THEORY OF HEAT.

Et ignem regunt numeri.—PLATO[1].

CHAPTER I.

INTRODUCTION.

FIRST SECTION.

Statement of the Object of the Work.

1. THE effects of heat are subject to constant laws which cannot be discovered without the aid of mathematical analysis. The object of the theory which we are about to explain is to demonstrate these laws; it reduces all physical researches on the propagation of heat, to problems of the integral calculus whose elements are given by experiment. No subject has more extensive relations with the progress of industry and the natural sciences; for the action of heat is always present, it penetrates all bodies and spaces, it influences the processes of the arts, and occurs in all the phenomena of the universe.

When heat is unequally distributed among the different parts of a solid mass, it tends to attain equilibrium, and passes slowly from the parts which are more heated to those which are less; and at the same time it is dissipated at the surface, and lost in the medium or in the void. The tendency to uniform distribution and the spontaneous emission which acts at the surface of bodies, change continually the temperature at their different points. The problem of the propagation of heat consists in

[1] Cf. Plato, *Timæus*, 53, B.

ὅτε δ' ἐπεχειρεῖτο κοσμεῖσθαι τὸ πᾶν, πῦρ πρῶτον καὶ γῆν καὶ ἀέρα καὶ ὕδωρ........ διεσχηματίσατο [ὁ θεὸς] εἴδεσί τε καὶ ἀριθμοῖς. [A. F.]

determining what is the temperature at each point of a body at a given instant, supposing that the initial temperatures are known. The following examples will more clearly make known the nature of these problems.

2. If we expose to the continued and uniform action of a source of heat, the same part of a metallic ring, whose diameter is large, the molecules nearest to the source will be first heated, and, after a certain time, every point of the solid will have acquired very nearly the highest temperature which it can attain. This limit or greatest temperature is not the same at different points; it becomes less and less according as they become more distant from that point at which the source of heat is directly applied.

When the temperatures have become permanent, the source of heat supplies, at each instant, a quantity of heat which exactly compensates for that which is dissipated at all the points of the external surface of the ring.

If now the source be suppressed, heat will continue to be propagated in the interior of the solid, but that which is lost in the medium or the void, will no longer be compensated as formerly by the supply from the source, so that all the temperatures will vary and diminish incessantly until they have become equal to the temperatures of the surrounding medium.

3. Whilst the temperatures are permanent and the source remains, if at every point of the mean circumference of the ring an ordinate be raised perpendicular to the plane of the ring, whose length is proportional to the fixed temperature at that point, the curved line which passes through the ends of these ordinates will represent the permanent state of the temperatures, and it is very easy to determine by analysis the nature of this line. It is to be remarked that the thickness of the ring is supposed to be sufficiently small for the temperature to be sensibly equal at all points of the same section perpendicular to the mean circumference. When the source is removed, the line which bounds the ordinates proportional to the temperatures at the different points will change its form continually. The problem consists in expressing, by one equation, the variable

form of this curve, and in thus including in a single formula all the successive states of the solid.

4. Let z be the constant temperature at a point m of the mean circumference, x the distance of this point from the source, that is to say the length of the arc of the mean circumference, included between the point m and the point o which corresponds to the position of the source; z is the highest temperature which the point m can attain by virtue of the constant action of the source, and this permanent temperature z is a function $f(x)$ of the distance x. The first part of the problem consists in determining the function $f(x)$ which represents the permanent state of the solid.

Consider next the variable state which succeeds to the former state as soon as the source has been removed; denote by t the time which has passed since the suppression of the source, and by v the value of the temperature at the point m after the time t. The quantity v will be a certain function $F(x, t)$ of the distance x and the time t; the object of the problem is to discover this function $F(x, t)$, of which we only know as yet that the initial value is $f(x)$, so that we ought to have the equation $f(x) = F(x, o)$.

5. If we place a solid homogeneous mass, having the form of a sphere or cube, in a medium maintained at a constant temperature, and if it remains immersed for a very long time, it will acquire at all its points a temperature differing very little from that of the fluid. Suppose the mass to be withdrawn in order to transfer it to a cooler medium, heat will begin to be dissipated at its surface; the temperatures at different points of the mass will not be sensibly the same, and if we suppose it divided into an infinity of layers by surfaces parallel to its external surface, each of those layers will transmit, at each instant, a certain quantity of heat to the layer which surrounds it. If it be imagined that each molecule carries a separate thermometer, which indicates its temperature at every instant, the state of the solid will from time to time be represented by the variable system of all these thermometric heights. It is required to express the successive states by analytical formulæ, so that we

may know at any given instant the temperatures indicated by each thermometer, and compare the quantities of heat which flow during the same instant, between two adjacent layers, or into the surrounding medium.

6. If the mass is spherical, and we denote by x the distance of a point of this mass from the centre of the sphere, by t the time which has elapsed since the commencement of the cooling, and by v the variable temperature of the point m, it is easy to see that all points situated at the same distance x from the centre of the sphere have the same temperature v. This quantity v is a certain function $F(x, t)$ of the radius x and of the time t; it must be such that it becomes constant whatever be the value of x, when we suppose t to be nothing; for by hypothesis, the temperature at all points is the same at the moment of emersion. The problem consists in determining that function of x and t which expresses the value of v.

7. In the next place it is to be remarked, that during the cooling, a certain quantity of heat escapes, at each instant, through the external surface, and passes into the medium. The value of this quantity is not constant; it is greatest at the beginning of the cooling. If however we consider the variable state of the internal spherical surface whose radius is x, we easily see that there must be at each instant a certain quantity of heat which traverses that surface, and passes through that part of the mass which is more distant from the centre. This continuous flow of heat is variable like that through the external surface, and both are quantities comparable with each other; their ratios are numbers whose varying values are functions of the distance x, and of the time t which has elapsed. It is required to determine these functions.

8. If the mass, which has been heated by a long immersion in a medium, and whose rate of cooling we wish to calculate, is of cubical form, and if we determine the position of each point m by three rectangular co-ordinates x, y, z, taking for origin the centre of the cube, and for axes lines perpendicular to the faces, we see that the temperature v of the point m after the time t, is a function of the four variables x, y, z, and t. The quantities of heat

which flow out at each instant through the whole external surface of the solid, are variable and comparable with each other; their ratios are analytical functions depending on the time t, the expression of which must be assigned.

9. Let us examine also the case in which a rectangular prism of sufficiently great thickness and of infinite length, being submitted at its extremity to a constant temperature, whilst the air which surrounds it is maintained at a less temperature, has at last arrived at a fixed state which it is required to determine. All the points of the extreme section at the base of the prism have, by hypothesis, a common and permanent temperature. It is not the same with a section distant from the source of heat; each of the points of this rectangular surface parallel to the base has acquired a fixed temperature, but this is not the same at different points of the same section, and must be less at points nearer to the surface exposed to the air. We see also that, at each instant, there flows across a given section a certain quantity of heat, which always remains the same, since the state of the solid has become constant. The problem consists in determining the permanent temperature at any given point of the solid, and the whole quantity of heat which, in a definite time, flows across a section whose position is given.

10. Take as origin of co-ordinates x, y, z, the centre of the base of the prism, and as rectangular axes, the axis of the prism itself, and the two perpendiculars on the sides: the permanent temperature v of the point m, whose co-ordinates are x, y, z, is a function of three variables $F(x, y, z)$: it has by hypothesis a constant value, when we suppose x nothing, whatever be the values of y and z. Suppose we take for the unit of heat that quantity which in the unit of time would emerge from an area equal to a unit of surface, if the heated mass which that area bounds, and which is formed of the same substance as the prism, were continually maintained at the temperature of boiling water, and immersed in atmospheric air maintained at the temperature of melting ice.

We see that the quantity of heat which, in the permanent state of the rectangular prism, flows, during a unit of time, across a certain section perpendicular to the axis, has a determinate ratio to the quantity of heat taken as unit. This ratio is not the same for all sections: it is a function $\phi(x)$ of the distance x, at which the section is situated. It is required to find an analytical expression of the function $\phi(x)$.

11. The foregoing examples suffice to give an exact idea of the different problems which we have discussed.

The solution of these problems has made us understand that the effects of the propagation of heat depend in the case of every solid substance, on three elementary qualities, which are, its capacity for heat, its own conducibility, and the exterior conducibility.

It has been observed that if two bodies of the same volume and of different nature have equal temperatures, and if the same quantity of heat be added to them, the increments of temperature are not the same; the ratio of these increments is the ratio of their capacities for heat. In this manner, the first of the three specific elements which regulate the action of heat is exactly defined, and physicists have for a long time known several methods of determining its value. It is not the same with the two others; their effects have often been observed, but there is but one exact theory which can fairly distinguish, define, and measure them with precision.

The proper or interior conducibility of a body expresses the facility with which heat is propagated in passing from one internal molecule to another. The external or relative conducibility of a solid body depends on the facility with which heat penetrates the surface, and passes from this body into a given medium, or passes from the medium into the solid. The last property is modified by the more or less polished state of the surface; it varies also according to the medium in which the body is immersed; but the interior conducibility can change only with the nature of the solid.

These three elementary qualities are represented in our formulæ by constant numbers, and the theory itself indicates experiments suitable for measuring their values. As soon as they are determined, all the problems relating to the propagation of heat depend only on numerical analysis. The knowledge of these specific properties may be directly useful in several applications of the physical sciences; it is besides an element in the study and description of different substances. It is a very imperfect knowledge of bodies which ignores the relations which they have with one of the chief agents of nature. In general, there is no mathematical theory which has a closer relation than this with public economy, since it serves to give clearness and perfection to the practice of the numerous arts which are founded on the employment of heat.

SECTION V.

Law of the permanent temperatures in a prism of small thickness.

73. We shall easily apply the principles which have just been explained to the following problem, very simple in itself, but one whose solution it is important to base on exact theory.

A metal bar, whose form is that of a rectangular parallelopiped infinite in length, is exposed to the action of a source of heat which produces a constant temperature at all points of its extremity A. It is required to determine the fixed temperatures at the different sections of the bar.

The section perpendicular to the axis is supposed to be a square whose side $2l$ is so small that we may without sensible error consider the temperatures to be equal at different points of the same section. The air in which the bar is placed is maintained at a constant temperature 0, and carried away by a current with uniform velocity.

Within the interior of the solid, heat will pass successively all the parts situate to the right of the source, and not exposed directly to its action; they will be heated more and more, but the temperature of each point will not increase beyond a certain limit. This maximum temperature is not the same for every section; it in general decreases as the distance of the section from the origin increases: we shall denote by v the fixed temperature of a section perpendicular to the axis, and situate at a distance x from the origin A.

Before every point of the solid has attained its highest degree of heat, the system of temperatures varies continually, and approaches more and more to a fixed state, which is that which we consider. This final state is kept up of itself when it has once been formed. In order that the system of temperatures may be permanent, it is necessary that the quantity of heat which, during unit of time, crosses a section made at a distance x from the origin, should balance exactly all the heat which, during the same time, escapes through that part of the external surface of the prism which is situated to the right of the same section. The lamina whose thickness is dx, and whose external surface is $8ldx$, allows the escape into the air, during unit of time, of a quantity of heat expressed by $8hlv \, . \, dx$, h being the measure of the external conducibility of the prism. Hence taking the integral $\int 8hlv \, . \, dx$ from $x = 0$ to $x = \infty$, we shall find the quantity of heat which escapes from the whole surface of the bar during unit of time; and if we take the same integral from $x = 0$ to $x = x$, we shall have the quantity of heat lost through the part of the surface included between the source of heat and the section made at the distance x. Denoting the first integral by C, whose value is constant, and the variable value of the second by $\int 8hlv \, . \, dx$; the difference $C - \int 8hlv \, . \, dx$ will express the whole quantity of heat which escapes into the air across the part of the surface situate to the right of the section. On the other hand, the lamina of the solid, enclosed between two sections infinitely near at distances x and $x + dx$, must resemble an infinite solid, bounded by two parallel planes, subject to fixed temperatures v and $v + dv$, since, by hypothesis, the temperature does not vary throughout the whole extent of the same section. The thickness of the solid is dx, and the area of the section is $4l^2$: hence the quantity of heat which flows uniformly, during unit of time, across a section of this solid, is, according to the preceding principles, $-4l^2k\frac{dv}{dx}$, k being the specific internal conducibility: we must therefore have the equation

$$-4l^2k\frac{dv}{dx} = C - \int 8hlv\,.\,dx,$$

whence $$kl\frac{d^2v}{dx^2} = 2hv.$$

74. We should obtain the same result by considering the equilibrium of heat in a single lamina infinitely thin, enclosed between two sections at distances x and $x + dx$. In fact, the quantity of heat which, during unit of time, crosses the first section situate at distance x, is $-4l^2k\frac{dv}{dx}$. To find that which flows during the same time across the successive section situate at distance $x + dx$, we must in the preceding expression change x into $x + dx$, which gives $-4l^2k\,.\left[\frac{dv}{dx} + d\left(\frac{dv}{dx}\right)\right]$. If we subtract the second expression from the first we shall find how much heat is acquired by the lamina bounded by these two sections during unit of time; and since the state of the lamina is permanent, it follows that all the heat acquired is dispersed into the air across the external surface $8ldx$ of the same lamina: now the last quantity of heat is $8hlvdx$: we shall obtain therefore the same equation

$$8hlvdx = 4l^2kd\left(\frac{dv}{dx}\right), \text{ whence } \frac{d^2v}{dx^2} = \frac{2h}{kl}v.$$

75. In whatever manner this equation is formed, it is necessary to remark that the quantity of heat which passes into the lamina whose thickness is dx, has a finite value, and that its exact expression is $-4l^2k\frac{dv}{dx}$. The lamina being enclosed between two surfaces the first of which has a temperature v,

and the second a lower temperature v', we see that the quantity of heat which it receives through the first surface depends on the difference $v - v'$, and is proportional to it: but this remark is not sufficient to complete the calculation. The quantity in question is not a differential: it has a finite value, since it is equivalent to all the heat which escapes through that part of the external surface of the prism which is situate to the right of the section. To form an exact idea of it, we must compare the lamina whose thickness is dx, with a solid terminated by two parallel planes whose distance is e, and which are maintained at unequal temperatures a and b. The quantity of heat which passes into such a prism across the hottest surface, is in fact proportional to the difference $a - b$ of the extreme temperatures, but it does not depend only on this difference: all other things being equal, it is less when the prism is thicker, and in general it is proportional to $\frac{a-b}{e}$. This is why the quantity of heat which passes through the first surface into the lamina, whose thickness is dx, is proportional to $\frac{v - v'}{dx}$.

We lay stress on this remark because the neglect of it has been the first obstacle to the establishment of the theory. If we did not make a complete analysis of the elements of the problem, we should obtain an equation not homogeneous, and, *a fortiori*, we should not be able to form the equations which express the movement of heat in more complex cases.

It was necessary also to introduce into the calculation the dimensions of the prism, in order that we might not regard, as general, consequences which observation had furnished in a particular case. Thus, it was discovered by experiment that a bar of iron, heated at one extremity, could not acquire, at a distance of six feet from the source, a temperature of one degree (octogesimal[1]); for to produce this effect, it would be necessary for the heat of the source to surpass considerably the point of fusion of iron; but this result depends on the thickness of the prism employed. If it had been greater, the heat would have been propagated to a greater distance, that is to say, the point of the bar which acquires a fixed temperature of one degree is

[1] Reaumur's Scale of Temperature. [A. F.]

much more remote from the source when the bar is thicker, all other conditions remaining the same. We can always raise by one degree the temperature of one end of a bar of iron, by heating the solid at the other end; we need only give the radius of the base a sufficient length: which is, we may say, evident, and of which besides a proof will be found in the solution of the problem (Art. 78).

76. The integral of the preceding equation is

$$v = Ae^{-x\sqrt{\frac{2h}{kl}}} + Be^{+x\sqrt{\frac{2h}{kl}}},$$

A and B being two arbitrary constants; now, if we suppose the distance x infinite, the value of the temperature v must be infinitely small; hence the term $Be^{+x\sqrt{\frac{2h}{kl}}}$ does not exist in the integral: thus the equation $v = Ae^{-x\sqrt{\frac{2h}{kl}}}$ represents the permanent state of the solid; the temperature at the origin is denoted by the constant A, since that is the value of v when x is zero.

This law according to which the temperatures decrease is the same as that given by experiment; several physicists have observed the fixed temperatures at different points of a metal bar exposed at its extremity to the constant action of a source of heat, and they have ascertained that the distances from the origin represent logarithms, and the temperatures the corresponding numbers.

77. The numerical value of the constant quotient of two consecutive temperatures being determined by observation, we easily deduce the value of the ratio $\frac{h}{k}$; for, denoting by v_1, v_2 the temperatures corresponding to the distances x_1, x_2, we have

$$\frac{v_1}{v_2} = e^{-(x_1-x_2)\sqrt{\frac{2h}{kl}}}, \text{ whence } \sqrt{\frac{2h}{k}} = \frac{\log v_1 - \log v_2}{x_2 - x_1}\sqrt{l}.$$

As for the separate values of h and k, they cannot be determined by experiments of this kind: we must observe also the varying motion of heat.

78. Suppose two bars of the same material and different dimensions to be submitted at their extremities to the same tem-