and the pipe are assumed to be in proportion as p is to π, the impulse of the water against the base will be to the residual impulse as $p + \pi$ is to p; for the impulse is distributed equally over all the material of the water as well as the pipe, and only the fluid reacts on the base.

But now let us assume a very small orifice m in the base BA; through this, nevertheless, water is considered to flow very freely; thus we understand that a particle of water will be ejected through the small orifice m during the impulse; however, the quantity of that water cannot be determined, for it depends upon the rigidity of the material AP receiving the impulse: indeed, if that material is very rigid, a greater pressure is to be substituted for the impetus, but lasting for less time; for example, let the same impetus be considered for two different cases: moreover, in one let the pressure be quadrupled, in the other let the duration of the pressure be quadrupled, which can happen when the material is more rigid in the former case than in the latter; thus, approximately double the quantity will flow out in the impulse of the lesser pressure and greater duration than in the other case. In this way the rigidities of materials can be explored: but they can be found as well from sound.

TWELFTH CHAPTER

Which shows the Statics of Moving Fluids, which I call Hydraulico-Statics

§1. Among those who gave measurements of the pressure of fluids existing within vessels, few have gone beyond the common rules of Hydrostatics which we showed in Chapter II; nevertheless, there are many other rules which pertain to the appropriately named Hydrostatics, such as whenever a centrifugal force or the force of inertia is united with the action of gravity, each of which we discussed in the preceding chapter; dead forces of this type can be devised and combined in infinitely many other ways. But these are not the things which seem to me to be most desirable, since it is not difficult to give general rules for this procedure. I desire, rather, [to treat] the statics of fluids which are moved within vessels in a progressive motion, such as of water flowing through conduits to leaping fountains: indeed, this is of multiple use, and it has not been treated by anyone, or, if some people can be said to have made mention of it, it was not at all properly explained by them; indeed, those who have spoken about the pressure of water flowing through aqueducts and the strength required of the latter for sustaining that pressure did not hand down any laws other than those for extended fluids with no motion.

§2. It is singular in this *hydraulico-statics* that the pressure of water cannot be defined unless the motion has been known correctly, which is the reason that this doctrine escaped notice for so long; indeed, up to now Authors were hardly anxious to investigate the motion of water, and they estimated velocities almost everywhere from the height of the water alone; however, although the motion often tends so quickly toward this velocity that the accelerations clearly cannot be distinguished by observation, and all the motion seems to be generated in an instant; nevertheless, it is of interest to understand these accelerations correctly, because otherwise the pressures of the flowing water often cannot be defined, and on that account I estimated that it is a matter of greatest moment to consider those changes, however *instantaneous*, from the beginning of motion up to a given limit with all care, and to confirm them by experiments, which I did at different places in this treatise, but especially in Chapter III.

§3. If the motion could be defined everywhere, it would be easy to develop the most general statics in moving fluids: indeed, if one assumes an orifice which is infinitely small in that very place at which the pressure of the water is desired, one will seek to learn first at what velocity the water would erupt through that tiny orifice and to what height that velocity would be due; moreover, one understands that the pressure which is sought is proportional to this very height.

From this principle the pressure is to be sought which the horizontal plate LQ in Fig. 43 sustains if it has not been perforated. Indeed, since it has been shown by us in the second corollary of §31, Chapter VIII, that, if the orifice H is infinitely small in proportion to the orifices M and N, and the ratio of these orifices M and N is indicated by α and γ, then the height due to the velocity of the water erupting through H will be $\frac{\alpha\alpha(LB) - \gamma\gamma(NQ)}{\alpha\alpha + \gamma\gamma}$, we will thence judge that the pressure of the water against the nonperforated plate LQ is proportional to this very height. We gave the same proof in another way in §19 of the cited chapter. Hence it follows that it can occur that the section LQ experiences no pressure, however great the height of the water above it may be, as for example when $\gamma = \alpha\sqrt{LB/NQ}$; indeed, the pressure can even be changed into suction.

§4. Similarly, the pressure of the water against the section LQ is obtained if, for instance, the latter is perforated by an orifice of finite size in proportion to the two remaining [orifices]. For if the section is perforated by an infinitely small orifice with respect to that which exists at H, the water cannot but erupt at a common velocity through either one. And since this velocity is known (from §30, Chapter VIII) for the orifice H, the velocity is also obtained at which the water must erupt through the tiny orifice which we conceive, and thus we know the pressure of the water. For example, let the orifices M, H, and N be equal to one another, and also let the height BL have a ratio to the height LQ as 10 is to 3, and the pressure against the plate LQ will be one-tenth of what it is with the orifices H and N closed off.

Finally, if one should desire the pressure of the water in another location, he will simply add the height by which the section LQ exceeds that point to the height of the thrust through the orifice H. The same method serves for determining water pressures in the rest

of the vessels which we treated in Chapter VIII. But all these matters differ from those which pertain to the motion of fluids through conduits, because the water, on account of the infinite size of the vessels assumed by us, is as if at rest in cavities, and nevertheless it exerts a far different pressure from what is otherwise customary. Moreover, in conduits the water changes its pressure more, the greater the velocity at which it flows through, and it exerts almost all its customary pressure if that velocity is very small.

This is so whenever the velocities of fluids can be determined by the methods presented by us just above. But the matter must be handled by a singular method when the water flows through conduits, and I comprehend this doctrine especially under the title of *hydraulico-statics*. Here, not so much can the pressure be defined from the velocity as, reciprocally, the velocity from the pressure, if a small orifice is made in the walls of the conduit. And in the present chapter I decided to treat especially that *hydraulico-statics*, the application of which is very broad.

Problem

§5. The very wide vessel $ACEB$ (Fig. 72), with the cylindrical and horizontal pipe ED attached, is to be kept constantly full of water;

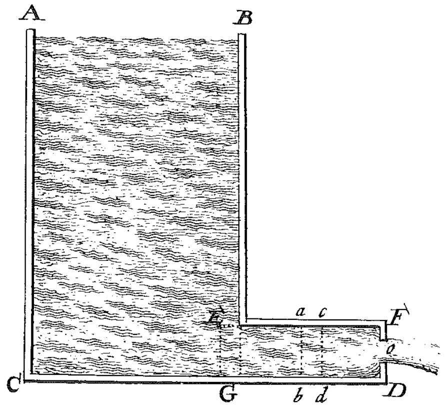

Figure 72

and at the extremity of the pipe let there be the orifice o emitting water at a uniform velocity; the pressure of the water against the walls of the pipe ED is sought.

SOLUTION. Let the height of the aqueous surface AB above the orifice o be a; the velocity of the water flowing out at o, if one excludes the first instants of flow, will have to be considered uniform and equal to $\sqrt{a}$, because we assume the vessel to be kept constantly full; and, with the ratio of the areas of the pipe and its orifice assumed equal to $\frac{n}{1}$, the velocity of the water in the pipe will be $\frac{\sqrt{a}}{n}$. But if the entire base FD were missing, the ultimate velocity of the water in the pipe itself would be $\sqrt{a}$, which is greater than $\frac{\sqrt{a}}{n}$. Therefore, the water in the pipe tends to greater motion, but its pressure is impeded by the added base FD. By this pressure and repressure the water is compressed, which very compression is confined by the walls of the pipe, and hence these sustain a like pressure. Thus it appears that the pressure of the walls is proportional to the acceleration, or the increment of velocity which the water would receive if the entire obstacle to motion would vanish in an instant so that [the water] might be ejected immediately into the air.

Therefore, the problem is now changed into this: if during the flow of water through o the pipe ED were broken at cd at an instant, one seeks the magnitude of the acceleration the volume element $acbd$ would thence be about to obtain; indeed, the particle ac taken at the walls of the pipe will sense that much pressure from the water flowing through. To this end the vessel $ABEcdC$ is to be considered, and with regard to it the acceleration of an aqueous particle close to efflux is to be found, if this would have the velocity $\frac{\sqrt{a}}{n}$. We handled that matter very generally in §3, Chapter V. Nevertheless, because the calculation is short in this particular case, we will here again subject the motion in the shortened vessel $ABEcdC$ to evaluation.

Let the velocity in the pipe Ed, which [velocity] is now to be considered as variable, be v; let the area of the pipe, as before, be n, the length $Ec = c$; let the length of the aqueous particle ac, infinitely small and about to flow out, be indicated by dx. There will be an equal volume element at E entering the pipe at the same instant that the other, $acdb$, is ejected; moreover, while the volume element at E, the mass of which is $n\,dx$, enters the pipe, it acquires the velocity v and the *live force* $nvv\,dx$, which entire *live force* was generated anew; indeed, the volume element at E, not yet having entered the pipe, had no motion on account of the infinite size of the vessel AE; to this *live force*, $nvv\,dx$, is to be added the increment of *live force* which the water at Eb receives while the volume element ad flows out, namely, $2ncv\,dv$; the sum is due to the *actual descent* of the volume element $n\,dx$ through the height BE or a; therefore, one obtains $nvv\,dx + 2ncv\,dv = na\,dx$, or $\frac{v\,dv}{dx} = \frac{a - vv}{2c}$.

Moreover, in all motion the increment of velocity dv is proportional to the pressure multiplied by the differential time, which here is $\frac{dx}{v}$; therefore, in our case the pressure which the volume element ad experiences is proportional to the quantity $\frac{v\,dv}{dx}$, that is, to the quantity $\frac{a - vv}{2c}$.

But at that instant at which the pipe is broken, $v = \frac{\sqrt{a}}{n}$, or $vv = \frac{a}{nn}$; therefore, this value is to be substituted in the expression $\frac{a - vv}{2c}$, which thus is transformed into $\frac{nn - 1}{2nnc}\,a$. And this is the quantity to which the pressure of the water against the portion ac of the pipe is proportional, whatever area the pipe may have, or by whatever orifice its base may be perforated. Therefore, if in a particular case the pressure of the water would be known, it would be understood at the same time in all remaining [cases]: but, indeed, we have this [pressure] when the orifice is infinitely small or n is infinitely large with respect to unity: for then it is evident from itself that the water exerts its entire pressure, which conforms to the total height a, and this pressure we will designate by a; but when n is infinite, unity vanishes with respect to the number nn, and the quantity to which the pressure is proportional becomes $\frac{a}{2c}$. Therefore, if we wish to know in general how great the pressure is when n is any number whatever, the following analogy must be used. If the pressure a conforms to the quantity $\frac{a}{2c}$, what then will be the pressure for the quantity $\frac{nn - 1}{2nnc}\,a$? And thus the desired pressure is found equal to $\frac{nn - 1}{nn}\,a$. Q.E.I.

§6. COROLLARY 1. Because the letter c vanishes from the calculation, it follows that all portions of the pipe, those which are nearer to the vessel AG as well as those which are more remote, are pressed equally by the water flowing through, and certainly less than the elements of the base CG, and the difference is the greater, the larger is the orifice o; and, further, the walls of the pipe do not sustain any pressure if in the latter the entire barrier FD is missing, so that the water flows out from a full orifice.

§7. Corollary 2. If the pipe is perforated somewhere by a very small orifice that is necessarily in some ratio to the orifice o, the water will spring forth at the velocity by which it could ascend to the height $\frac{nna - a}{nn}$ if only no foreign hindrances were interfering. Indeed, this will be the height of the thrust in Fig. 73, or $ln = \frac{nna - a}{nn}$. But

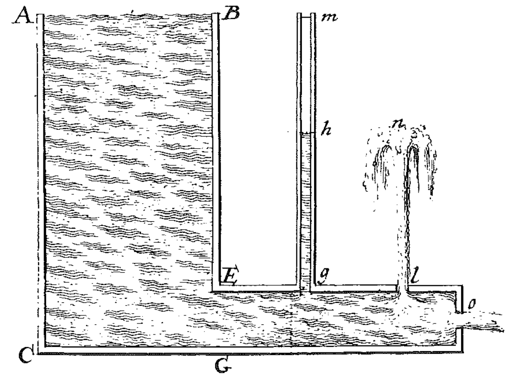

Figure 73

if the small tube gm is attached, vertical or even inclined in some way, connecting with the horizontal pipe, but so, nevertheless, that the extremity of the inserted tube does not project into the cavity of the horizontal pipe lest the water flowing past strike against that extremity, the vertical height gh of the water standing in the inserted tube will also be equal to $\frac{nna - a}{nn}$; and it is not necessary in this latter case that the tube gm be very narrow.

Scholium

§8. Therefore, this theory can be confirmed very easily by experiment, and this will be of more importance because up to this time no one has defined equilibria of this sort, the use of which is very widely evident, because by the same method the pressure of water flowing through conduits can be obtained very generally for aqueducts inclined in any way whatever, curved, of varied area, and at any velocity of water whatever; then, as well, because not only this [theory] of pressures, but the entire theory of motions besides, which we gave above, is confirmed by experiments of this sort, because they prove that the accelerations of the water were defined correctly by us. But one must take care in the experiment that the horizontal pipe is very smooth on the interior, perfectly cylindrical and horizontal, and that it is wide enough so that no noticeable decrement of motion can arise from the adhesion of the water to the walls of the pipe; let the vessel itself be very wide and be kept full continuously. Also one must observe how great is the characteristic of elevating standing water in the glass tube gm, which characteristic pertains to all capillary or rather narrow tubes; for this elevation is to be subtracted from the height gh; or, rather, a pipe of equal thickness is to be assumed with the orifice o blocked off, the point m is to be noted, and then, with the water allowed to flow, the point h is also to be noted; moreover, according to the theory the descent will be $mh = \frac{1}{nn} a = \frac{1}{nn} (EB)$.

Finally, one must pay attention as well to the stream of water flowing out at o, for its contraction also causes the water in the horizontal pipe to flow through at a velocity less than $\frac{\sqrt{a}}{n}$. I treated that contraction and the method of preventing it in Chapter IV. But although it can happen with these inconveniences that no noticeable error remains in the experiment, nevertheless, if we wish to apply greater accuracy, the quantity of water flowing out in a given time will have to be discovered by experiment, which [quantity], compared with the area of the pipe, will give very correctly the velocity of the water flowing within the pipe, which in the calculation we have set equal to $\frac{\sqrt{a}}{n}$. But if in the experiment it will be found to be less, such, for example, as is due to the height b, then the pressure of the water flowing by will be approximately $a - b$.

§9. Corollary 3. If the orifice at o is blocked off at first by a finger, and afterwards the water is allowed to flow, the pressure a at the first moment of flow is changed into the pressure $\frac{nna - a}{nn}$, but that change of pressures does not occur in an instant; if, indeed, one is to speak accurately, it occurs at last after an infinite time, because, as we saw in Chapter V, the entire velocity of the water, which was assumed by us in the calculation to correspond to the whole height a, is never present exactly; nevertheless, it tends toward this velocity with an incredible acceleration immediately after the first drops have been ejected, so that it seems to have acquired the total [velocity], as much as can be judged by observation, without any noticeable delay, unless the aqueducts are very long, for then the accelerations of the water can be discerned clearly by eye, an example of which I gave in §13, Chapter V. Therefore, in those conduits bearing water to a leaping fountain from a reservoir located very far away, if the pressures are investigated at some point by experiment in the manner that I mentioned above, it is found that the pressure is diminished quickly indeed, nevertheless not in an instant, and it will be possible to distinguish the differences of the pressures.

But in order to define the force of the water generally, one must assume for v that velocity which the water has at that same place and that same instant at which the force is desired, and if this velocity is known to conform to the height b, the force of the water will be $a - b$. Hence, since those things which were offered in Chapter V have agreed with the present, it will be possible to define what the pressure will be at any moment.

For these [statements] it is not difficult to anticipate the laws of this *hydraulico-statics* if both the shape of the vessel and the velocity of the water flowing through the conduits are assumed at will as anything whatever. Indeed, the pressure of the water will always be $a - b$, where by a is understood the height due to the velocity at which water will flow out of an abrupt conduit and vessel kept constantly full after an infinite time, and by b the height due to the velocity at which the water actually flows through. It is clearly amazing that this very simple rule, which nature affects, could remain unknown up to this time. Therefore, I will now show it more expressly.

Problem

§**10.** To find the pressure of water flowing at any uniform velocity whatever through a conduit arbitrarily formed and inclined.

Solution. Let there be a conduit ACD (Fig. 74), through the orifice o of which water is considered to flow at a uniform velocity due to the vertical height oS; let the line SN be drawn, and let the infinitely wide vessel $NMQP$ be assumed full of water right up to NP, from which the conduit draws its water perpetually and equally; I assume these things accordingly in order that a cause be present, or a uniform propelling force, which propels the water at a given velocity or maintains an equal flow of water. And without this hypothesis our problem would be indeterminate, because the same velocity in the same conduit pertaining to any instant can be generated in infinitely many ways, and therefore, in order that a measure of the cause propelling the water be obtained, uniformity must be assumed in the motion of the water.

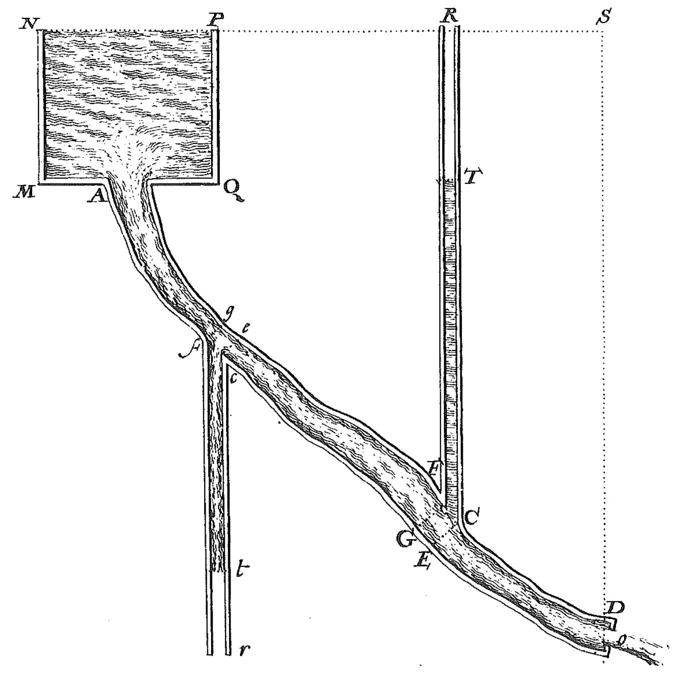

Figure 74

Now the pressure of the water is to be defined at CF (or cf); and to this end we will consider again that the conduit is broken at the section CE (or ce) perpendicular to the conduit, and we will examine what acceleration or retardation the volume element $CEGF$ (or $cegf$) will receive after the first instant of rupture; for this reason we have to define generally the *instantaneous* motion through the shortened vessel $NMECAQP$ (or $NMceAQP$). Therefore, let the velocity of the infinitely small volume element $CEGF$ (or $cegf$) at that very point of cutting off be v, and let its mass be dx; the *live force* of the water moving in the shortened vessel will be proportional to the quantity vv; hence we will set it equal to αvv, understanding by the letter α some constant quantity which depends upon the areas of the suddenly broken conduit; however, its precise determination is not required here. Let it be