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VIII. *On Faraday's Lines of Force.*

[Read *Dec.* 10, 1855, and *Feb.* 11, 1856.]

THE present state of electrical science seems peculiarly unfavourable to speculation. The laws of the distribution of electricity on the surface of conductors have been analytically deduced from experiment; some parts of the mathematical theory of magnetism are established, while in other parts the experimental data are wanting; the theory of the conduction of galvanism and that of the mutual attraction of conductors have been reduced to mathematical formulæ, but have not fallen into relation with the other parts of the science. No electrical theory can now be put forth, unless it shews the connexion not only between electricity at rest and current electricity, but between the attractions and inductive effects of electricity in both states. Such a theory must accurately satisfy those laws, the mathematical form of which is known, and must afford the means of calculating the effects in the limiting cases where the known formulæ are inapplicable. In order therefore to appreciate the requirements of the science, the student must make himself familiar with a considerable body of most intricate mathematics, the mere retention of which in the memory materially interferes with further progress. The first process therefore in the effectual study of the science, must be one of simplification and reduction of the results of previous investigation to a form in which the mind can grasp them. The results of this simplification may take the form of a purely mathematical formula or of a physical hypothesis. In the first case we entirely lose sight of the phenomena to be explained; and though we may trace out the consequences of given laws, we can never obtain more extended views of the connexions of the subject. If, on the other hand, we adopt a physical hypothesis, we see the phenomena only through a medium, and are liable to that blindness to facts and rashness in

assumption which a partial explanation encourages. We must therefore discover some method of investigation which allows the mind at every step to lay hold of a clear physical conception, without being committed to any theory founded on the physical science from which that conception is borrowed, so that it is neither drawn aside from the subject in pursuit of analytical subtleties, nor carried beyond the truth by a favourite hypothesis.

In order to obtain physical ideas without adopting a physical theory we must make ourselves familiar with the existence of physical analogies. By a physical analogy I mean that partial similarity between the laws of one science and those of another which makes each of them illustrate the other. Thus all the mathematical sciences are founded on relations between physical laws and laws of numbers, so that the aim of exact science is to reduce the problems of nature to the determination of quantities by operations with numbers. Passing from the most universal of all analogies to a very partial one, we find the same resemblance in mathematical form between two different phenomena giving rise to a physical theory of light.

The changes of direction which light undergoes in passing from one medium to another, are identical with the deviations of the path of a particle in moving through a narrow space in which intense forces act. This analogy, which extends only to the direction, and not to the velocity of motion, was long believed to be the true explanation of the refraction of light; and we still find it useful in the solution of certain problems, in which we employ it without danger, as an artificial method. The other analogy, between light and the vibrations of an elastic medium, extends much farther, but, though its importance and fruitfulness cannot be over-estimated, we must recollect that it is founded only on a resemblance *in form* between the laws of light and those of vibrations. By stripping it of its physical dress and reducing it to a theory of "transverse alternations," we might obtain a system of truth strictly founded on observation, but probably deficient both in the vividness of its conceptions and the fertility of its method. I have said thus much on the disputed questions of Optics, as a preparation for the discussion of the almost universally admitted theory of attraction at a distance.

We have all acquired the mathematical conception of these attractions. We can reason about them and determine their appropriate forms or formulæ. These formulæ have a distinct mathematical significance, and their results are found to be in accordance with natural phenomena. There is no formula in applied

mathematics more consistent with nature than the formula of attractions, and no theory better established in the minds of men than that of the action of bodies on one another at a distance. The laws of the conduction of heat in uniform media appear at first sight among the most different in their physical relations from those relating to attractions. The quantities which enter into them are *temperature, flow of heat, conductivity*. The word *force* is foreign to the subject. Yet we find that the mathematical laws of the uniform motion of heat in homogeneous media are identical in form with those of attractions varying inversely as the square of the distance. We have only to substitute *source of heat* for *centre of attraction*, *flow of heat* for *accelerating effect of attraction* at any point, and *temperature* for *potential*, and the solution of a problem in attractions is transformed into that of a problem in heat.

This analogy between the formulæ of heat and attraction was, I believe, first pointed out by Professor William Thomson in the *Camb. Math. Journal*, Vol. III.

Now the conduction of heat is supposed to proceed by an action between contiguous parts of a medium, while the force of attraction is a relation between distant bodies, and yet, if we knew nothing more than is expressed in the mathematical formulæ, there would be nothing to distinguish between the one set of phenomena and the other.

It is true, that if we introduce other considerations and observe additional facts, the two subjects will assume very different aspects, but the mathematical resemblance of some of their laws will remain, and may still be made useful in exciting appropriate mathematical ideas.

It is by the use of analogies of this kind that I have attempted to bring before the mind, in a convenient and manageable form, those mathematical ideas which are necessary to the study of the phenomena of electricity. The methods are generally those suggested by the processes of reasoning which are found in the researches of Faraday*, and which, though they have been interpreted mathematically by Prof. Thomson and others, are very generally supposed to be of an indefinite and unmathematical character, when compared with those employed by the professed mathematicians. By the method which I adopt, I hope to render it evident that I am not attempting to establish any physical theory of a science in which I have hardly made a single experiment, and that the limit of my design is to shew how, by a strict application of the ideas and

* See especially Series XXXVIII. of the *Experimental Researches*, and *Phil. Mag.* 1852.

methods of Faraday, the connexion of the very different orders of phenomena which he has discovered may be clearly placed before the mathematical mind. I shall therefore avoid as much as I can the introduction of anything which does not serve as a direct illustration of Faraday's methods, or of the mathematical deductions which may be made from them. In treating the simpler parts of the subject I shall use Faraday's mathematical methods as well as his ideas. When the complexity of the subject requires it, I shall use analytical notation, still confining myself to the development of ideas originated by the same philosopher.

I have in the first place to explain and illustrate the idea of "lines of force."

When a body is electrified in any manner, a small body charged with positive electricity, and placed in any given position, will experience a force urging it in a certain direction. If the small body be now negatively electrified, it will be urged by an equal force in a direction exactly opposite.

The same relations hold between a magnetic body and the north or south poles of a small magnet. If the north pole is urged in one direction, the south pole is urged in the opposite direction.

In this way we might find a line passing through any point of space, such that it represents the direction of the force acting on a positively electrified particle, or on an elementary north pole, and the reverse direction of the force on a negatively electrified particle or an elementary south pole. Since at every point of space such a direction may be found, if we commence at any point and draw a line so that, as we go along it, its direction at any point shall always coincide with that of the resultant force at that point, this curve will indicate the direction of that force for every point through which it passes, and might be called on that account a *line of force*. We might in the same way draw other lines of force, till we had filled all space with curves indicating by their direction that of the force at any assigned point.

We should thus obtain a geometrical model of the physical phenomena, which would tell us the *direction* of the force, but we should still require some method of indicating the *intensity* of the force at any point. If we consider these curves not as mere lines, but as fine tubes of variable section carrying an incompressible fluid, then, since the velocity of the fluid is inversely as the section of the tube, we may make the velocity vary according to any given law, by regulating the section of the tube, and in this way we might represent the

intensity of the force as well as its direction by the motion of the fluid in these tubes. This method of representing the intensity of a force by the velocity of an imaginary fluid in a tube is applicable to any conceivable system of forces, but it is capable of great simplification in the case in which the forces are such as can be explained by the hypothesis of attractions varying inversely as the square of the distance, such as those observed in electrical and magnetic phenomena. In the case of a perfectly arbitrary system of forces, there will generally be interstices between the tubes; but in the case of electric and magnetic forces it is possible to arrange the tubes so as to leave no interstices. The tubes will then be mere surfaces, directing the motion of a fluid filling up the whole space. It has been usual to commence the investigation of the laws of these forces by at once assuming that the phenomena are due to attractive or repulsive forces acting between certain points. We may however obtain a different view of the subject, and one more suited to our more difficult inquiries, by adopting for the definition of the forces of which we treat, that they may be represented in magnitude and direction by the uniform motion of an incompressible fluid.

I propose, then, first to describe a method by which the motion of such a fluid can be clearly conceived; secondly to trace the consequences of assuming certain conditions of motion, and to point out the application of the method to some of the less complicated phenomena of electricity, magnetism, and galvanism; and lastly to shew how by an extension of these methods, and the introduction of another idea due to Faraday, the laws of the attractions and inductive actions of magnets and currents may be clearly conceived, without making any assumptions as to the physical nature of electricity, or adding anything to that which has been already proved by experiment.

By referring everything to the purely geometrical idea of the motion of an imaginary fluid, I hope to attain generality and precision, and to avoid the dangers arising from a premature theory professing to explain the cause of the phenomena. If the results of mere speculation which I have collected are found to be of any use to experimental philosophers, in arranging and interpreting their results, they will have served their purpose, and a mature theory, in which physical facts will be physically explained, will be formed by those who by interrogating Nature herself can obtain the only true solution of the questions which the mathematical theory suggests.

I. *Theory of the Motion of an incompressible Fluid.*

(1) The substance here treated of must not be assumed to possess any of the properties of ordinary fluids except those of freedom of motion and resistance to compression. It is not even a hypothetical fluid which is introduced to explain actual phenomena. It is merely a collection of imaginary properties which may be employed for establishing certain theorems in pure mathematics in a way more intelligible to many minds and more applicable to physical problems than that in which algebraic symbols alone are used. The use of the word "Fluid" will not lead us into error, if we remember that it denotes a purely imaginary substance with the following property :

The portion of fluid which at any instant occupied a given volume, will at any succeeding instant occupy an equal volume.

This law expresses the incompressibility of the fluid, and furnishes us with a convenient measure of its quantity, namely its volume. The unit of quantity of the fluid will therefore be the unit of volume.

(2) The direction of motion of the fluid will in general be different at different points of the space which it occupies, but since the direction is determinate for every such point, we may conceive a line to begin at any point and to be continued so that every element of the line indicates by its direction the direction of motion at that point of space. Lines drawn in such a manner that their direction always indicates the direction of fluid motion are called *lines of fluid motion*.

If the motion of the fluid be what is called *steady motion*, that is, if the direction and velocity of the motion at any fixed point be independent of the time, these curves will represent the paths of individual particles of the fluid, but if the motion be variable this will not generally be the case. The cases of motion which will come under our notice will be those of steady motion.

(3) If upon any surface which cuts the lines of fluid motion we draw a closed curve, and if from every point of this curve we draw a line of motion, these lines of motion will generate a tubular surface which we may call a *tube of fluid motion*. Since this surface is generated by lines in the direction of fluid

motion no part of the fluid can flow across it, so that this imaginary surface is as impermeable to the fluid as a real tube.

(4) The quantity of fluid which in unit of time crosses any fixed section of the tube is the same at whatever part of the tube the section be taken. For the fluid is incompressible, and no part runs through the sides of the tube, therefore the quantity which escapes from the second section is equal to that which enters through the first.

If the tube be such that unit of volume passes through any section in unit of time it is called a *unit tube of fluid motion*.

(5) In what follows, various units will be referred to, and a finite number of lines or surfaces will be drawn, representing in terms of those units the motion of the fluid. Now in order to define the motion in every part of the fluid, an infinite number of lines would have to be drawn at indefinitely small intervals; but since the description of such a system of lines would involve continual reference to the theory of limits, it has been thought better to suppose the lines drawn at intervals depending on the assumed unit, and afterwards to assume the unit as small as we please by taking a small submultiple of the standard unit.

(6) To define the motion of the whole fluid by means of a system of unit tubes.

Take any fixed surface which cuts all the lines of fluid motion, and draw upon it any system of curves not intersecting one another. On the same surface draw a second system of curves intersecting the first system, and so arranged that the quantity of fluid which crosses the surface within each of the quadrilaterals formed by the intersection of the two systems of curves shall be unity in unit of time. From every point in a curve of the first system let a line of fluid motion be drawn. These lines will form a surface through which no fluid passes. Similar impermeable surfaces may be drawn for all the curves of the first system. The curves of the second system will give rise to a second system of impermeable surfaces, which, by their intersection with the first system, will form quadrilateral tubes, which will be tubes of fluid motion. Since each quadrilateral of the cutting surface transmits unity of fluid in unity of time, every tube in the system will transmit unity of fluid through any of its sections in unit of time. The motion of the fluid at every part of the space it occupies

is determined by this system of unit tubes; for the direction of motion is that of the tube through the point in question, and the velocity is the reciprocal of the area of the section of the unit tube at that point.

(7) We have now obtained a geometrical construction which completely defines the motion of the fluid by dividing the space it occupies into a system of unit tubes. We have next to shew how by means of these tubes we may ascertain various points relating to the motion of the fluid.

A unit tube may either return into itself, or may begin and end at different points, and these may be either in the boundary of the space in which we investigate the motion, or within that space. In the first case there is a continual circulation of fluid in the tube, in the second the fluid enters at one end and flows out at the other. If the extremities of the tube are in the bounding surface, the fluid may be supposed to be continually supplied from without from an unknown source, and to flow out at the other into an unknown reservoir; but if the origin of the tube or its termination be within the space under consideration, then we must conceive the fluid to be supplied by a *source* within that space, capable of creating and emitting unity of fluid in unity of time, and to be afterwards swallowed up by a *sink* capable of receiving and destroying the same amount continually.

There is nothing self-contradictory in the conception of these sources where the fluid is created, and sinks where it is annihilated. The properties of the fluid are at our disposal, we have made it incompressible, and now we suppose it produced from nothing at certain points and reduced to nothing at others. The places of production will be called *sources*, and their numerical value will be the number of units of fluid which they produce in unit of time. The places of reduction will, for want of a better name, be called *sinks*, and will be estimated by the number of units of fluid absorbed in unit of time. Both places will sometimes be called sources, a source being understood to be a sink when its sign is negative.

(8) It is evident that the amount of fluid which passes any fixed surface is measured by the number of unit tubes which cut it, and the direction in which the fluid passes is determined by that of its motion in the tubes. If the surface be a closed one, then any tube whose terminations lie on the same side of the surface must cross the surface as many times in the one direction as in the other, and therefore must carry as much fluid out of the surface as

it carries in. A tube which begins within the surface and ends without it will carry out unity of fluid; and one which enters the surface and terminates within it will carry in the same quantity. In order therefore to estimate the amount of fluid which flows out of the closed surface, we must subtract the number of tubes which end within the surface from the number of tubes which begin there. If the result is negative the fluid will on the whole flow inwards.

If we call the beginning of a unit tube a unit source, and its termination a unit sink, then the quantity of fluid produced within the surface is estimated by the number of unit sources minus the number of unit sinks, and this must flow out of the surface on account of the incompressibility of the fluid.

In speaking of these unit tubes, sources and sinks, we must remember what was stated in (5) as to the magnitude of the unit, and how by diminishing their size and increasing their number we may distribute them according to any law however complicated.

(9) If we know the direction and velocity of the fluid at any point in two different cases, and if we conceive a third case in which the direction and velocity of the fluid at any point is the resultant of the velocities in the two former cases at corresponding points, then the amount of fluid which passes a given fixed surface in the third case will be the algebraic sum of the quantities which pass the same surface in the two former cases. For the rate at which the fluid crosses any surface is the resolved part of the velocity normal to the surface, and the resolved part of the resultant is equal to the sum of the resolved parts of the components.

Hence the number of unit tubes which cross the surface outwards in the third case must be the algebraical sum of the numbers which cross it in the two former cases, and the number of sources within any closed surface will be the sum of the numbers in the two former cases. Since the closed surface may be taken as small as we please, it is evident that the distribution of sources and sinks in the third case arises from the simple superposition of the distributions in the two former cases.

II. *Theory of the uniform motion of an imponderable incompressible fluid through a resisting medium.*

(10) The fluid is here supposed to have no inertia, and its motion is opposed by the action of a force which we may conceive to be due to the resistance of a