Even in the principle of the composition of forces its properties are found.

direction of p in r and s. The work done in an element of time is proportional to the squares of the forces, and if there be no connections is expressible by $p^2 + q^2 = p^2 + r^2 + s^2$. If now r act directly counter to the force p, a diminution of work will be effected and the sum mentioned becomes $(p - r)^2 + s^2$. Even in the principle of the composition of forces, or of the mutual independence of forces, the properties are contained which Gauss's principle makes use of. This will best be perceived by imagining all the accelerations simultaneously performed. If we discard the obscure verbal form in which the principle is clothed, the metaphysical impression which it gives also vanishes. We see the simple fact; we are disillusioned, but also enlightened.

The elucidations of Gauss's principle here presented are in great part derived from the paper of Scheffler cited above. Some of his opinions which I have been unable to share I have modified. We cannot, for example, accept as new the principle which he himself propounds, for both in form and in import it is *identical* with the D'Alembert-Lagrangian.

VIII.

THE PRINCIPLE OF LEAST ACTION.

The original, obscure form of the principle of least action.

1. MAUPERTUIS enunciated, in 1747, a principle which he called "*le principe de la moindre quantité d'action*," the principle of *least action*. He declared this principle to be one which eminently accorded with the wisdom of the Creator. He took as the measure of the "action" the product of the mass, the velocity, and the space described, or mvs. *Why*, it must be confessed, is not clear. By mass and velocity definite quantities may be understood; not so, however, by

space, when the time is not stated in which the space is described. If, however, unit of time be meant, the distinction of space and velocity in the examples treated by Maupertuis is, to say the least, peculiar. It appears that Maupertuis reached this obscure expression by an unclear mingling of his ideas of *vis viva* and the principle of virtual velocities. Its indistinctness will be more saliently displayed by the details.

2. Let us see how Maupertuis applies his principle. If M, m be two inelastic masses, C and c their velocities before impact, and u their common velocity after impact, Maupertuis requires, (putting here velocities for spaces,) that the "action" expended in the change of the velocities in impact shall be a minimum. Hence, $M(C-u)^2+m(c-u)^2$ is a minimum; that is, $M(C-u)+m(c-u)=0$; or

Determination of the laws of impact by this principle.

$$u=\frac{MC+mc}{M+m}.$$

For the impact of elastic masses, retaining the same designations, only substituting V and v for the two velocities after impact, the expression $M(C-V)^2+m(c-v)^2$ is a minimum; that is to say,

$$M(C-V)\,dV+m(c-v)\,dv=0. \quad . \quad . \quad . \quad . \quad (1)$$

In consideration of the fact that the velocity of approach before impact is equal to the velocity of recession after impact, we have

$$C-c=-(V-v) \text{ or}$$
$$C+V-(c+v)=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and

$$dV-dv=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The combination of equations (1), (2), and (3) readily gives the familiar expressions for V and v. These two cases may, as we see, be viewed as processes in which the least change of *vis viva* by reaction takes place, that is, in which the *least counter-work* is done. They fall, therefore, under the principle of Gauss.

Maupertuis's deduction of the law of the lever by this principle.

3. Peculiar is Maupertuis's deduction of the *law of the lever*. Two masses M and m (Fig. 188) rest on a bar a, which the fulcrum divides into the portions x and $a - x$. *If the bar be set in rotation,* the velocities and the spaces described will be proportional to the lengths of the lever-arms, and $Mx^2 + m(a - x)^2$ is the quantity to be made a minimum, that is $Mx - m(a - x) = 0$; whence $x = ma/\overline{M + m}$,—a condition that in the case of *equilibrium* is actually fulfilled. In criticism of this, it is to be remarked, first, that masses not subject to gravity or other forces, as Maupertuis here tacitly assumes, are *always* in equilibrium, and, secondly, that the inference from Maupertuis's deduction is that the principle of least action is fulfilled *only* in the case of equilibrium, a conclusion which it was certainly not the author's intention to demonstrate.

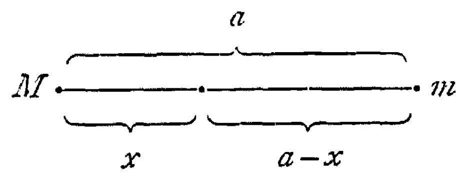

Fig. 188.

The correction of Maupertuis's deduction.

If it were sought to bring this treatment into approximate accord with the preceding, we should have to assume that the *heavy* masses M and m constantly produced in each other during the process the least possible change of *vis viva*. On that supposition, we should get, designating the arms of the lever briefly by a, b, the velocities acquired in unit of time by u, v, and the acceleration of gravity by g, as our minimum expression, $M(g - u)^2 + m(g - v)^2$; whence $M(g - u)du + m(g - v)dv = 0$. But in view of the connection of the masses as lever,

$$\frac{u}{a} = - \frac{v}{b}, \text{ and}$$

$$du = - \frac{a}{b} dv;$$

whence these equations correctly follow

$$u = a \frac{Ma - mb}{Ma^2 + mb^2} g, \quad v = - b \frac{Ma - mb}{Ma^2 + mb^2} g,$$

and for the case of equilibrium, where $u = v = 0$,

$$Ma - mb = 0.$$

Thus, this deduction also, when we come to rectify it, leads to Gauss's principle.

Treatment of the motion of ligh by the prin ciple of least action.

4. Following the precedent of Fermat and Leibnitz, Maupertuis also treats by his method the *motion of light*. Here again, however, he employs the notion "least action" in a totally different sense. The expression which for the case of refraction shall be a minimum, is $m \cdot AR + n \cdot RB$, where AR and RB denote the paths described by the light in the first and second media respectively, and m and n the corresponding velocities. True, we really do obtain here, if R be determined in conformity with the minimum condition, the result $\sin\alpha/\sin\beta = n/m = const.$ But before, the "action" consisted in the *change* of the expressions mass $\times$ velocity $\times$ distance; now, however, it is constituted of the *sum* of these expressions. Before, the spaces described in unit of time were considered; in the present case the *total* spaces traversed are taken. Should not $m \cdot AR - n \cdot RB$ or $(m - n)(AR - RB)$ be taken as a minimum, and if not, why not? But

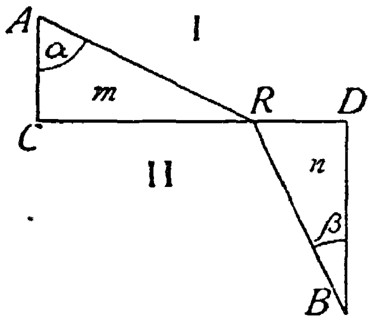

Fig. 189.

even if we accept Maupertuis's conception, the reciprocal values of the velocities of the light are obtained, and not the actual values.

Characterisation of Maupertuis's principle.

It will thus be seen that Maupertuis really had no principle, properly speaking, but only a vague formula, which was forced to do duty as the expression of different familiar phenomena not really brought under one conception. I have found it necessary to enter into some detail in this matter, since Maupertuis's performance, though it has been unfavorably criticised by all mathematicians, is, nevertheless, still invested with a sort of historical halo. It would seem almost as if something of the pious faith of the church had crept into mechanics. However, the mere *endeavor* to gain a more extensive view, although beyond the powers of the author, was not altogether without results. Euler, at least, if not also Gauss, was stimulated by the attempt of Maupertuis.

Euler's contributions to this subject.

5. Euler's view is, that the *purposes* of the phenomena of nature afford as good a basis of explanation as their *causes*. If this position be taken, it will be presumed *a priori* that all natural phenomena present a maximum or minimum. Of what character this maximum or minimum is, can hardly be ascertained by metaphysical speculations. But in the solution of mechanical problems by the ordinary methods, it is possible, if the requisite attention be bestowed on the matter, to find the expression which in all cases is made a maximum or a minimum. Euler is thus not led astray by any metaphysical propensities, and proceeds much more scientifically than Maupertuis. He seeks an expression whose variation put $= 0$ gives the ordinary equations of mechanics.

For a *single* body moving under the action of forces

Euler finds the requisite expression in the formula $\int v\, ds$, where ds denotes the element of the path and v the corresponding velocity. This expression is smaller for the path *actually* taken than for any other infinitely adjacent neighboring path between the same initial and terminal points, which the body may be *constrained* to take. Conversely, therefore, by *seeking* the path that makes $\int v\, ds$ a minimum, we can also determine the path. The problem of minimising $\int v\, ds$ is, of course, as Euler assumed, a permissible one, only when v depends on the position of the elements ds, that is to say, when the principle of *vis viva* holds for the forces, or a force-function exists, or what is the same thing, when v is a simple function of coördinates. For a motion in a plane the expression would accordingly assume the form

The form which the principle assumed in Euler's hands.

$$\int \varphi(x, y) \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, . \, dx$$

In the simplest cases Euler's principle is easily verified. If no forces act, v is constant, and the curve of motion becomes a straight line, for which $\int v\, ds = v \int ds$ is unquestionably *shorter* than for any other curve between the same terminal points. Also, a body moving on a curved surface without the action of forces or friction, preserves its velocity, and describes on the surface a *shortest* line.

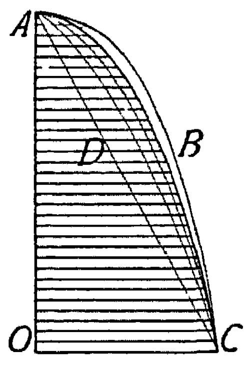

Fig. 190

Euler's principle applied to the motion of a projectile.

The consideration of the motion of a projectile in a parabola ABC (Fig. 190) will also show that the quantity $\int v\, ds$ is smaller for the parabola than for any other neighboring curve; smaller, even, than for the *straight* line ABC between the same terminal points. The velocity, here, depends solely on the

Mathematical development of this case.

vertical space described by the body, and is therefore the same for all curves whose altitude above OC is the same. If we divide the curves by a system of horizontal straight lines into elements which severally correspond, the elements to be multiplied by the same v's, though in the upper portions smaller for the straight line AD than for AB, are in the lower portions just the reverse; and as it is here that the larger v's come into play, the sum upon the whole is smaller for ABC than for the straight line.

Putting the origin of the coördinates at A, reckoning the abscissas x vertically downwards as positive, and calling the ordinates perpendicular thereto y, we obtain for the expression to be minimised

$$\int_0^x \sqrt{2g(a+x)}\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,.\,dx,$$

where g denotes the acceleration of gravity and a the distance of descent corresponding to the initial velocity. As the condition of minimum the calculus of variations gives

$$\frac{\sqrt{2g(a+x)}\,\frac{dy}{dx}}{\sqrt{1+\left(\frac{dy^2}{dx}\right)}} = C \text{ or}$$

$$\frac{dy}{dx} = \frac{C}{\sqrt{2g(a+x)-C^2}} \text{ or}$$

$$y = \int \frac{C\,dx}{\sqrt{2g(a+x)-C^2}},$$

and, ultimately,

$$y = \frac{C}{g}\sqrt{2g(a+x)-C^2} + C',$$

where C and C' denote constants of integration that pass into $C = \sqrt{2ga}$ and $C' = 0$, if for $x = 0$, $dx/dy = 0$ and $y = 0$ be taken. Therefore, $y = 2\sqrt{ax}$. By this method, accordingly, the path of a projectile is shown to be of parabolic form.

6. Subsequently, Lagrange drew *express* attention to the fact that Euler's principle is applicable only in cases in which the principle of *vis viva* holds. Jacobi pointed out that we cannot assert that $\int v\,ds$ for the actual motion is a *minimum*, but simply that the *variation* of this expression, in its passage to an infinitely adjacent neighboring path, is $= 0$. Generally, indeed, this condition coincides with a maximum or minimum, but it is possible that it should occur *without* such; and the minimum property in particular is subject to certain limitations. For example, if a body, constrained to move on a spherical surface, is set in motion by some impulse, it will describe a great circle, generally a shortest line. But if the length of the arc described exceeds 180°, it is easily demonstrated that there exist shorter infinitely adjacent neighboring paths between the terminal points. The additions of Lagrange and Jacobi.

7. So far, then, this fact only has been pointed out, that the ordinary equations of motion are obtained by equating the variation of $\int v\,ds$ to zero. But since the properties of the motion of bodies or of their paths may always be defined by differential expressions equated to zero, and since furthermore the condition that the variation of an integral expression shall be equal to zero is likewise given by differential expressions equated to zero, unquestionably *various other* integral expressions may be devised that give by variation the ordinary equations of motion, without its following that the Euler's expression but one of many which give the equations of motion.