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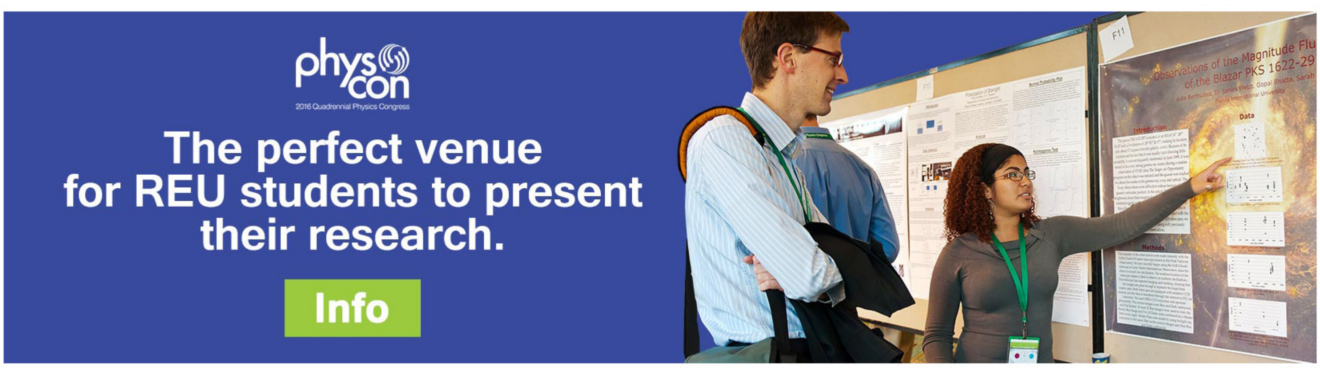

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Is Newton's second law really Newton's?

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When we call the equation $\mathbf{f} = m\mathbf{a}$ "Newton's second law," how much historical truth lies behind us? Many textbooks on introductory physics and classical mechanics claim that the *Principia*'s second law becomes $\mathbf{f} = m\mathbf{a}$, once Newton's vocabulary has been translated into more familiar terms. But there is nothing in the *Principia*'s second law about acceleration and nothing about a rate of change. If the *Principia*'s second law does not assert $\mathbf{f} = m\mathbf{a}$, what does it assert, and is there some other axiom or some proposition in the *Principia* that does assert $\mathbf{f} = m\mathbf{a}$? Is there any historical truth behind us when we call $\mathbf{f} = m\mathbf{a}$ "Newton's second law"? This article answers these questions. © *2011 American Association of Physics Teachers.*
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I. INTRODUCTION

In courses and textbooks on classical mechanics, the name "Newton's second law" generally refers to the equation $\mathbf{f} = m\mathbf{a}$. In the history of science, on the other hand, "Newton's second law" generally refers to the second of the three laws of motion that Newton records in his great work, the *Principia*:[1]

> LAW II. *A change in motion is proportional to the motive force impressed and takes place along the straight line in which that force is impressed.*[2]

In this article, to reduce the risk of confusion, we will distinguish these different senses of "Newton's second law," the modern sense and the historical sense, with different names: the "equation $\mathbf{f} = m\mathbf{a}$" and the "*Principia*'s second law." Applying this clarification to our ambiguous title, uncovers the real question: "Is the equation $\mathbf{f} = m\mathbf{a}$ really Newton's?"

If we assume the equation $\mathbf{f} = m\mathbf{a}$ is called Newton's second law for good reason, then it would be natural to expect the *Principia*'s second law to be a Newtonian version of that equation, a version disguised in the *Principia*'s unfamiliar vocabulary. But this is not the case. Well then, what exactly does the *Principia*'s second law assert? And if not the *Principia*'s second law, then which of the *Principia*'s assertions, if any, is Newton's version of $\mathbf{f} = m\mathbf{a}$? In the pages ahead, we shall answer these two questions.

Now you might think the first question—What exactly does the *Principia*'s second law assert?—would be a simple one to answer; after all, it should be sufficient to look up Newton's definitions for "change in motion" and "motive force" in the *Principia*. Yet, the definition for "motive force" is confusing and the definition for "change in motion" is missing. The unclear and incomplete account of the second law in the *Principia* has left the intended meaning of the *Principia*'s second law an unsettled question for over three centuries. Not that historians of science haven't settled on a standard interpretation, a received view, of the *Principia*'s second law—they have[3]—it's just that this standard interpretation is a serious misinterpretation, even more mistaken, but in a different way, than the misinterpretation found in so many textbooks on introductory physics and Newtonian mechanics: that the *Principia*'s second law asserts the *rate of change of momentum is proportional to the force impressed*.[4] The interpretation presented here, although not (yet) the standard interpretation, has behind it overwhelming evidence, including the direct testimony of Newton, who, in an unpublished manuscript from the early 1690s, carefully wrote out the precise meaning of the second law as he understood it.[5]

II. THE EXPECTED SECOND LAW IN THE *PRINCIPIA*

In this section, we will not make historical arguments backed by historical evidence, but rather plausibility arguments that strive to make the meaning of the second law as Newton understood it so plausible and natural that when we do supply historical evidence in Sec. III A, namely Newton's own testimony on the meaning of the second law, the reader's reaction will be, "Well, of course, that's the assertion we expected Newton to choose."

To see whether we can anticipate Newton's own interpretation of his second law, we ask the same question—What would we expect Newton to take as his second law in the *Principia*?—in two different ways.

A. The natural sequel to the first law

Here is the first way: What would be the natural sequel to the first law in the *Principia*?

> LAW I. *Every body perseveres in its state of being at rest*[6] *or of moving uniformly straight forward except in so far as it is compelled to change its state by forces impressed.*[2]

A statement of the logical form p except in so far as q would generally be taken as meaning if not q then p, or equivalently, if not p then q. It follows that we may rewrite the first law in the equivalent form:

LAW I (equivalent version). *If a body deviates from its state of rest or of uniform straight line motion, then some impressed force compels that deviation.*

Let us introduce notation for these deviations which, according to the first law, signal the presence of an impressed force. Suppose a body in motion, arriving at the point P, would have gone on to traverse the line segment $\overrightarrow{PL}$ in a given small time h, had its speed and direction at P been uniformly continued. But instead, suppose the body traverses the arc segment (or possibly line segment) PQ in that same time h. The directed line segment $\overrightarrow{LQ}$ is then the

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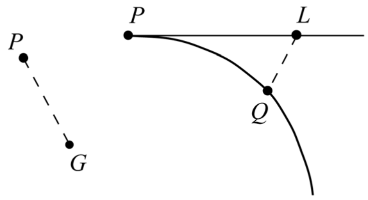

Fig. 1. Resting and moving deflections, $\overrightarrow{PG}$ and $\overrightarrow{LQ}$, generated in a given time.

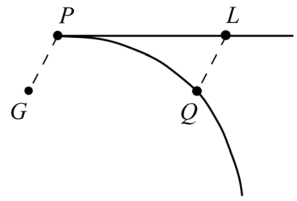

Fig. 2. The equality of the deflections $\overrightarrow{LQ}$ and $\overrightarrow{PG}$, generated by a given force in a given time on a given body, when the body is in motion at P and at rest at P, respectively.

deviation (or, to use Newton's word, *deflection*) from uniform straight line motion generated in the time h. We call $\overrightarrow{LQ}$ a *moving deflection*. Now imagine that a body, starting from rest at P, traverses the directed line segment[7] $\overrightarrow{PG}$ in a given time. We call $\overrightarrow{PG}$ a *resting deflection*[8] (see Fig. 1).

We now rephrase the first law in terms of these deflections:

LAW I (equivalent version). *For any body, if either a moving deflection or a resting deflection*, $\overrightarrow{LQ}$ or $\overrightarrow{PG}$, *is nonzero, then some impressed force compels that deviation.*

In Newton's mechanics, the word "force" may refer, depending on the context, to a source or mechanism (gravity or magnetism, say), to an action (a specific "thrust" or "pull") produced by that mechanism, or to the observed effect (a moving or resting deflection, for example) generated by that action on a given body. In the *Principia*, an "impressed force," for instance the "impressed force" of the first and second laws, always refers to an action, that is, a particular "thrust" or "pull." As such, the "impressed force" in these laws has a particular magnitude and direction.

According to the first law, a nonzero moving or resting deflection signals the presence of an impressed force. What could be more natural for Newton than to characterize the magnitude and direction of such an impressed force in terms of the length and direction of the observed effect (that is, the deflection) generated by that impressed force on the given body? But to be able to characterize the impressed force in terms of the observed deflection, the same force acting on the same body must generate in the same time the same deflection, no matter the speed or direction of the body. For without this constraint, the same given force (intuitively, the same "thrust") could end up having one magnitude and direction when it acts on a given body in motion and a different magnitude and direction when it acts on the same body at rest. Without this constraint, the same given impressed force (or "thrust") could even generate a finite acceleration on a body in motion, but an infinite acceleration (as in an instantaneous impulse) on the same body at rest: if the deflections $\overrightarrow{LQ}$ and $\overrightarrow{PG}$, generated by the same force, could be different, then the limits of $2(\overrightarrow{LQ}/h^2)$ and $2(\overrightarrow{PG}/h^2)$ as $h \to 0$ could be different, the first finite and the second infinite, for example, and these limits (as we will show in Sec. IV A) are the vector accelerations generated by the given force on the body in motion and at rest.

It would be natural, in a sequel to the first law, for Newton to rule out such unruly behavior, in order to ensure that a given impressed force may be characterized in terms of the observed deflection:

NATURAL SEQUEL TO THE FIRST LAW. *The same force (or instantaneous impulse) acting on the same body generates in the same time the same deflection, whether the body is in motion or at rest*: $\overrightarrow{LQ} = \overrightarrow{PG}$.[9]

Thus, while (the contrapositive of) the first law announces the existence of an impressed force when a nonzero moving or resting deflection has been observed, its natural sequel (see Fig. 2) supplies the license to characterize that impressed force in terms of those nonzero deflections. As we shall see, this natural sequel is the *Principia*'s second law as Newton understood it.

B. The natural generalization of Huygens's hypotheses

"What books will prepare me for reading your *Principia*?" When asked this question in 1691 by Dr. Richard Bentley, a classical scholar who later became Master of Trinity College, Cambridge, Newton recommended a daunting list of mathematical and scientific texts, but singled out one in particular: "These are sufficient for understanding my book: but if you can procure Hugenius's *Horologium oscillatorium*, the perusal of that will make you much more ready."[10] As his own well-thumbed copy attests, Newton greatly admired the *magnum opus* of the Dutch scientist Christiaan Huygens: *Horologium Oscillatorium,* published in 1673, 14 years before the first edition of the *Principia*. Part II of the *Horologium Oscillatorium*, on "The falling of heavy bodies and their motion in a cycloid," opens with three "hypotheses" of motion:[11]

> HYPOTHESIS I. *If there were no gravity, and if the air did not impede the motion of bodies, then any body will continue its given motion with uniform velocity in a straight line.*
>
> HYPOTHESIS II. *By the action of gravity, whatever its sources, it happens that bodies are moved by a motion composed both of a uniform motion in one direction or another and of a motion downward due to gravity.*
>
> HYPOTHESIS III. *These two motions can be considered separately, with neither being impeded by the other.*

To illustrate these hypotheses, Huygens drew the lovely Fig. 3, where we see the motion $\overrightarrow{CB}$ generated by (uniform) gravity on the body at rest "composed" with the uniform motion $\overrightarrow{CD}$ that would have taken place in the same time had the body been projected from the point C (with various speeds and directions) in the absence of gravity, the two motions composing "separately, with neither being impeded by the other," and producing the actual motion along the arc $\overrightarrow{CE}$.

So far we have been concerned in Sec. II, not with historical evidence, but with the question, "What would we expect

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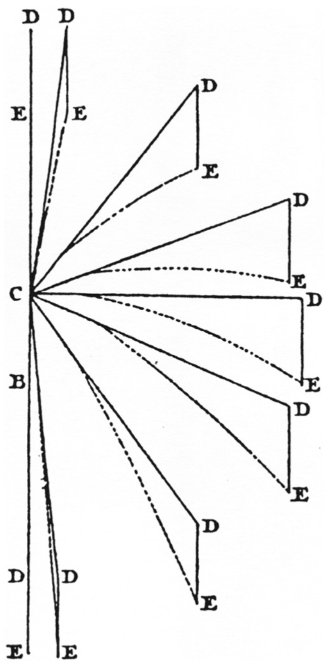

Fig. 3. The figure drawn by Huygens in *Horologium Oscillatorium* to illustrate his three hypotheses of motion, showing the same body given different initial projections $\overrightarrow{CD}$ that in each case compound independently with the effect $\overrightarrow{CB}$ due to (uniform) gravity on the same body at rest: $\overrightarrow{CE} = \overrightarrow{CD} + \overrightarrow{CB}$ or, equivalently, $\overrightarrow{DE} = \overrightarrow{CB}$.

Newton to write down as the *Principia*'s second law?" In Sec. II A, we asked this question in one way: What would be the natural sequel to the first law? Now we ask it in a second way: What would be the natural generalization of Huygens's hypotheses II and III?

As he came to write the *Principia*, Newton would have been very familiar with both the form and fruitfulness of these hypotheses. But restricted as they were to uniform gravity near the surface of the Earth, Huygens's hypotheses had a limited scope that could never have encompassed Newton's much broader investigation: to deduce "the motions of the planets, the comets, the moon, and the sea."[12] What could be more natural than for Newton to fashion axioms of motion for the *Principia* by generalizing the hypotheses in the *Horologium Oscillatorium*? Recognizing the role of Hypothesis III, to ensure the motions in Hypothesis II "compose" independently, and understanding that "a motion due to gravity" in the second hypothesis means the motion due to gravity on the body at rest, we make the obvious replacements— "any force" for "gravity," "any motion uniformly continued" for "uniform motion," and "any direction" for the direction "downward" — to obtain the natural generalization of Huygens's hypotheses II and III: *By the action of any force in any direction, whatever its sources, a body is moved by a motion independently compounded of the motion of the body uniformly continued and of a motion due to the same force on the same body at rest.*

It is now a simple matter to translate this generalization into the language of deflections using our P, L, G, Q notation. Suppose a body, arriving at the place P, would have gone on to traverse the segment $\overrightarrow{PL}$, had its motion (direction and speed) at P been uniformly continued for a given time h, but instead, under the influence of a force, suppose the body traverses the arc segment PQ in that same time. Suppose, under the influence of this same force, this same body, now resting at P, would have traversed the segment $\overrightarrow{PG}$ in the time h. Then, because the inertial motion compounds independently with the "motion due to the same force on the same body at rest," we have $\overrightarrow{PQ} = \overrightarrow{PL} + \overrightarrow{PG}$ or, equivalently, $\overrightarrow{LQ} = \overrightarrow{PG}$, which yields the following translation (see Fig. 2):

THE NATURAL GENERALIZATION OF HUYGENS'S HYPOTHESES II AND III. *The same force acting on the same body generates in the same time the same deflection, whether the body is in motion or at rest*: $\overrightarrow{LQ} = \overrightarrow{PG}$.

This generalization is the same axiom we found in Sec. II A to be the natural sequel to the first law in the *Principia*.

III. THE *PRINCIPIA*'S SECOND LAW ACCORDING TO NEWTON

As the natural sequel to the *Principia*'s first law as well as the natural generalization of Huygens's hypotheses II and III in the *Horologium Oscillatorium*, the equality $\overrightarrow{LQ} = \overrightarrow{PG}$, between the moving and resting deflections (generated in a given time by a given force on a given body), becomes the assertion we would expect Newton to choose for the *Principia*'s second law. But did he? Is there convincing historical evidence that the equality $\overrightarrow{LQ} = \overrightarrow{PG}$ is the meaning of the *Principia*'s second law as Newton understood it?

A. Newton's testimony

Concerning arguments over the interpretation of a textual passage, it is rare when the author of the passage weighs in, the author being generally unknown, mute, or dead. In the case of scientific passages, such as the *Principia*'s second law—*A change in motion is proportional to the motive force impressed and takes place along the straight line in which that force is impressed*—the author's interpretation generally silences debate. Did Newton ever record his own interpretation of the second law with sufficient clarity to make its meaning plain? Yes.

By the time the Scottish mathematician David Gregory arrived at Cambridge for a visit in May of 1694, Newton had been hard at work for some months on revisions planned for the second edition of the *Principia*. During their time together, Newton was uncommonly expansive, describing for Gregory the outline of his more radical plans—a new mathematical and logical structure, based on the notion of curvature, for the early portions of Book I—and sharing with him manuscript pages that revealed details of this new scheme. In the end, whether for lack of time, energy, or interest, most of these planned revisions, not just the major large-scale changes but most of the merely stylistic improvements as well, never made it into print, although bits and pieces of the curvature scheme appeared as new corollaries or alternate solutions in the second 1713 edition.

Manuscript pages found in the Portsmouth Collection of the Cambridge University Library[13] and composed during the years 1692–1693 preserve this flurry of projected revisions to Book I of the 1687 *Principia*. On four loose sheets, we find Newton at work not on his more radical revisions, but on purely stylistic matters, as he tries out eight (!) different rewordings of the second law.[14] Seven of these are crossed out; the one rewording not cancelled—

> LAW II. *All new motion by which the state of a body is changed is proportional to the motive force*

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impressed, and occurs from the place which the body would otherwise occupy towards the goal at which the impressed force aims.

—is accompanied by the following explanatory passage and figure (see Fig. 4), which together make the meaning of the *Principia*'s second law crystal clear:

> NEWTON'S INTERPRETATION OF HIS SECOND LAW. If the body A should, at its place A where a force is impressed upon it, have a motion by which, when uniformly continued, it would describe the straight line Aa, but by the impressed force be deflected from this line into another one Ab and, when it ought to be located at the place a, be found at the place b, then, because the body, free of the impressed force, would have occupied the place a and is thrust out from this place by that force and transferred therefrom to the place b, the translation of the body from the place a to the place b will, in the meaning of this Law, be proportional to this force and directed to the same goal toward which this force is impressed.
>
> Whence, if the same body deprived of all motion and impressed by the same force with the same direction, could in the same time be transported from the place A to the place B, the two straight lines AB and ab will be parallel and equal. For the same force, by acting with the same direction and in the same time on the same body whether at rest or carried on with any motion whatever, will in the meaning of this Law achieve an identical translation towards the same goal, and in the present case the translation is AB where the body is at rest before the force was impressed and ab where it was there in a state of motion.

Note the phrase "in the meaning of this Law," which Newton inserted not once but twice to make clear that this passage describes no mere corollary or application of the second law, but the very meaning of that law,[15] and note as well the sentence which captures that meaning most concisely: "the two straight lines AB and ab will be parallel and equal. For the same force, by acting with the same direction and in the same time on the same body whether at rest or carried on with any motion whatever, will in the meaning of this Law achieve an identical translation towards the same goal." Even more concisely, *the moving deflection equals the resting deflection*: $\overrightarrow{ab} = \overrightarrow{AB}$. According to his own testimony, this equality is the *Principia*'s second law as Newton understood it.[16]

Of course, even disguised by Newton's different notation for the deflections ($\overrightarrow{AB}$ for the resting deflection $\overrightarrow{PG}$, $\overrightarrow{ab}$ for the moving deflection $\overrightarrow{LQ}$), we recognize this law at once: it was the natural sequel to the first law and the natural generalization of Huygens's hypotheses II and III, and as such it was the natural candidate for the second law in the *Principia*. Apparently, the law we would expect Newton to write down as the *Principia*'s second law is the very law that he did write down.

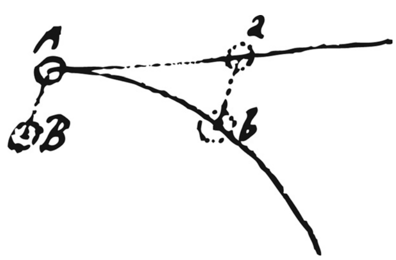

Fig. 4. Newton's original hand-drawn figure illustrating the continuous force case of his second law: the moving deflection equals the resting deflection, $\overrightarrow{ab} = \overrightarrow{AB}$.

In the standard interpretation of the *Principia*'s second law, the law applies to an instantaneous impulse directly and to a (continuous) force only indirectly, through a complicated approximation by impulses in series. But Newton's hand-drawn figure (Fig. 4), with its continuously turning tangent, tells us that the law was actually intended to apply directly to a (continuous) force as well. Hence, we have the following statement (illustrated in Fig. 5) of the *Principia*'s second law as Newton understood it. We call it the Compound Second Law, using the word "compound" for two reasons: the law applies directly to both an instantaneous impulse and a continuous force, and the law tells us the inertial motion of the body compounds independently with the motion that would have been generated by the given force on the body at rest.

COMPOUND SECOND LAW (the *Principia's* second law as Newton understood it). *The same force (or instantaneous impulse) acting on the same body generates in the same time the same deflection, whether the body is in motion or at rest:* $\overrightarrow{LQ} = \overrightarrow{PG}$. *In other words, the motion (speed and direction) of the body uniformly continued compounds independently with the motion that would have been generated by the given force on the same body at rest:* $\overrightarrow{PQ} = \overrightarrow{PL} + \overrightarrow{PG}$.[17]

Observe that the mass (Newton's "quantity of matter") does not figure into the compound second law. As Newton understood his three laws of motion, mass enters nontrivially only in the third law:

> LAW III. *To any action there is always an opposite and equal reaction; in other words, the actions of two bodies upon each other are always equal and always opposite in direction.*

This law, translated into the Newtonian language of deflections, could take the form $M(\overrightarrow{LQ}/h^2) = -m(\overrightarrow{lq}/h^2)$, where M, $\overrightarrow{LQ}$, m, and $\overrightarrow{lq}$ are the quantities of matter and moving deflections (generated in the time h) of the two interacting bodies. Newton's "massless" understanding of the second law might seem odd, especially to readers used to the mass being an integral component of the equation $\mathbf{f} = m\mathbf{a}$. On the other hand, although "massless," the *Principia*'s second law still applies to the many problems in the *Principia* where mass plays no significant role, the fundamental problem of Book I—to investigate the motion of a body in orbit about a fixed center—being the primordial example.

B. Motive force? change in motion?

As delighted as we are to know how Newton understood the second law, we still have a problem. Newton has told us

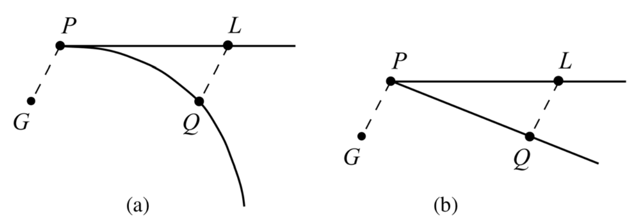

Fig. 5. Illustrating the *Principia*'s second law as Newton understood it, $\overrightarrow{LQ} = \overrightarrow{PG}$: (a) a (continuous) force and (b) an instantaneous impulse.

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that the Compound Second Law is the meaning of the *Principia*'s second law; yet you wouldn't know it from the way he stated that second law:

> LAW II. *A change in motion is proportional to the motive force impressed and takes place along the straight line in which that force is impressed.*

To make this statement become one with its meaning, we must uncover the intended meanings for the "change in motion" and the "motive force." As we noted, the *Principia*'s definition of the former is absent, while its definition of the latter is ambiguous, which has made it difficult for commentators to unearth the intended meaning of the second law. But we have an advantage: knowing the meaning of the second law, we can work backward to the correct definitions for "motive force" and "change in motion."

The *Principia* begins with a section called *Definitions*. From that section, here are the three statements—two definitions and one comment—relevant to the meaning of the "motive force":[18]

> DEFINITION VII. *The accelerative quantity of centripetal force [the accelerative force, "for the sake of brevity"] is the measure of this force that is proportional to the velocity which it generates in a given time.*
>
> DEFINITION VIII. *The motive quantity of centripetal force [the motive force] is the measure of this force that is proportional to the motion which it generates in a given time.*
>
> … motive force [arises] from accelerative force and quantity of matter jointly.

If the motive force is the product of the accelerative force and quantity of matter,[19] and the accelerative force is the "velocity which it generates in a given time," then the motive force has the form $M\Delta V$, where M is the "quantity of matter" and ΔV stands for some kind of Newtonian change in speed or velocity generated in a given time by the given force. Now we see the ambiguity: Is this ΔV generated on the body at rest? In motion? Is ΔV a scalar or does it have a direction? How exactly is ΔV measured?

To resolve this ambiguity, we recall Newton's understanding of the second law—that the observed effect generated in a given time by a given force on a given body does not depend on the speed or direction of that body—and then note that the measure of that effect called the "motive force," because its definition appears before the laws of motion in the *Principia*, is under no such constraint. Thus, unless it is defined in a way that prevents it, the motive force $M\Delta V$ could in principle depend, not just on the given force and given body, but also on the speed or direction of the body, which would make it a poor measure of the given force. To prevent such a dependence, the generated velocity ΔV must be defined for a body having some standard speed and direction, the only reasonable standard being rest. This argument suggests the following clarification of Definition VIII, where we assume that ΔV, Newton's "velocity … generate[d] in a given time" in Definition VII, refers to $\overrightarrow{PG}/h$, the (average) velocity generated from rest:

DEFINITION VIII (clarified). *The motive quantity of centripetal force (or, more briefly, the motive force) is the motion which that centripetal force generates in a given time from rest. In other words, if under the influence of a centripetal force a body with quantity of matter M moves from rest at P, describing the line segment PG in a given time h, then the motive force is the quantity* $M(PG/h)$ *taking place along the line PG, that is, the quantity* $M(\overrightarrow{PG}/h)$.

We should have expected Newton to characterize (the direction and magnitude of) a given centripetal force using the observed effect on the given body at rest, for this is just how scientists in the 17th century characterized gravity. Mersenne, Riccioli, and Huygens, for example, all used the distance fallen in the first second to measure surface gravity.[20]

What about the other ingredient in the statement of the second law, the "change in motion"? Oddly, the *Principia* gives no definition for this quantity, a real stumbling block for commentators over the years, but not for us: if the motive force is the quantity $M(\overrightarrow{PG}/h)$, and the underlying meaning of the *Principia*'s second law is the equality $\overrightarrow{LQ} = \overrightarrow{PG}$, then there is no choice, for the "change in motion" must be $M(\overrightarrow{LQ}/h)$.

DEFINITION. *Suppose a body with quantity of matter M, had its speed and direction at P been uniformly continued, would have gone on to describe the line segment PL in the time h, but instead is deflected from this line and describes a curve (or a line) PQ in that same time h. Then the change in motion is the quantity* $M(LQ/h)$ *taking place along the line LQ, that is, the quantity* $M(\overrightarrow{LQ}/h)$.

Given these two definitions, the meaning of the second law as stated in the *Principia* now matches the meaning of the second law as described by Newton in the early 1690s:

COMPOUND SECOND LAW (The *Principia*'s second law as understood by Newton). *By the action of any force (or instantaneous impulse), a change in motion equals the motive force impressed and takes place in the direction of that motive force:* $M\overrightarrow{LQ}/h = M\overrightarrow{PG}/h$. *Equivalently, the same force acting on the same body generates in the same time the same deflection, whether the body is in motion or at rest: the moving deflection equals the resting deflection,* $\overrightarrow{LQ} = \overrightarrow{PG}$. *In other words, the deflection that would have been generated by a given force on a body at rest compounds independently with the inertial motion of the body:* $\overrightarrow{PQ} = \overrightarrow{PL} + \overrightarrow{PG}$.

IV. WHERE IS F = *M*A?

A. Relative and absolute measures of force

We have now justified the claim we made in Sec. I that the *Principia*'s second law (the compound second law) is neither identical to nor even a Newtonian version of the equation $\mathbf{f} = m\mathbf{a}$. But where then is this equation? To handle all the difficult problems in the *Principia*, Newton surely must have needed a law that relates the force to the acceleration. After all, the *Principia*'s second law, namely the equality $\overrightarrow{LQ} = \overrightarrow{PG}$, guarantees only that the moving and resting deflections $\overrightarrow{LQ}$ and $\overrightarrow{PG}$ can be used in measures of the force; it does not guarantee that these deflections by themselves, nor the change in motion $M(\overrightarrow{LQ}/h)$ or motive force $M(\overrightarrow{PG}/h)$, would necessarily make good measures of the force on their own. Indeed, in the continuous force case, the change in motion and the motive force both approach zero as the generating time interval $h \to 0$, which might seem to disqualify them as measures of the force at P.

As we shall see, Newton needed and stated a Newtonian version of the equation $\mathbf{f} = m\mathbf{a}$, but even without this

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equation, working with just the compound second law, Newton was still able to solve nontrivial problems in mechanics. Yes, the change in motion $M(\overrightarrow{LQ}/h)$ and motive force $M(\overrightarrow{PG}/h)$ do approach zero as $h \to 0$, but there is information nonetheless in the rate with which they approach zero, and as a result they may be used as relative measures of the force, where their values for one force are compared in a ratio to their values for another force, and the ratio is examined as $h \to 0$.[21]

Consider, for example, Proposition X (Problem III) in Book II:

> PROPOSITION X. *Let a uniform force of gravity tend straight toward the plane of the horizon [that is, straight downward], and let the resistance be as the density of the medium and the square of the velocity jointly; it is required to find, in each individual place, the density of the medium that makes the body move in any given curved line and also the velocity of the body and resistance of the medium.*[22]

In Newton's solution—where we will leave his notation unexplained, because we are interested only in the structure of his approach—he argues that

> … since gravity generates the velocity $2NI/t$ in the same time in a falling body, the resistance will be to the gravity as $GH/T - HI/t + (2MI \times NI)/(t \times HI)$ to $2NI/t$ ….

Here the relative measure of resistance $GH/T - HI/t + (2MI \times NI)/(t \times HI)$ and the relative measure of gravity $2NI/t$ each approach zero as the brief time intervals T and t approach zero, and the Newtonian code "will be … as" signals that their ratio approaches a nonzero, finite limit. From this and further analysis, he finds, correctly, that the "resistance will now be to gravity … as $3S\sqrt{1+Q^2}$ to $4R^2$," the square of the "velocity is … $(1+Q^2)/R$," and "therefore the density of the medium … is as $S/R\sqrt{1+Q^2}$."

Still, despite his success with such relative measures of force, we would expect Newton to need an absolute measure as well, one that approximates the force at a given point P more and more precisely as the generating time interval approaches zero, one that therefore involves the (vector) acceleration. But before we start combing through the *Principia* looking for a Newtonian version of $m\mathbf{a}$ as a measure of the force $\mathbf{f}$, we should understand that Newton did not calculate vector acceleration the way we do.

We all know what we do. Given a vector function $\mathbf{r}(t)$ that assigns to each time t a position $\mathbf{r}(t)$ in space, we define the vector velocity $\mathbf{v} = \mathbf{r}'$ and vector acceleration $\mathbf{a} = \mathbf{v}'$ by taking vector derivatives, so that $\mathbf{a}(t_0)$ is the limit of $\Delta\mathbf{v}/h$ as $h \to 0$, where $\Delta\mathbf{v}$ is the difference $\mathbf{v}(t_0+h) - \mathbf{v}(t_0)$. For Newton, in contrast, a deviation from uniform straight line motion or from rest signaled a nonzero force (and a nonzero acceleration), and he calculated that force (and acceleration) directly from that signal, that is, directly from the moving or resting deflection, $\overrightarrow{LQ}$ or $\overrightarrow{PG}$. How? By dividing by the square of the time. In one of the *Principia*'s preliminary mathematical lemmas, Newton put it this way:[23]

> LEMMA 10. *The spaces which a body describes when urged by any finite force, whether that force is determinate and immutable or is continually increased or continually decreased, are at the very beginning of the motion in the squared ratio of the times.*

Here, "the spaces which a body describes" is the deflection $\overrightarrow{LQ}$ (or $\overrightarrow{PG}$ if the body started from rest) generated in a given time h and to say that these "spaces … are at the very beginning of the motion in the squared ratio of the times" is Newtonian limit language telling us the ratio $\overrightarrow{LQ}/h^2$ has a finite, nonzero limit as $h \to 0$. In other words, a finite force (or acceleration) is identified by the ratio $\overrightarrow{LQ}/h^2$ having a finite limit. The *Principia* gives a less than persuasive geometric argument for Lemma 10, but we can easily verify Newton's claim in a more contemporary way using Taylor series for vector functions: with $\overrightarrow{SL} = \mathbf{r}(t_0) + \mathbf{v}(t_0)h$ and $\overrightarrow{SQ} = \mathbf{r}(t_0+h) = \mathbf{r}(t_0) + \mathbf{v}(t_0)h + \frac{1}{2}\mathbf{a}(t_0)h^2 + \cdots$, where S is the origin, we have $\overrightarrow{LQ} = \frac{1}{2}\mathbf{a}(t_0)h^2 + \cdots$, and therefore $2(\overrightarrow{LQ}/h^2) = \mathbf{a}(t_0) + \cdots \to \mathbf{a}(t_0)$ as $h \to 0$.[24]

Calculating the acceleration via the ratio $\overrightarrow{LQ}/h^2$ is not an idiosyncrasy on Newton's part. Through the end of the 18th century, scientists continued to use (twice) the moving or resting deflection over the square of the time to calculate the acceleration and hence the force. For example, in his *Mécanique Analytique* (first published in 1788, with a second expanded edition in two volumes, the first appearing in 1811, the second in 1815) Lagrange observed that

> the value of the applied force on a body at any instant of time can always be determined by comparing … the distance [namely the resting deflection $\overrightarrow{PG}$] the body traverses with the square [h^2] of the duration of that instant. It is not even necessary that the body actually traverses this distance. It is sufficient that it can be imagined to be traversed by a composite motion [namely the moving deflection $\overrightarrow{LQ}$, which is the composite motion $\overrightarrow{PQ} - \overrightarrow{PL}$] since the effect is the same in one case [$\overrightarrow{PG}$] as in the other [$\overrightarrow{LQ}$] according to the principles of motion discussed above.[25]

Note that Lagrange assumes the Compound Second Law, that $\overrightarrow{LQ} = \overrightarrow{PG}$, without realizing that he should have cited the second law of the *Principia*.[26]

B. Newton's version of $\mathbf{f} = m\mathbf{a}$

As we scan the early portions of the *Principia* for a version of the equation $\mathbf{f} = m\mathbf{a}$, it seems we should look for a statement of the form, "The force is given by $M\vec{A}$," where M is Newton's "quantity of matter" and $\vec{A}$ is the (Newtonian) acceleration, the limit of $(\overrightarrow{LQ}/h^2)$ as $h \to 0$. But Newton preferred proportions to equations, and he normally expressed a limit with his "will be as" language, which means we really should be looking for a statement of the form: "The force will be as the moving deflection directly and as the time squared inversely," where ideally the limit predicted by this statement—that the force divided by the ratio LQ/h^2 approaches a nonzero and finite limit—turns out to be the quantity of matter M.

Knowing what to look for shortens the search:

> PROPOSITION VI. *…the centripetal force in the middle of the arc will be as the sagitta directly and as the time twice [i.e., as the square of the time] inversely.*[27]

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Provided the mysterious "sagitta" is equivalent to the moving deflection, this proposition certainly has the right form. But the "sagitta" (Latin for "arrow") of a given small arc qQ (traversed by a body moving from q to Q in the time $2h$ under the influence of a centripetal force) is the directed line segment $\overrightarrow{PX}$ (shown in Fig. 6) from the center P of that arc (the center with respect to time) to the center X of the chord of that arc (the center with respect to distance). This sagitta is equivalent to the moving deflection $\overrightarrow{LQ}$ of the arc from P to Q, in the sense that $LQ/PX \to 1$ as $h \to 0$, which means we can restate Proposition VI with the moving deflection replacing the sagitta:

> PROPOSITION VI (equivalent form). *…the centripetal force in the middle of the arc will be as the [moving deflection] directly and as the time twice [i.e. as the square of the time] inversely.*

It seems that Proposition VI must represent the *Principia*'s version of the equation $\mathbf{f} = m\mathbf{a}$.

Ah, but not so fast: what does the $\mathbf{f}$ become in this version, and what happened to the mass? Although, strangely, the *Principia* provides no explicit definition for the "centripetal force" that appears in Proposition VI, we can infer the meaning that Newton must have in mind,[28] namely $M(\overrightarrow{PG}/h^2)$, which is just the motive force $M(\overrightarrow{PG}/h)$ per unit time. Using this definition, we can figure out where the "quantity of matter" M is hiding: Proposition VI asserts, in part, that the ratio of the centripetal force $M(PG/h^2)$ to the quantity LQ/h^2 approaches a finite, nonzero constant, and that constant is easily seen to be M.[29] As a result, if we let $h \to 0$ and let F stand for the limit of the centripetal force, we could even write Proposition VI as $\vec{F} = M\vec{A}$.

With Proposition VI, Newton could calculate the centripetal force from the observed motion of the body, and in a series of such calculations (Propositions VII through XIII) he derives the force laws that correspond to specific pairings of the trajectory and force center, Propositions XI, XII, and XIII being especially important: *when a body moves in a conic, under the influence of a centripetal force directed toward a focus, the centripetal force will be inversely as the square of distance to that focus*. From this result, together with a uniqueness theorem, Newton then sketched a proof of the converse: a body moving under the influence of a "centripetal force that is inversely proportional to the square of the distance … from the center … will move in some one of the conics having a focus in the center of force."[30]

No surprise then that Proposition VI has long been celebrated as a fundamental theorem in the *Principia*'s development of mechanics. Now we have another reason to celebrate: Proposition VI justifies us every time we call the equation $\mathbf{f} = m\mathbf{a}$ Newton's.[31]

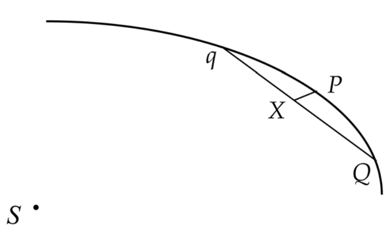

Fig. 6. The sagitta ("arrow") from the center (with respect to time) of the arc qQ to the center (with respect to distance) of the chord qQ.

V. THE MORAL

When we use the phrase "Newton's second law," referring to the equation $\mathbf{f} = m\mathbf{a}$, we are, historically speaking, only half right: it is definitely Newton's, yet it is not the *Principia*'s second law (which is the equality $\overrightarrow{LQ} = \overrightarrow{PG}$, between the moving and resting deflections), but its sixth proposition. However, because Proposition VI follows from the *Principia*'s second law, the half that's wrong is really not all that wrong.

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[1]Isaac Newton, *Philosophiae Naturalis Principia Mathematica* or *Mathematical Principles of Natural Philosophy* [*Natural Philosophy* meaning *Physics*], first published in 1687, with a second edition in 1713, and a third, just a year before Newton died, in 1726. Devoted to "rational mechanics"—"the science," as Newton puts it, "expressed in exact propositions and demonstrations, of the motions that result from any forces whatever and of the forces that are required for any motions whatever"—the *Principia* consists of two preliminary sections (*Definitions* and *Axioms, or the Laws of Motion*) followed by three books: Books I and II (both titled *The Motion of Bodies*, but with Book II concentrating on *resisted* motion) together present mathematical principles, while Book III (*The System of the World*) applies these principles to planets and moons of our solar system. All quotations from the *Principia* come from the recent English translation: I. Newton, *The Principia, Mathematical Principles of Natural Philosophy*, translated into English from the Latin of the third (1726) edition by I. Bernard Cohen and Anne Whitman, assisted by Julia Budenz, and preceded by "A guide to Newton's *Principia*" by I. Bernard Cohen (University of California Press, Berkeley, 1999).

[2]Reference 1, p. 416.

[3]According to this standard interpretation, the "change in motion" is the change $\Delta mv = m\Delta v$ in linear momentum, where Δv stands for a generated change in speed, and the second law applies directly only to an instantaneous impulse, which produces an instantaneous change in either speed or direction. Generally, the meaning of the "motive force" remains a mystery in this reading. Under this interpretation, the second law would apply to a (continuous) force only indirectly, through a complicated limit argument involving impulses in series, and to an oblique force only after finding the component parallel to the direction of motion. See, for instance, I. B. Cohen, "Newton's Concept of Force and Mass, With Notes on the Laws of Motion," in *The Cambridge Companion to Newton*, edited by I. B. Cohen and G. E. Smith (Cambridge U.P., Cambridge, 2002), pp. 65–67; M. Blay, "Force, Continuity, and the Mathematization of Motion at the End of the Seventeenth Century," in *Isaac Newton's Natural Philosophy*, edited by J. Z. Buchwald and I. B. Cohen (The MIT Press, Cambridge, MA, 2001), pp. 226–227; and H. Erlichson, "Motive Force and Centripetal Force in Newton's Mechanics," Am. J. Phys. **59**, 842–849 (1991), p. 844.

[4]See, for example, these excellent mechanics texts: J. M. Knudsen and P. G. Hjorth, *Elements of Newtonian Mechanics, Including Nonlinear Dynamics*, revised and enlarged 3rd ed. (Springer-Verlag, Berlin, 2000), p. 28; J. B. Marion and S. T. Thornton, *Classical Dynamics of Particles and Systems*, 4th ed. (Saunders, Fort Worth, 1995), p. 49; K. R. Symon, *Mechanics*, 3rd ed. (Addison-Wesley, Reading, MA, 1971), p. 7.

[5]For more detail on the interpretation of the *Principia*'s second law presented here, see B. Pourciau, "Newton's interpretation of Newton's second law," Arch. Hist. Exact Sci. **60**, 157–207 (2006).

[6]"At rest" in Newton's sense of being at rest relative to "that immovable space in which the bodies truly move." Newton was well aware of the weaknesses in his own theory—that it depends on "immovable space … [which] makes no impression on the senses," (Ref. 1, p. 414) for example,

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or that it provides no explanation for the proportionality of inertial and gravitational mass.

[7]Newton applies the *Principia*'s second law only to forces (such as centripetal forces) that would move a body at rest along a line.

[8]Today we would detect any deviation from uniform straight line motion or rest with a nonzero (vector) acceleration. Working instead with the moving and resting deflections, Newton could accommodate deviations generated by instantaneous impulses, when the acceleration, being infinite, does not exist. (For an instantaneous impulse, the arc segment *PQ* would be a line segment.) Such impulses play a central role early in the *Principia*, where they are used in series to approximate the motion of a body under the influence of a (continuous) centripetal force in the argument for Proposition I.

[9]To clarify a possible confusion: A charged particle at rest at a point *P* in a magnetic field will remain at rest, while the same particle projected through *P* with a nonzero velocity (not parallel to the magnetic field) will be deflected from its uniform straight line motion. This difference does not violate the principle that the resting and moving deflections must be the same (for a given force on a given body). Although the origin or mechanism (the magnetic field) remains the same for the particle in both cases, the force (or "thrust") experienced by the particle is not the same. If we were to imagine applying the same "thrust" experienced by the moving particle to the resting particle, then the resting deflection would equal the moving deflection, according to this natural sequel to the first law. That this natural sequel is relativistically false does not worry us, because we are working within an earlier paradigm: the mechanics of the *Principia*.

[10]*The Correspondence of Isaac Newton*, 1688–1694, edited by H. Turnbull (Cambridge U.P., Cambridge, 1961), Vol. **3**, pp. 155–156.

[11]*Christiaan Huygens' The Pendulum Clock or Geometrical Demonstrations Concerning the Motion of Pendula as Applied to Clocks*, translated and with notes by R. Blackwell, based on the original 1673 edition (Iowa State U.P., Ames, 1986), p. 33.

[12]Reference 1, p. 382.

[13]*The Mathematical Papers of Isaac Newton,* edited and with extensive commentaries by D. T. Whiteside (Cambridge U.P., Cambridge, 1974), Vol. **6**, pp. 538–609.

[14]See Ref.13, pp. 539–543, or I. B. Cohen, "Newton's Second Law and the Concept of Force in the *Principia*," in *The Annus Mirabilis of Sir Isaac Newton 1666–1966*, edited by R. Palter (The MIT Press, Cambridge, MA, 1970), pp. 143–185.

[15]Nowhere else in his work on mechanics, published or unpublished, does Newton use the phrase "in the meaning of this law" in a passage that describes the second law of motion. Indeed, nowhere else does he describe how he interprets the second law. The passage we have quoted is the only known record, in his own words, of Newton's understanding of his second law.

[16]Newton often writes that two line segments are "parallel and equal," or that a quantity "takes place along" a specified line. Such directed line segments are all over the *Principia*. The anachronistic overhead arrow we have used captures this notion perfectly, with no distortion of Newton's intended meaning.

[17]In Newton's mechanics, a given force and its observable sign are conceptually distinct, the observable sign being the deflection (or the acceleration): intuitively, a thrust is seen as distinct from the observed effect of that thrust. In any development of classical mechanics where this distinction disappears, the Compound Second Law becomes a vacuous tautology. Within such developments, a meaningful analogue may be phrased in terms of reference frames and Galilean relativity: Two observers, moving with constant relative (vector) velocity, record the same deflection….

[18]Reference 1, p. 407.

[19]Because the mathematics of the *Principia,* for the most part, is a geometrical version of limits and calculus, Newton preferred to work with proportions rather than equations. But we lose nothing and we gain a more modern presentation treating these proportions as equalities. Hence, we write the "motive force *is*…," rather than the "motive force is proportional to…"

[20]A. Koyré, "An experiment in measurement," Proc. Am. Philos. Soc. **97**, 222–237 (1953), reprinted in A. Koyré, *Metaphysics and Measurement* (Chapman and Hall, London, 1968), pp. 89–117. Also see J. G. Yoder, *Unrolling Time: Christiaan Huygens and the Mathematization of Nature* (Cambridge U.P., Cambridge, 1988), Chaps. II–IV.

[21]For example, we can gauge how fast $1 - \cos h$ approaches 0 as $h \to 0^+$ by comparing it to the quantities h, h^2, and h^3, which approach 0 increasingly fast: $(1 - \cos h)/h \to 0$, $(1 - \cos h)/h^2 \to \frac{1}{2}$, and $(1 - \cos h)/h^3 \to +\infty$. These limits tell us that $1 - \cos h$ approaches 0 faster than h, at the same rate as h^2, and slower than h^3. Newton would write that $1 - \cos h$ "will be as" h^2 (as $h \to 0$). We have the good approximation $1 - \cos h \approx \frac{1}{2}h^2$ for h sufficiently small; so we can also think of the Newtonian code "will be as" as signaling an "ultimate proportion": $1 - \cos h$ is nearly proportional to h^2 for sufficiently small h. It does not matter that the limit of the ratio is 1/2, only that the limit is nonzero and finite.

[22]Reference 1, p. 655.

[23]Reference 1, pp. 437–438.

[24]A (finite) force, by definition, generates a finite acceleration: thus $2(\overrightarrow{LQ}/h^2) = 2(\overrightarrow{LQ}/h)/h$ approaches the finite limit $\mathbf{a}(t_0)$. When a ratio has a finite limit and its denominator $[h]$ approaches zero, then the numerator $[\overrightarrow{LQ}/h]$ must also approach zero. The same is true for $\overrightarrow{PG}/h$. Thus, as we claimed earlier, the motive force $M(\overrightarrow{PG}/h)$ and the change in motion $M(\overrightarrow{LQ}/h)$ are relative, not absolute, measures of the force.

[25]J. Lagrange, *Analytical Mechanics*, translated and edited by Auguste Boissonnade and Victor N. Vagliente (Kluwer Academic Publishers, Boston, 1997), p. 171.

[26]The major treatises on mechanics in the 18th century—by Varignon, Hermann, Euler, MacLaurin, d'Alembert, Euler (again), Lagrange, and Laplace—all acknowledge the enormous influence of Newton's *Principia*, yet none of these works states or cites the second law as it appears in the *Principia*. Given that Newton evidently regarded the *Principia*'s second law as a fundamental axiom of mechanics, this omission is odd. But the explanation is easy. Confused by the ambiguous account of the second law in the *Principia*, these scientists misinterpreted the second law in ways that made it appear irrelevant to their developments of mechanics, as a law applying to impulses only, for example. Nevertheless, all of them did apply the property asserted by the *Principia*'s second law (as Newton understood it, in the form of the Compound Second Law), using it as a fundamental assumption of their own mechanics, without realizing that they should have cited the second law in the *Principia*. See, for example, J. Hermann, *Phoronomia, sive de viribus et motibus corporum solidorum et fluidorum, libri duo* (R. & G. Wetstenios, Amsterdam, 1716), pp. 68–69; C. MacLaurin, *A Treatise of Fluxions in Two Books* (T. W. and T. Ruddimans, Edinburgh, 1742), Vol. **1**, p. 347; and L. Euler, *Theoria motus corporum solidorum seu rigidorum: ex primus nostrae cognitionis principiis stabilita et ad omnes motus, qui in huis modi corpora cadere possunt, accommodata*, Leonhardi Euleri Opera Omnia, Series II: Opera Mechanica et Astronomica Vol. **3**, edited by C. Blanc, originally published by Orell Füssli in 1765 (Birkhäuser, Berlin, 1948), pp. 68–69. For more along these lines, read B. Pourciau, "Newton's second law (as Newton understood it) from Galileo to Laplace," unpublished.

[27]Reference 1, p. 454.

[28]For details on this and on Proposition VI more generally, read B. Pourciau, "Force, deflection, and time: Proposition VI of Newton's *Principia*," Hist. Math. **34**, 140–172 (2007).

[29]See Ref. 28, p. 160. Our $\overrightarrow{PG}$ and $\overrightarrow{LQ}$ notation in Proposition VI hides a technical point: if these two deflections were generated by exactly the same force (in direction and magnitude) during the brief time h, then we would have $\overrightarrow{LQ} = \overrightarrow{PG}$ by the Compound Second Law and the ratio would be M even for $h > 0$ and not just in the limit. But in Proposition VI, the force which generates $\overrightarrow{LQ}$ is only approximately the same as the force that generates $\overrightarrow{PG}$. Nevertheless, we still obtain M in the limit as $h \to 0$.

[30]Reference 1, p. 467.

[31]We have concluded that Proposition VI is the *Principia*'s version of $\mathbf{f} = m\mathbf{a}$. Although alike underneath, Proposition VI and $\mathbf{f} = m\mathbf{a}$ are not alike on the surface; hence the need for the modifier "version of." A natural question arises: Who was the first to write the equation $\mathbf{f} = m\mathbf{a}$ in something like its present form? The answer, like so many answers in the history of mathematics and physics, is Leonhard Euler. His "Discovery of a new principle in mechanics," published in 1750, recorded the equation in Cartesian coordinates: $2Mddx = Pdt^2$, $2Mddy = Qdt^2$, $2Mddz = Rdt$ (L. Euler, "Découverte d'un nouveau principe de mécanique," Mémoires de l'Académie des Sciences de Berlin **6**, 185–217 (1750). Also see *Leonhardi Euleri Opera Omnia*, Serie II, Vol. **5**, pp. 81–108.) Euler could legitimately regard this principle as new, even though its ancestor had appeared 63 years earlier as Newton's Proposition VI, because, for the first time, it was written in a fixed orthogonal coordinate system and applied, not just to point-masses, but to extended bodies. For further details, read G. Maltese, "The Ancients' Inferno: The Slow and Tortuous Development of 'Newtonian' Principles of Motion in the Eighteenth Century," in *Essays on the History of Mechanics: In Memory of Clifford Ambrose Truesdell and Edoardo Benvenuto*, edited by Antonio Becchi (Birkhäuser Verlag, Basel, 2003), pp. 199–221. Also see C. Truesdell, *The Rational Mechanics of Flexible or Elastic Bodies 1638–1788*, Introduction to Leonhardi Euleri Opera Omnia, Series II, Vol. **10** and **11** (Orell Füssli, Zürich, 1960), p. 251.

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