PRESENTATION OF THE PRINCIPLE

" Bodies only act on each other in three ways that are known to us— either by immediate impulse as in ordinary impact; or by means of some body interposed between them and to which they are attached ; or finally, by a reciprocal property of attraction, as they do in the Newtonian system of the Sun and the Planets. Since the effects of this last mode of action have been sufficiently investigated, I shall confine myself to a treatment of bodies which collide in any manner whatever, and of those which are acted upon be means of threads or rigid rods. I shall dwell on this subject even more readily because the greatest geometers have only so far (1742) solved a small number of problems of this kind, and because I hope, by means of the general method which I am going to present, to equip all those who are familar with the calculations and principles of mechanics so that they can solve the most difficult problems of this kind.

DEFINITION

" In what follows, I shall call *motion* of a body the velocity of this same body and shall take account of its direction. And by *quantity of motion*, I shall understand, as is customary, the product of the mass and the velocity.

GENERAL PROBLEM

" *Let there be given a system of bodies arranged in any way with respect to each other ; and suppose that a particular motion is imparted to each of these bodies, which it cannot follow because of the action of the other bodies— to find the motion that each body must take.*

SOLUTION

" Let A, B, C, etc. . . . be the bodies that constitute the system and suppose that the motions a, b, c, etc. . . . are impressed on them ; let there be forces, arising from their mutual action, which change these into the motions a, $\bar{\mathrm{b}}$, c, etc. . . . It is clear that the motion a impressed on the body A can be compounded of the motion a which it acquires and another motion α. In the same way the motions b, c, etc. . . . can be regarded as compounded of the motions $\bar{\mathrm{b}}$ and β, c and $\varkappa$, etc. . . . From this it follows that the motions of the bodies A, B, C, etc. . . . would be the same, among themselves, if instead of their having been given the impulses *a*, *b*, *c*, etc. . . . they had been simultaneously given the twin impulsions a and α, $\bar{\text{b}}$ and β, c and $\varkappa$, etc. . . . Now, by supposition, the bodies *A*, *B*, *C*, etc. . . . have assumed, by their own action, the motions a, $\bar{\text{b}}$, c, etc. . . . Therefore the motions α, β, $\varkappa$, etc. . . . must be such that they do not disturb the motions a, $\bar{\text{b}}$, c, etc. . . . in any way. That is to say, that if the bodies had only received the motions α, β, $\varkappa$, etc. . . . these motions would have been cancelled out among themselves, and the system would have remained at rest.

"From this results the following principle for finding the motion of several bodies which act upon each other. *Decompose each of the motions a, b, c, etc. . . . which are impressed on the bodies into two others,* a *and* α, $\bar{\text{b}}$ *and* β, c *and* $\varkappa$, *etc. . . . which are such that if the motions* a, $\bar{\text{b}}$, c, *etc. . . . had been impressed on the bodies, they would have been retained unchanged ; and if the motions* α, β, $\varkappa$, *etc. . . . alone had been impressed on the bodies, the system would have remained at rest. It is clear that* a, $\bar{\text{b}}$, c, *etc. . . . will be the motions that the bodies will take because of their mutual action. This is what it was necessary to find.*"

5. D'Alembert's solution of the problem of the centre of oscillation.

Although d'Alembert's principle is perfectly clear, its application is difficult, and the *Traité de Dynamique* remains a difficult book to read.

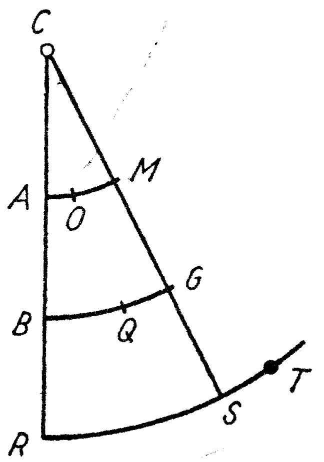

Fig. 88

As a concrete example of its application, we shall give d'Alembert's solution of the celebrated problem of the centre of oscillation.[1]

"*Problem.* — *To find the velocity of a rod CR fixed at C, and loaded with as many weights as may be desired, under the supposition that these bodies, if the rod had not prevented them, would have described infinitely short lines AO, BQ, RT, perpendicular to the rod, in equal times.*

"All the difficulty reduces to finding the line *RS* travelled by one of the bodies, *R*, in the time that it would have travelled *RT*. For then the velocities *BG*, *AM*, of all the other bodies are known.

"Now regard the impressed velocities, *RT*, *BQ*, *AO* as being composed of the velocities *RS* and *ST* ; *BG* and —*GQ* ; *AM* and —*MO*. By our principle, the lever *CAR* would have

[1] *Traité de Dynamique*, p. 96.

remained in equilibrium if the bodies R, B, A had received the motions ST, $-GQ$, $-MO$ alone.

" Therefore

$$A\cdot MO\cdot AC + B\cdot QG\cdot BC = R\cdot ST\cdot CR.$$

" Denoting AO by a, BQ by b, RT by c, CA by r, CB by r', CR by ϱ and RS by z, we will have

$$R\,(c\text{-}z)\,\varrho = Ar\left(\frac{zr}{\varrho} - a\right) + Br'\left(\frac{zr'}{\varrho} - b\right).$$

" Consequently

$$z = \frac{Aar\varrho + Bbr'\varrho + Rc\varrho^2}{Ar^2 + Br'^2 + R\varrho^2}.$$

" *Corollary.* — Let F, f, φ be the motive forces of the bodies A, B, R. The accelerating force will be found to be

$$\frac{Fr + fr' + \varphi\varrho}{Ar^2 + Br^2 + R\varrho^2} \times \varrho$$

on giving a, b, c, their values $\frac{F}{A}, \frac{f}{b}, \frac{\varphi}{R}$. Therefore, if the element of arc described by the radius CR is taken to be ds and the velocity of R to be u, then, in general,

$$\frac{Fr + fr' + \varphi\varrho}{Ar^2 + Br^2 + R\varrho^2}\,\varrho ds = udu$$

whatever the forces F, f, φ may be. It is easy, by this means, to solve the problem of centres of oscillation under any hypothesis.

6. The priority of Herman and Euler in the matter of d'Alembert's principle.

After recalling Jacques Bernoulli's solution of the problem of the centre of oscillation, d'Alembert remarks that Euler, in Volume III of the old Commentaries of the Academy of Petersbourg (1740), had used the principle according to which the powers $R\cdot RS$, $B\cdot BG$, $A\cdot AM$ must be equivalent to the powers $R\cdot RT$, $B\cdot BQ$, $A\cdot AO$. " But M. Euler has in no way demonstrated this principle and this, it seems to me, can only be done by means of ours. Moreover, the author has only applied this principle to the solution of a small number of problems concerning the oscillation of flexible or inflexible bodies, and the solution that he has given to one of these problems is not correct. [This was the problem of the oscillation of a solid body on a plane.] This shows to what extent