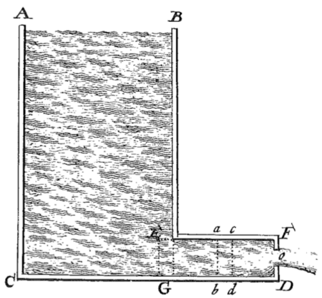
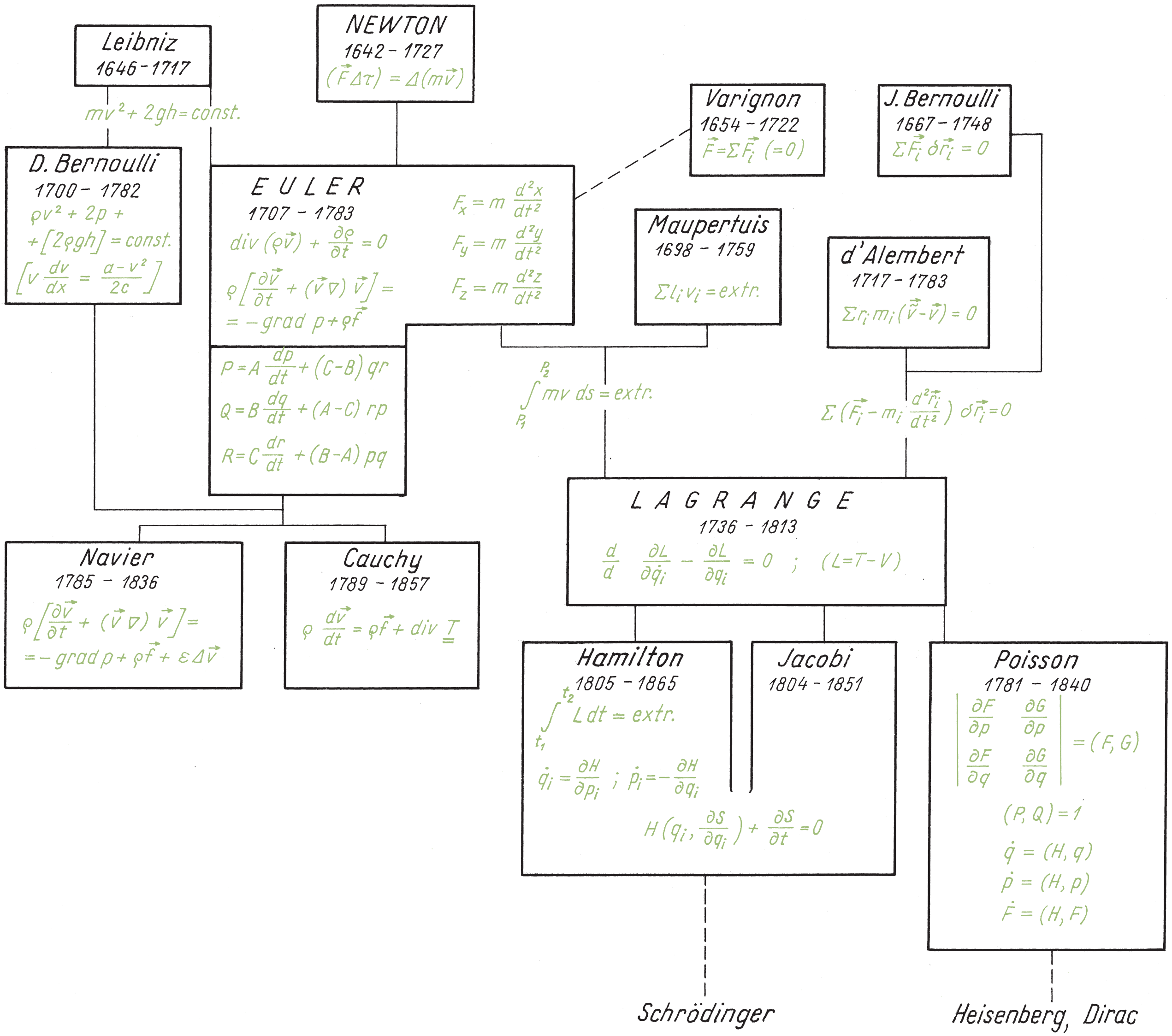
Module 3
Fluid Dynamics



UNIVERSITY OF COPENHAGEN

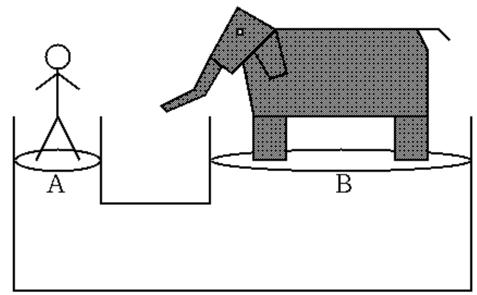
Mechanics’ Formalism over 2 centuries

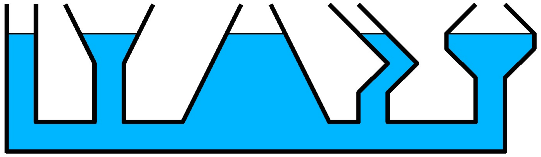


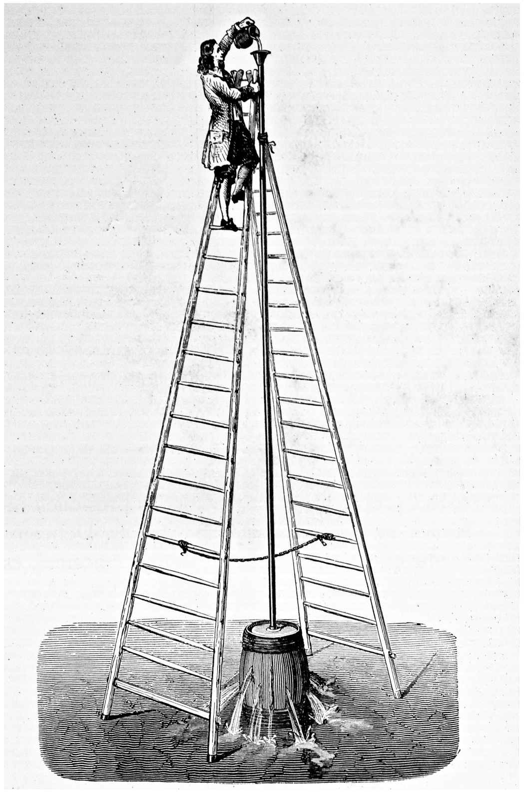
Source: Simonyi (2012)

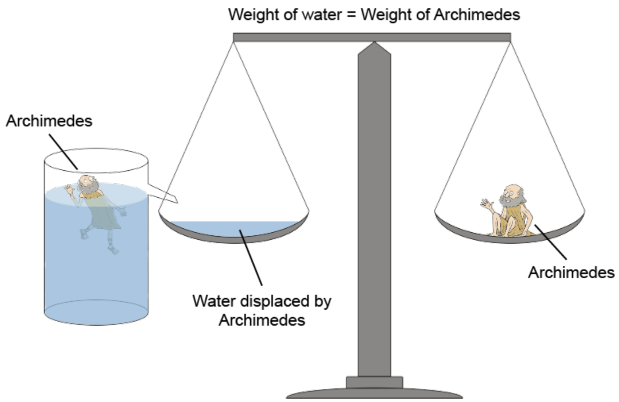
Before Daniel Bernoulli's *Hydrodynamica* (1738)

- Hydrostatics: Archimedes, Static pressure (Stevin, Pascal)

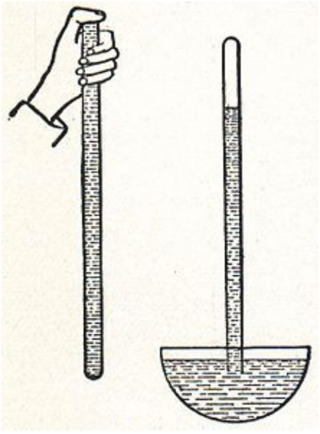






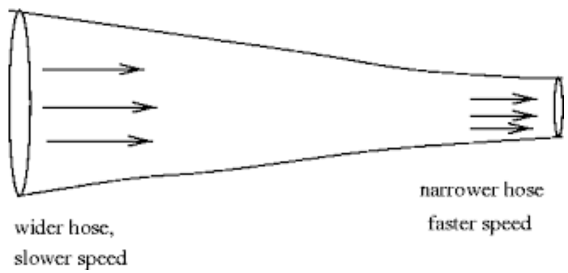


$$\Delta p = \rho g \Delta h$$



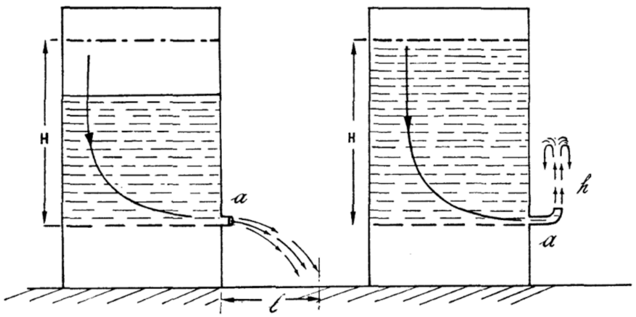
Before Daniel Bernoulli's Hydrodynamica (1738)

- Hydrodynamics: Velocity-Area law, Efflux, Resistance



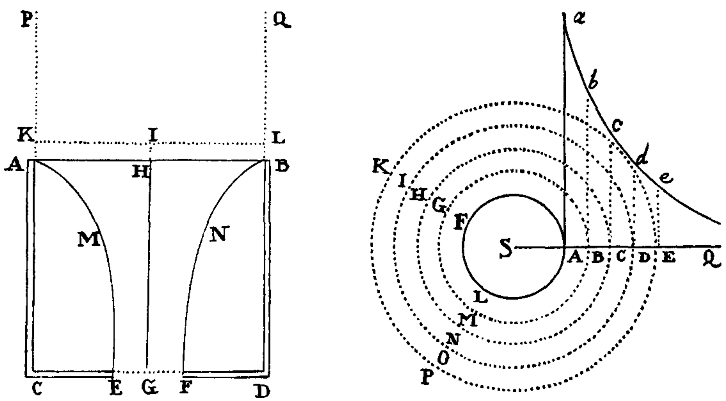
"where the river becomes shallower, the water becomes faster" (da Vinci)

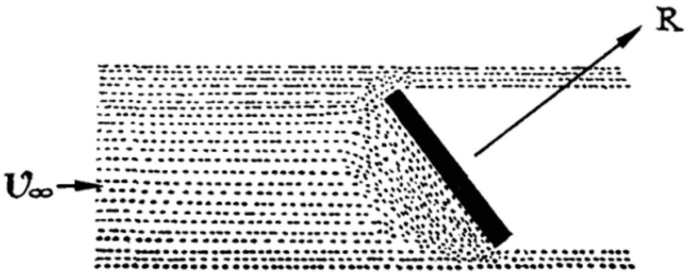
$$S_1 v_1 = S_2 v_2$$



$\ell \propto \sqrt{H}$ $v = \sqrt{2gh}$

Newton's Principia (Book 2)





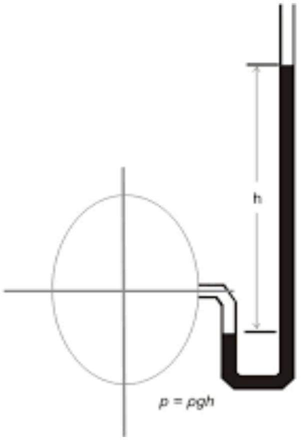
"Newton's" Sine-Square law

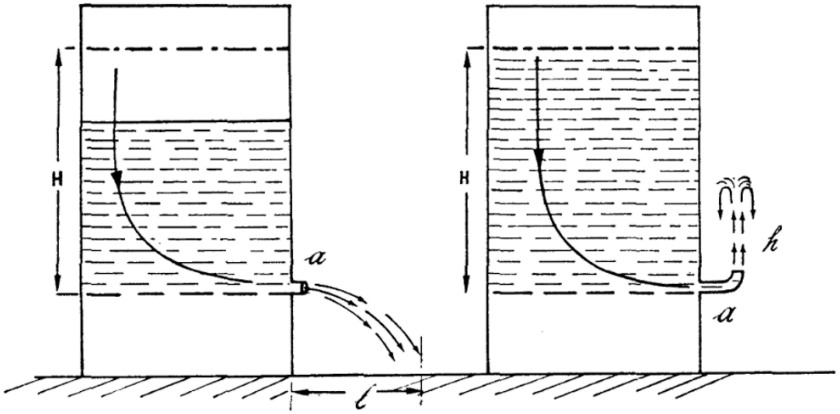
Context and Motivation

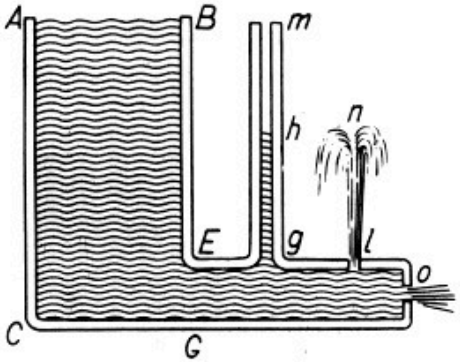
- Daniel Bernoulli was a doctor (blood pressure)
- *Hydraulico-statics*

The pressure of water at rest must be clearly distinguished from the pressure of flowing waters, although no one, as far I know, has been aware of this.

Stephen Hales

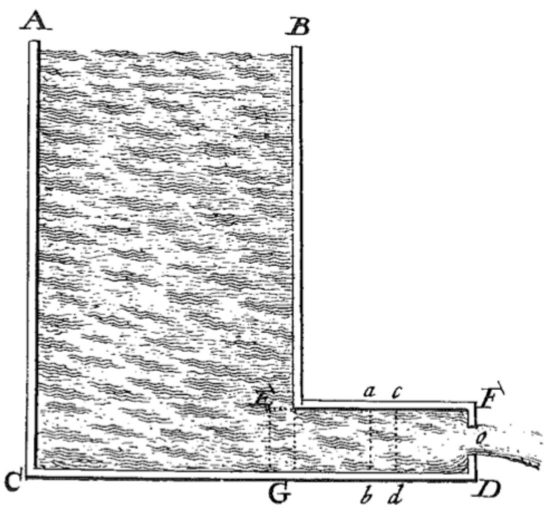






Original derivation of the Bernoulli equation

Problem (§ 5 – Chapter XII): Find the water pressure against the walls of pipe *ED*



Speed through orifice *o* $v_o \propto \sqrt{a}$

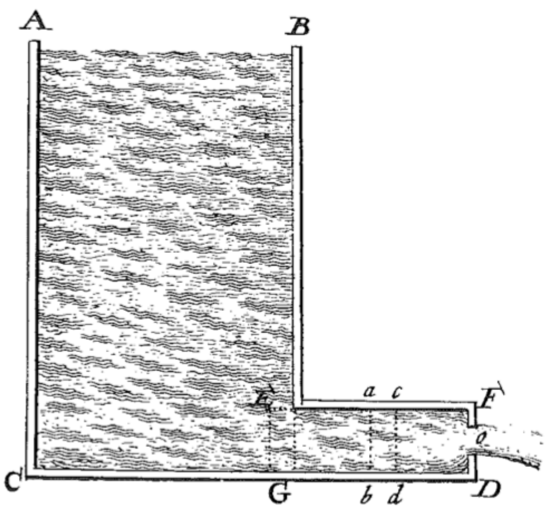
$A_{pipe}/A_o = n$ Speed in the pipe $v_{pipe} \propto \dfrac{\sqrt{a}}{n}$

Thus, the water in the pipe "tends to a greater motion"

The effect of the perforated plug can be interpreted as if its presence were compressing and retaining the water, pressing it against the walls of the reservoir and preventing it from expanding. This retention pressure will be greater as the velocity of the water circulating through the tube is slower (qualitative explanation)

Original derivation of the Bernoulli equation

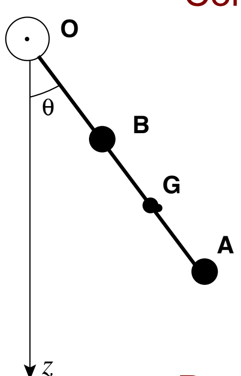
Problem (§ 5 – Chapter XII): Find the water pressure against the walls of pipe *ED*



It seems that the pressure of the lateral walls is proportional to the acceleration which the water would receive if the entire obstacle to motion were to vanish in an instant, so that [the water] might pour out directly into the air. Therefore, the problem is now: if during the flow of water through *o* the pipe *ED* were broken at *cd* at an instant, we seek the magnitude of the acceleration the volume element *acbd* would thence be about to obtain.

(How to determine the acceleration?)

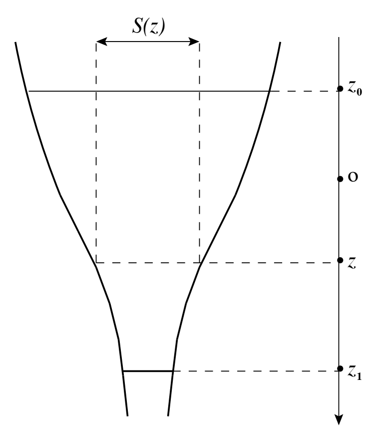
Compound pendulum



$$\frac{m_{\mathrm{A}}(v_{\mathrm{A}}^2/2g)+m_{\mathrm{B}}(v_{\mathrm{B}}^2/2g)}{m_{\mathrm{A}}+m_{\mathrm{B}}}=z_{\mathrm{G}}$$

$$\Sigma mv^2 = 2g\Sigma mh$$

Potential ascent = Actual descent



Analogy with efflux

$$\int_{z_0}^{z_1}\frac{v^2(z)}{2g}S(z)\,\mathrm{d}z=\int_{z_0}^{z_1}zS(z)\,\mathrm{d}z$$

Details in Darrigol (2000)

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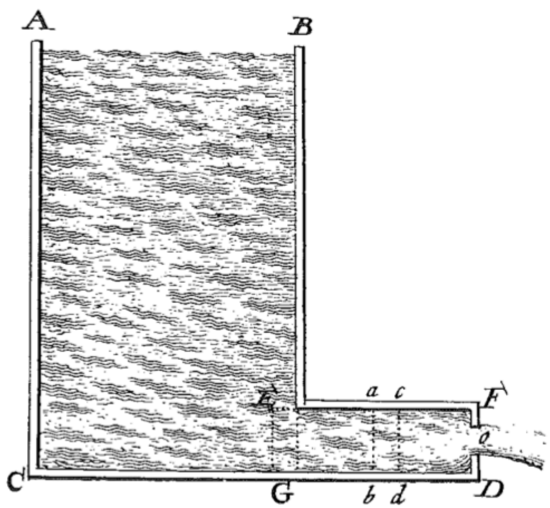
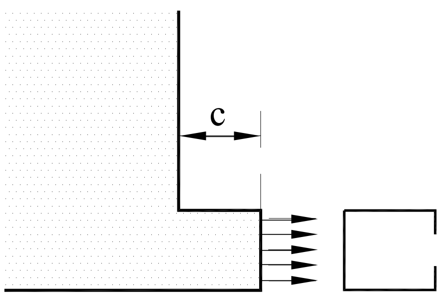
Problem (§ 5 – Chapter XII): Find the water pressure against the walls of pipe *ED*

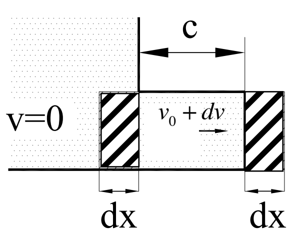


It seems that the pressure of the lateral walls is proportional to the acceleration which the water would receive if the entire obstacle to motion were to vanish in an instant, so that [the water] might pour out directly into the air. Therefore, the problem is now: if during the flow of water through *o* the pipe *ED* were broken at *cd* at an instant, we seek the magnitude of the acceleration the volume element *acbd* would thence be about to obtain.

Total increment of live force (kin energy) $\rho S v^2 dx + 2\rho S c v dv$

Equal to actual descent (potential energy) $2g\rho S a dx$





m = ρSdx

$$\Sigma m v^2 = 2g\Sigma m h$$

$$\frac{vdv}{dx} = \frac{2ga - v^2}{2c}$$

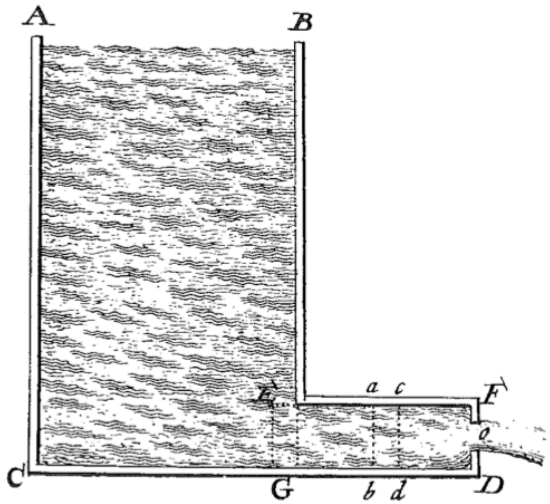
Pressure ∝ acceleration

$$p \propto \frac{vdv}{vdt} = \frac{2ga - v^2}{2c}$$

Details in Calero (2000)

Original derivation of the Bernoulli equation

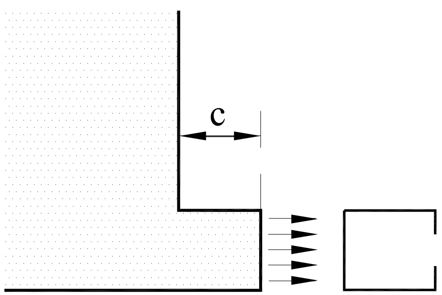
Problem (§ 5 – Chapter XII): Find the water pressure against the walls of pipe *ED*

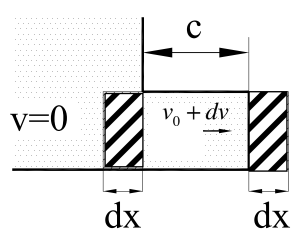


the other, *acdb*, is ejected; moreover, while the volume element at *E*, the mass of which is *n dx*, enters the pipe, it acquires the velocity *v* and the *live force nvv dx*, which entire *live force* was generated anew; indeed, the volume element at *E*, not yet having entered the pipe, had no motion on account of the infinite size of the vessel *AE*; to this *live force*, *nvv dx*, is to be added the increment of *live force* which the water at *Eb* receives while the volume element *ad* flows out, namely, $2ncv\, dv$; the sum is due to the *actual descent* of the volume element *n dx* through the height *BE* or *a*; therefore, one obtains $nvv\, dx + 2ncv\, dv = na\, dx$, or $\frac{v\, dv}{dx} = \frac{a - vv}{2c}$.

Total increment of live force (kin energy) $\rho S v^2 dx + 2\rho S c v dv$

Equal to actual descent (potential energy) $2 g \rho S a dx$





m = ρSdx

$$\Sigma m v^2 = 2 g \Sigma m h \qquad \frac{vdv}{dx} = \frac{2ga - v^2}{2c}$$

Pressure ∝ acceleration $\quad p \propto \frac{vdv}{vdt} = \frac{2ga - v^2}{2c}$

Details in Calero (2000)

Original derivation of the Bernoulli equation

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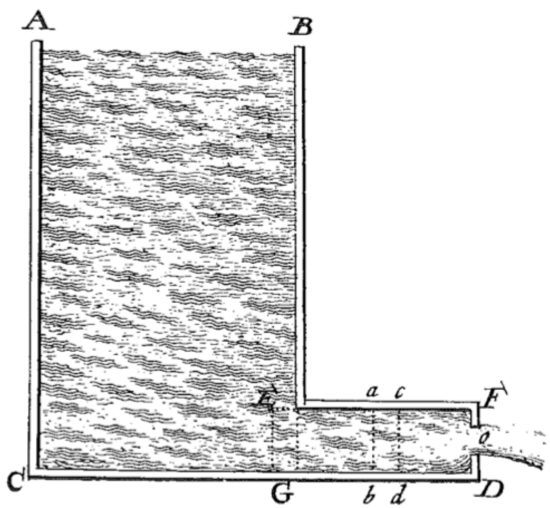
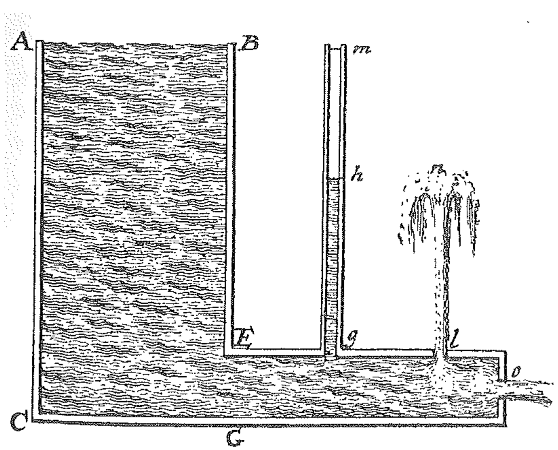


Pressure ∝ acceleration $p \propto \dfrac{vdv}{vdt} = \dfrac{2ga - v^2}{2c}$

Since $v = \dfrac{\sqrt{a}}{n}$ $\quad p \propto \dfrac{2ga}{2c}\left(1 - \dfrac{1}{n^2}\right)$ $\quad \dfrac{2ga}{2c}\left(1 - \dfrac{1}{n^2}\right) = kp = \rho g h$

If the orifice is infinitesimal $n \to \infty$ Pressure is hydrostatic $p = \rho g a$



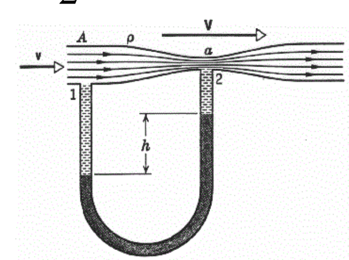
$$h = \left(\frac{n^2 - 1}{n^2}\right) a$$

That is Bernoulli's equation in the original

Today we would write

$$p = p_0 - \frac{1}{2}\rho v^2 \quad \text{or} \quad p + \frac{1}{2}\rho v^2 = \text{constant}$$

Can you see/show that they are equivalent?



Questions for discussion

- As an experimentalist, I due fancy that his theories can be shown easily by experiments! :-) For example, the system as in Figure 73 would not be that difficult to set up.
- Why are the units so strange? Did they simply not care about the units in the same way at this time? No matter what it makes some of the derivations kind of nonsensical.
- Why does Bernoulli not mention the quantity energy?

Activity for discussion

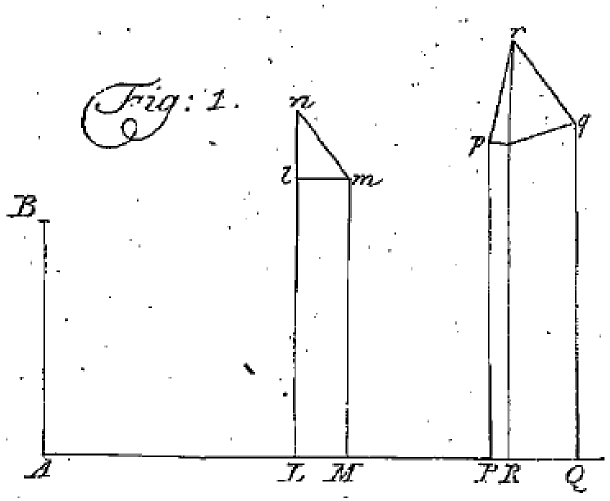
- Compare a modern derivation of Bernoulli's equation with the original one. What are the differences and similarities? Considering the learning perspective, could there be advantages of understanding the original?

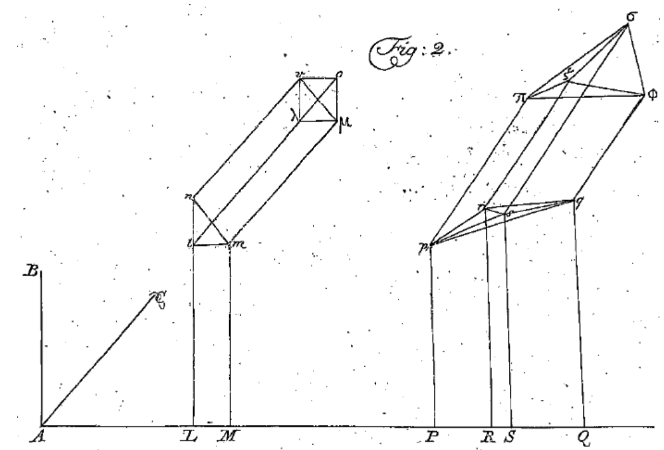
Euler's *Principia motus fluidorum* (1752)

Here are treated the elements of the theory of the motion of fluids in general, the whole matter being reduced to this: given a mass of fluid, either free or confined in vessels, upon which an arbitrary motion is impressed, and which in turn is acted upon by arbitrary forces, to determine the motion carrying forward each particle, and at the same time to ascertain the pressure exerted by each part, acting on it as well as on the sides of the vessel. At first in this memoir, before undertaking the investigation of these effects of the forces, the Most Famous Author[1] carefully evaluates all the possible motions which can actually take place in the fluid. Indeed, even if the individual particles of the fluid are free from each other, motions in which the particles interpenetrate are nevertheless excluded, since we are dealing with fluids that do not permit any compression into a narrower volume. Thus it is clear that an arbitrary small portion of fluid cannot receive a motion other than the one which constantly conserves the same volume; even though meanwhile the shape is changed in any way. It would hold indeed, as long as no elementary portion would be compressed at any time into a smaller volume; furthermore[2] if the portion expanded into a larger volume, the continuity of the particles was violated, these were dispersed and no longer clinged together, such a motion would no longer pertain to the science of the motion of fluids; but individual droplets would separately perform their motion. Therefore, this case being excluded, the motion of the fluids must be restricted by this rule that each small portion must retain for ever the same volume; and this principle restricts the general expressions of motion for elements of the fluid. Plainly, considering an arbitrary small portion of the fluid, its individual points have to be carried by such a motion that, when at a moment of time they arrive at the next location, till then they occupy a volume equal to the previous one; thus if, as usual, the motion of a point is decomposed parallel to fixed orthogonal directions, it is necessary that a certain established relation hold between these three velocities, which the Author has determined in the first part.

Euler’s *Principia motus fluidorum* (1752)

Part 1: Constant volume of the fluid particle (incompressibility)





Euler's *Principia motus fluidorum* (1752)

Part 2: Forces (pressure, gravity) and motion (Newton's 2nd law)

Acceleration (kinematics)

$u\ (x,\ y,\ t) \quad v\ (x,\ y,\ t)$

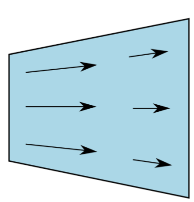
$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial t}dt$$

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy + \frac{\partial v}{\partial t}dt$$

$$\frac{du}{dt} = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \frac{\partial u}{\partial t}$$

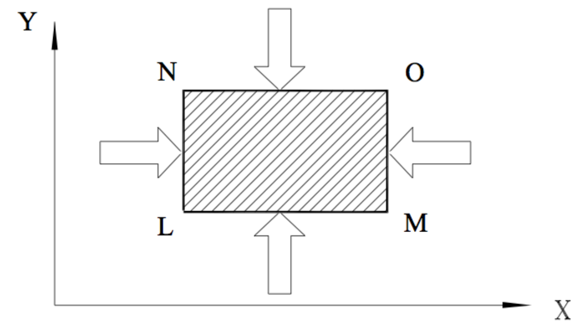
$$\frac{dv}{dt} = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + \frac{\partial v}{\partial t}$$

Convective acceleration



Forces (dynamics)

Internal pressure



$\mathbf{L}\!: \ p \qquad \mathbf{M}\!: \ p + \frac{\partial p}{\partial x}dx$

$\mathbf{N}\!: \ p + \frac{\partial p}{\partial y}dy \qquad \mathbf{O}\!: \ p + \frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy$

$\textbf{OX}\text{ Axis: } -\frac{\partial p}{\partial x}dxdy \qquad \textbf{OY}\text{ Axis: } -\frac{\partial p}{\partial y}dxdy$

Force due to pressure gradient

If gravity (g) along OX

"Accelerating forces"

Direction OX

$$g - \frac{1}{\rho}\frac{\partial p}{\partial x} = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \frac{\partial u}{\partial t}$$

Direction OY

$$-\frac{1}{\rho}\frac{\partial p}{\partial y} = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + \frac{\partial v}{\partial t}$$

Pressure can also change with time

$$dp = \frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy + \frac{\partial p}{\partial t}dt$$

Substituting

$$\frac{dp}{\rho} = gdx - \left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \frac{\partial u}{\partial t}\right)dx - \left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + \frac{\partial v}{\partial t}\right)dy + \frac{1}{\rho}\frac{\partial p}{\partial t}dt$$

Euler's *Principia motus fluidorum* (1752)

Part 2: Forces (pressure, gravity) and motion (Newton's 2nd law)

$$\boxed{\frac{dp}{\rho} = g\,dx - \left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \frac{\partial u}{\partial t}\right)dx - \left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + \frac{\partial v}{\partial t}\right)dy + \frac{1}{\rho}\frac{\partial p}{\partial t}dt}$$

That (!) it must be integrable, i.e. *dp* is an exact differential

- *g* is *per se* integrable
- No restrictions for ∂p/∂t, thus integrable
- Therefore, equality of cross-derivatives yields $\frac{\partial}{\partial y}\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \frac{\partial u}{\partial t}\right) = \frac{\partial}{\partial x}\left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + \frac{\partial v}{\partial t}\right)$

After manipulations $\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + \frac{\partial}{\partial t}\right)\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) = 0$

Which implies

$$\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = 0$$

From part 1 $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$ incompressible

From part 2 $\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = 0$ "irrotational"

Ideal Fluid
Potential Flow
Velocity Potential

Euler's *Principia motus fluidorum* (1752)

"Now we shall be able to ascertain the pressure p itself, which is absolutely necessary for the perfect determination of the motion of the fluid"

Combining $\frac{dp}{\rho} = gdx - \left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \frac{\partial u}{\partial t} \right) dx - \left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + \frac{\partial v}{\partial t} \right) dy + \frac{1}{\rho}\frac{\partial p}{\partial t} dt$ and $\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$

leads to $\frac{dp}{\rho} = gdx - udu - vdv - \frac{\partial u}{\partial t} dx - \frac{\partial v}{\partial t} dy$ *static pressure field ∂p/∂t = 0*

Introducing the function S (velocity potential) $dS = udx + vdy$

yields $\frac{dp}{\rho} = gdx - udu - vdv - d\frac{\partial S}{\partial t}$

Integrating $\frac{p}{\rho} = gx - \frac{1}{2}(u^2 + v^2) - U + Cte$

Voilá: Bernoulli´s equation from Euler's!!!

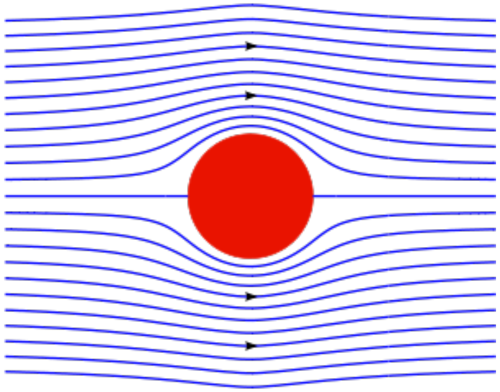
Read Euler, read Euler, he is the master of us all! (Laplace)

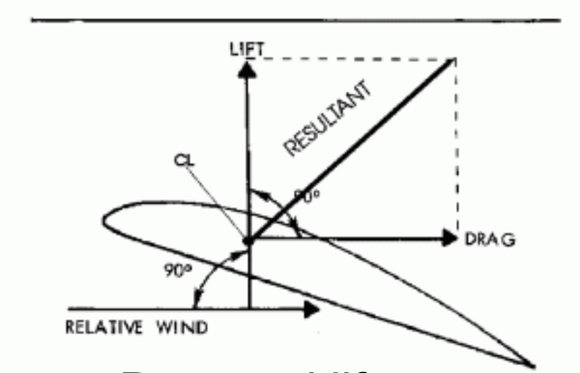
Be aware that we barely scratched the surface…


Turbulence


Viscosity


Vortices


Resistance



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End of module feedback

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- Enter the HISPHYSKU room
- Fill out the short (anonymous) survey
- Tak skal du have!