Methodus inveniendi/Additamentum II

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Appendix 2

Concerning the motion of particles in a non-resistant medium, determined by a method of maxima and minima

1. Since all natural phenomena obey a certain maximum or minimum law; there is no doubt that *some* property must be maximized or minimized in the trajectories of particles acted upon by external forces. However, it does not seem easy to determine *which* property is minimized from metaphysical principles known *a priori*. Yet if the trajectories can be determined by a direct method, the property being minimized or maximized by these trajectories can be determined, provided that sufficient care is taken. After considering the effects of external forces and the movements they generate, it seems most consistent with experience to assert that the **integrated momentum** (i.e., the sum of all momenta contained in the particle's movement) is the minimized quantity. This assertion is not sufficiently proven at present; however, if I can show it to be connected with some truth known *a priori*, it will carry such weight as to utterly vanquish every conceivable doubt. If indeed it's truth can be verified, this assertion will make it easier to investigate the deepest laws of Nature and their final causes, and also easier to identify a firmer rationale for this assertion.

2. Let the mass of a moving particle be M, and let its speed be v while being moved over an infinitesimal distance *ds*. The particle will have a momentum Mv that, when multiplied by the distance *ds*, gives $Mvds$, the momentum of the particle integrated over the distance *ds*. Now **I assert** that the true trajectory of the moving particle is the trajectory to be described (from among all possible trajectories connecting the same endpoints) that minimizes $\int Mvds$ or (since *M* is constant) $\int vds$. Since the speed v resulting from the external forces can be calculated *a posteriori* from the trajectory itself, a method of maxima and minima should suffice to determine the trajectory *a priori*. The minimized integral can be expressed in terms of the momentum (as above), but also in terms of the kinetic energy. For, given an infinitesimal time dt during which the element ds is traversed, we have $ds = vdt$. Hence, $\int Mvds = \int Mv^2dt$ is minimized, i.e., the true trajectory of a moving particle minimizes the integral over time of its instantaneous kinetic energies. Thus, this minimum principle should appeal both to those who favor momentum for mechanics calculations and to those who favor kinetic energy.

3. For our first example, consider a moving particle free of external forces, which has a constant speed, denoted b. By our principle, such a particle describes a trajectory that minimizes $\int bds$ or $\int ds = s$. Hence, the true path of a free particle has the minimum length of all paths connecting the same endpoints; this path is a straight line, just as the first principles of Mechanics postulate. I do not present this example as evidence for the general principle, since the integral of *any* function of the constant speed $v = b$ would, upon minimization, produce a straight line. I begin with this simple case merely to illustrate the reasoning.

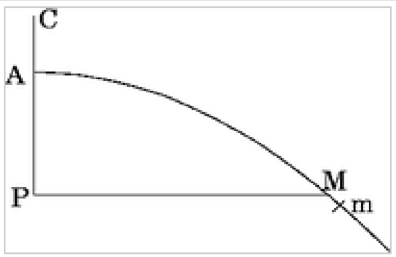

Figure 26

4. Let us proceed to the case of uniform gravity or, more generally, to the case in which a moving particle is acted upon by a downwards force of constant acceleration g. Let the trajectory of the particle under these conditions be AM (Figure 26), let Mm represent the infinitesimal distance ds, and let x and y represent the vertical coordinate along AP and the horizontal coordinate along the perpendicular axis PM, respectively. The external force produces an acceleration described by $dv^2 = 2gdy$, which integrates to $v^2 = v_0^2 + 2gy$. Hence, we seek the trajectory that minimizes the integral $\int ds\sqrt{v_0^2 + 2gy}$. Defining $p \equiv \frac{dx}{dy}$, we have $ds = dy\sqrt{1+p^2}$; in other words, we seek the trajectory that minimizes $\int dy\sqrt{v_0^2 + 2gy}\,\sqrt{1+p^2}$. Comparing this expression with our general formula $\int Zdy$, we see that $Z = \sqrt{v_0^2 + 2gy}\,\sqrt{1+p^2}$ and, given that $dZ \equiv Mdx + Ndy + Pdp$, we obtain $M = 0$ and $P = \dfrac{p\sqrt{v_0^2 + 2gy}}{\sqrt{1+p^2}}$. Minimization requires that $Mdy = dP$ and, since

$M = 0$ in this case, $dP = 0$ and, thus, $P = \sqrt{C}$, where C is a constant. Hence, we have the differential equation $\sqrt{C} = \dfrac{p\sqrt{v_0^2 + 2gy}}{\sqrt{1+p^2}} = \dfrac{dx\sqrt{v_0^2 + 2gy}}{ds}$. This may be rearranged to $Cdx^2 + Cdy^2 = dx^2\left(v_0^2 + 2gy\right)$ and separated $dx = \dfrac{dy\sqrt{C}}{\sqrt{v_0^2 - C + 2gy}}$. Integration yields the trajectory solution $x = \dfrac{1}{g}\sqrt{C\left(v_0^2 - C + 2gy\right)}$.

5. This solution is obviously a parabola, but we should consider more carefully whether it agrees with experience. The initial tangent to the trajectory is clearly horizontal, i.e., $dy = 0$ at the point where $v_0^2 - C + gy = 0$. Since the origin (0,0) of the coordinate system may be chosen at will, we set it at that tangent point, (i.e., at the highest point of the trajectory), which is equivalent to setting $C = v_0^2$. In this coordinate frame, the trajectory solution is $x = v_0\sqrt{\dfrac{2y}{g}}$, which is a parabola. Moreover, since the initial speed is evidently v_0, the height CA from which the falling body acquires the same speed from the same force g is $\dfrac{v_0^2}{2g}$, just as predicted by the direct methods of mechanics.

6. Now let the particle be acted upon by a downwards vertical force that depends on the height CP (Figure 27). We set y equal to the vertical coordinate along CP, and let $F_y(y)$ be a downwards vertical force that is a function of y. Similarly, let the horizontal coordinate along PM be denoted as x and let Mm represent the infinitesimal distance ds. Defining $p \equiv \dfrac{dx}{dy}$, we have $dv^2 = 2F_y\,dy$ and, thus, $v^2 = v_0^2 + \int 2F_y\,dy$. Hence, we seek the trajectory that minimizes the integral $\int dy\sqrt{v_0^2 + \int 2F_y\,dy}\,\sqrt{1+p^2}$. By arguments analogous to those used in paragraph 4, we obtain the trajectory equation $\sqrt{C} = \dfrac{p\sqrt{v_0^2 + \int 2F_y\,dy}}{\sqrt{1+p^2}}$ or, equivalently, $p = \dfrac{\sqrt{C}}{\sqrt{v_0^2 - C + \int 2F_y\,dy}}$, which may be simplified to $x = \int \dfrac{dy\sqrt{C}}{\sqrt{v_0^2 - C + \int 2F_y\,dy}}$. The tangent will be horizontal whenever $\int 2F_y\,dy = C - v_0^2$. These results agree with the trajectories predicted by direct methods of mechanics.

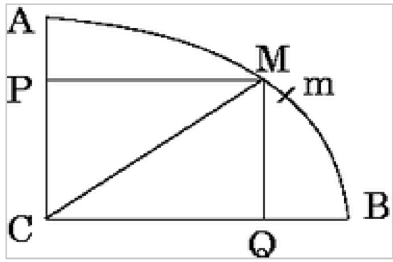

Figure 27

7. Now let the body be acted upon by two external forces, denoted as F_x in the horizontal direction MP and F_y in the vertical direction MQ (Figure 27). Let $F_x(x)$ be a function of the horizontal distance PM = x and $F_y(y)$ be a function of the vertical height MQ = CP = y. Defining $p \equiv \dfrac{dx}{dy}$ as before, we have $dv^2 = -2F_x dx - 2F_y dy$, which integrates to $v^2 = v_0^2 - \int 2F_x dx - \int 2F_y dy$. Hence, we seek the trajectory that minimizes the integral $\int dy\sqrt{1+p^2}\sqrt{v_0^2 - \int 2F_x dx - \int 2F_y dy}$. Differentiating the integrand $\sqrt{1+p^2}\sqrt{v_0^2 - \int 2F_x dx - \int 2F_y dy}$ yields

$$\frac{-F_x dx\sqrt{1+p^2}}{\sqrt{v_0^2 - \int 2F_x dx - \int 2F_y dy}} - \frac{F_y dy\sqrt{1+p^2}}{\sqrt{v_0^2 - \int 2F_x dx - \int 2F_y dy}} + \frac{pdp\sqrt{v_0^2 - \int 2F_x dx - \int 2F_y dy}}{\sqrt{1+p^2}}$$

Setting $M = \dfrac{-F_x\sqrt{1+p^2}}{\sqrt{v_0^2 - \int 2F_x dx - \int 2F_y dy}}$, and $P = \dfrac{p\sqrt{v_0^2 - \int 2F_x dx - \int 2F_y dy}}{\sqrt{1+p^2}}$, the minimal trajectory must satisfy $Mdy = dP$, i.e.,