (PV \times VG)/QV^2 = PC^2/CD^2$$

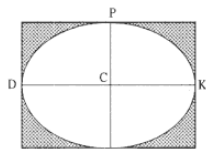


The diameter  $PG$  bisects the chords  $QQ'$  and  $DK$ . From Proposition 15 of Book 1 of Apollonius's *Conics*, the ratio of  $PV \times VG / QV^2$  is equal to the ratio  $PC^2 / DC^2$ .

$$1 \text{ and } 2: QT^2 = (PV \times VG) \cdot (CD^2/PC^2) \cdot (PF/PC)^2$$

$$QR/QT^2 = PC^4/(VG \times CD^2 \times PF^2)$$

3:  $BC \times CA = CD \times PF = \text{const.}$



### PARALLELOGRAM B

The area of parallelogram A is equal to the area of parallelogram B (Proposition 31, Book 7, of the *Conics* of Apollonius of Perga).

4:  $R \rightarrow P$      $VG \rightarrow 2PC$

$$F(r) \propto r$$

# Problem 3: Center in the focus of an ellipse (finally!)

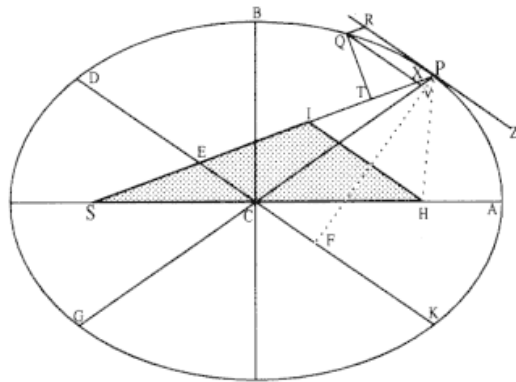
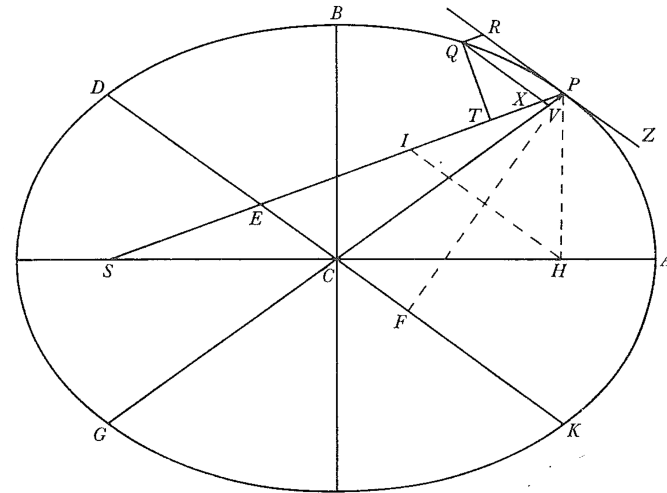
$$F \propto QR/(SP^2 \times QT^2)$$

Find QR



$$1: \Delta PXV \sim \Delta PEC \therefore (PE/PC) = (PX/PV)$$

$$\text{Since } PX = QR \therefore QR = PV (PE/PC)$$



$$2: PE = AC \quad \text{Why? "clearly"}$$

$$\Delta SIH \sim \Delta SEC \quad CS = CH \text{ (foci)}$$

$$SE = EI \quad PE = (PS + PI)/2 = (PS + PH)/2$$

$$3: (PV \times VG)/QV^2 = PC^2/CD^2$$

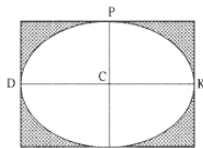
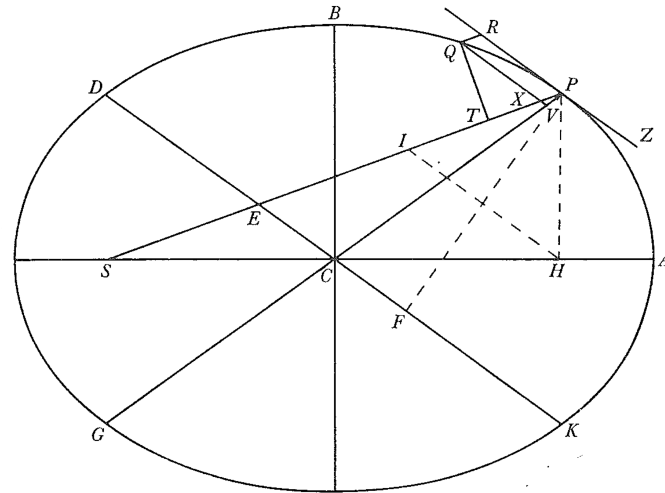
Sub (2) and (3) in (1)

$$QR = (QV^2/VG) \cdot (PC^2/CD^2) \cdot (AC/PC)$$

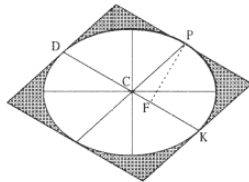
$$F \propto QR/(SP^2 \times QT^2)$$

$$1: \Delta PEF \sim \Delta QTX \therefore QT/QX = PF/PE$$

$$QT^2 = QX^2.(PF^2/PE^2)$$



### PARALLELOGRAM A



### PARALLELOGRAM B

$$2: AC \times BC = DC \times PF$$

Since  $PE = AC$

$$QT^2 = QX^2.(BC^2/DC^2)$$

$$QR/QT^2 = (QV^2/QX^2).(VG/PC)(AC/BC^2)$$

$$R \rightarrow P \quad QV \rightarrow QX \quad VG \rightarrow 2PC$$

From the definition of the constant *latus rectum*  $L = 2BC^2/AC$

$QR/QT^2 = (1/L)$  which is a constant!

$$QR/QT^2 = (QV^2/QX^2).(PC/NG)(2/L)$$

$$F(r) \propto 1/r^2$$

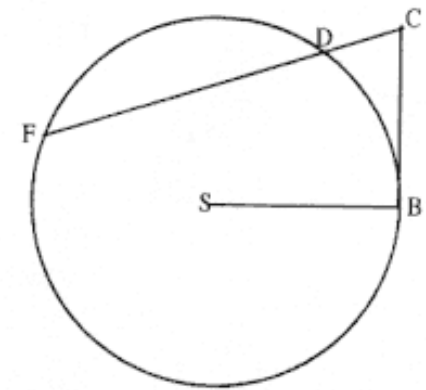
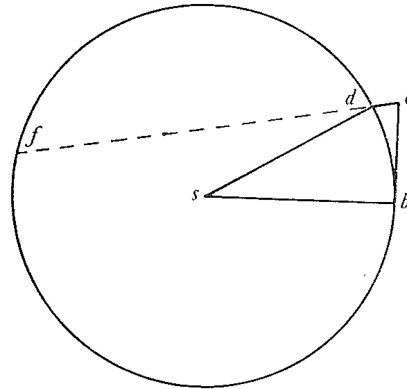
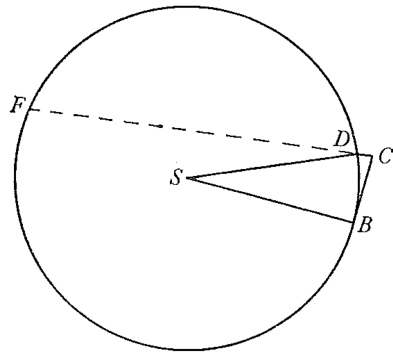
Figure 4.1  
The area of parallelogram A is equal to the area of parallelogram B (Proposition 31, Book 7, of the *Conics* of Apollonius of Perga).



# Questions for discussion

- In theorem 2, I do not understand why it is square? I cannot see why  $BC^2/CF$  makes more sense than  $BC/CF$ . And I cannot see why  $BC^2/CF$  can be written as  $BD^{2/1/2}CF$ .

# Theorem 2: Centripetal force $\propto \text{arc}^2/R$



$$CD / BC = BC / CF$$

C1:  $F_1/F_2 = (v_1^2/r_1)/(v_2^2/r_2)$

C2:  $F_1/F_2 = (T_2^2/r_2)/(T_1^2/r_1)$

C3: If  $T_2^2/r_2 = T_1^2/r_1$  then  $F_1 = F_2$

C4: If  $T_2^2/r_2^2 = T_1^2/r_1^2$  then  $F_1/F_2 = r_2/r_1$

C5: If  $T_2^2/r_2^3 = T_1^2/r_1^3$  then  $F_1/F_2 = r_2^2/r_1^2$

# Questions for discussion

- In theorem 2, I do not understand why it is square? I cannot see why  $BC^2/CF$  makes more sense than  $BC/CF$ . And I cannot see why  $BC^2/CF$  can be written as  $BD^2/1/2CF$ .
- It seems that many of Newton's formulations emerge from the idea that we can take motion, divide into small time steps, sum it all up (integrate) and voilà we have a model describing something continuous. Was this widely accepted as a method of derivation in the 1680's?
- I understand that this piece of work is prior to Newton's Principia, but I can't help to question why Newton uses lines and points to derive his theorems. Why is a vector (force vector) not used for these tasks? The force vector does not exist at this time?
- I would like to know if this paper was written before the term force defined or not? Or is this paper the origin of the definition of force ( $F=m*a$ )?

# Questions for discussion

- Why are the algebraic statements in his theorems and hypotheses **not written in algebraic notation**?
- **Was he only interested in the proportionality relations** or were the actual constants, equations and units also a priority for him as well?
- How much of physics is at Newton's time in history is **actually mathematical**?
- His *scholium* in the paragraph after theorem 4. I do not understand it... **Is he trying to say that we need loads of empiric data to be able to say something quantitative about some system?** (I mean; if we only take a snapshot of all the heavenly bodies in our solar system, we have no idea which orbit they have. We need to have some historical data (some knowledge of its circle arc) in order to be able to predict the planetary motion.)

## End of module feedback

- Please go to [b.socrative.com](https://b.socrative.com) (student login)
- Enter the HISPHYSKU room
- Fill out the short (anonymous) survey
- Tak skal du have!

P.S.: Remember to think about the topic for your seminar!