12.4 The d'Alembert–Lagrange Formulation of Mechanics

Jean-Baptiste le Rond d'Alembert was a remarkable savant—almost a man born out of his time, who possessed a deep understanding of technical and foundational issues. In subjects as diverse as the fundamental theorem of algebra, the metaphysics of the calculus, the nature of functions and the principles of mechanics he displayed an acute critical sense, grasping issues that would only become the focus of study much later. These gifts were evident early in his career in his seminal *Treatise on Dynamics* (1743). This book was an investigation of the constrained interaction of bodies: the collision of spheres, the motion of pendula, the movement of bodies as they slide past each other, and various other connected systems. Many of the problems would today be studied as part of engineering mechanics. His basic conception was that of a 'hard body.' Such a body is impenetrable and non-deformable. Assume a small hard sphere hits a wall with a velocity that is perpendicular to the wall. When the sphere hits the wall all motion ceases. The closest modern approximation to d'Alembert's conception is that of a perfectly inelastic body, although it must be emphasized that d'Alembert's point of view was different from the modern one. D'Alembert thought in a Cartesian way of hard bodies as geometrical solids in motion, whose laws of interaction could be determined by deductive reasoning from *a priori* postulates or principles. (Hankins (1970) documents the importance of Cartesian philosophy in d'Alembert's scientific thought.) In this conception dynamics is very similar to geometry, where the properties of the objects under study are derivable from a few postulates that are believed to be necessarily true.

Central to d'Alembert's dynamics was a principle that he enunciated at the beginning of the *Treatise* and which in various later forms became known as 'd'Alembert's principle.' (The account which follows is based on (Fraser 1985).) In its original and most basic form the principle may be illustrated by the example of a hard particle that strikes a wall obliquely with velocity u (Fig. 12.1). We must determine the velocity of the particle following impact. Decompose u into two components v and w, v being the post-impact velocity and w being the velocity that is 'lost' in the collision. D'Alembert's principle asserts that if the particle were animated by the lost velocity alone then equilibrium would subsist. From this condition it follows that w must be the component of u that is perpendicular to the wall. Hence v is the component of u that is parallel to the wall, and the collision problem is solved.

Assume now that two hard bodies m and M approach each other with velocities u and U along the line joining their centres. It is necessary to find the velocities after impact. We write $u = v + (u - v)$ and $U = V + (U - V)$, where v and V are the post-impact velocities of m and M. The quantities u, v and $u - v$ are the impressed, actual and 'lost' motions of the body m; a similar decomposition holds for M. Because v and V are followed unchanged v must equal V. In addition, the application of the lost velocities $u - v$ and $U - V$ to m and M must produce equilibrium. D'Alembert

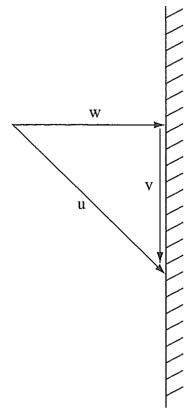

Fig. 12.1. D'Alembert's principle.

reasoned from the very concept of hard body itself that for this to happen we must have $m(u - v) + M(U - V) = 0$. Hence v or V is equal to $(mu + MU)/(M + m)$.

In modern dynamics we would analyse this collision using what are known as impulsive forces. It is assumed that in the collision very large forces act for a very short period of time. Integrating over these forces and using Newton's second law we are able to calculate the changes of velocity that result in the collision. In modern dynamics d'Alembert's principle involves a decomposition of forces or accelerations and is basically a statement combining Newton's second and third laws. By using impulsive forces in the modern principle and integrating, we can produce a decomposition of velocities that looks somewhat (and misleadingly) like the one d'Alembert originally presented. However, d'Alembert's point of view was very different. From the outset in his analysis of the collision of the two bodies he used a decomposition involving finite velocities. There are no forces, and the entire interaction is analysed using the conception of a hard body and the assumption—supported by *a priori* reasoning—that equilibrium would subsist if the bodies were animated by the motions they lose in the collision.

In examples involving forces that act continuously and in which the motion is continuous, d'Alembert analysed the system in a way that has some similarities to the model of instantaneous impulses set out by Newton in the opening proposition of Book One of the *Principia*. The motion is understood to consist of a succession of discrete impulses in which each impulse arises in an interaction of hard bodies or surfaces. The interaction is described in terms of finite velocities and infinitesimal velocity increments where the lost motions are governed by d'Alembert's principle. A clear example of d'Alembert's treatment of continuous forces is the tenth problem of the *Treatise*. Here all of the features of d'Alembert's theory come into play—the concept of hard body, d'Alembert's principle, and the Leibnizian differential calculus.

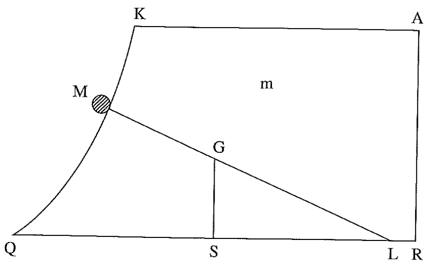

Fig. 12.2. D'Alembert's problem X.

We are given an irregularly shaped object $KARQ$ of mass m which is free to slide along a frictionless plane QR (Fig. 12.2). A body of mass M is situated on the curve KQ which forms the left edge of $KARQ$. A force acts on M in a direction perpendicular to QR. The two bodies possess given initial velocities (in the particular configuration adopted by d'Alembert it is assumed that the bodies initially possess a common motion to the right). The problem is to determine the motion of the system as M slides down KQ.

In d'Alembert's solution the trajectory traced by M is regarded as a polygon with an infinite number of sides in which the length of each side is infinitesimal. This geometric representation of M's trajectory corresponds to the physical analysis in which M's motion is understood to be the outcome of a succession of discrete dynamical events. The model of a continuous curve as an infinite-sided polygon is also used in the solution to represent the left edge of the body m.

Although d'Alembert's original principle involved a decomposition of velocities, he subsequently extended the principle in certain problems to what was in effect a decomposition of accelerations. He did so in some of the problems of his treatise involving pendulum motion, and adopted a similar formulation in his later researches in hydromechanics and theoretical astronomy. Assume that a body m is part of a constrained system of bodies and is acted upon by an external impressed force. At a given instant let $dv^{(I)}$ be the increment of velocity that would be imparted to the body by the impressed force if the body were free. Let v^{+} be the velocity of the body in the next instant, so that $dv = v^{+} - v$ is the actual increment of velocity experienced by the body. We have the decomposition of velocities $v + dv^{(I)} = v^{+} + w = (v + dv) + w$, or

$$v + dv^{(I)} = (v + dv) + w, \tag{12.11}$$

where $w = dv^{(I)} - dv$ is the lost velocity of the body. By d'Alembert's principle equilibrium will subsist if each body of the system were subjected to the motion $dv^{(I)} - dv$. In any given problem we now invoke a suitable statical law and obtain a relation among $-mdv$ and $mdv^{(I)}$. In the problems d'Alembert considered this procedure gave rise to a system of differential equations that described the motion of the system.

We may re-express eq. (12.11) in the form

$$dv^{(I)} = dv + w, \tag{12.12}$$

If eq. (12.12) is divided by dt and multiplied by m this becomes an equation involving forces:

$$\mathbf{F}_a + \mathbf{F}_c = m\mathbf{a}, \tag{12.13}$$

where $\mathbf{F}_a$ is the applied force acting on m, $\mathbf{F}_c$ is the constraint force on m, and $\mathbf{a}$ is the acceleration of m. (For convenience here and in what follows we use modern vector notation, although this notation was not used in the eighteenth century.) In this formulation d'Alembert's principle states that equilibrium would subsist if each of the bodies of the system were animated by the constraint force; that is, the constraint forces considered as a set of applied forces acting on the bodies result in a system in static equilibrium.

Although Lagrange was influenced by d'Alembert, his own development of mechanics occurred along lines that were significantly different from his older contemporary. Physical hypotheses about the ultimate nature of mechanical interactions were absent, and any adherence to a geometric-differential form of the calculus was rejected altogether. (Fraser (1983) and Panza (2003) explore the foundations of Lagrange's mechanics.) Lagrange's goal was to reduce mechanics to a branch of applied analysis in which the emphasis was primarily on the derivation and integration of differential equations to describe the motion of the system. Following his mathematical philosophy, he eschewed diagrams and geometric modes of representation in favour of a purely analytic approach involving operations, formulas and equations among variables. In his first extended memoir on the principles of mechanics Lagrange (1762) used what he called the principle of least action as the starting point for his analysis of the motion of a dynamical system. This principle was an integral variational law and its application was governed by the calculus of variations, a mathematical subject that Lagrange had pioneered in the very same volume. In subsequent investigations Lagrange abandoned the least action principle. Instead he combined d'Alembert's principle (in the form involving eq. (12.13)) and what is today called the principle of virtual work to arrive at the fundamental axiom of his presentation of mechanics. Thus in his mature theory of mechanics, he used methods and operations derived from the calculus of variations, but he did not develop the subject from an integral variational principle, as he had done in 1762.

Consider a set of external forces acting on a connected system of bodies, and suppose that the system is in static equilibrium under the action of these forces. Consider a small virtual velocity or displacement $\delta\mathbf{r}$ of a given body m of the system. Such a displacement is taken to be compatible with the constraints in the system. As before let $\mathbf{F}$ be the external or applied force acting on m. The principle of virtual velocities asserts that a general condition for equilibrium of the system is given by the equation

$$\sum_{m} \mathbf{F} \cdot \delta\mathbf{r} = \mathbf{0}. \tag{12.14}$$

Consider now any constrained mechanical system, not assumed to be in equilibrium. For each body m of the system we have the decomposition $\mathbf{F}_a + \mathbf{F}_c = m\mathbf{a}$ (eq. (12.13)). By d'Alembert's principle the given system would be in equilibrium if each m were animated by $\mathbf{F}_c$, where this force is now understood as an external force acting on m within the connected system. By the principle of virtual velocities we have

$$\sum_m \mathbf{F}_c \cdot \delta\mathbf{r} = \mathbf{0}, \tag{12.15}$$

which we may, using eq. (12.13), express as

$$\sum_m m\mathbf{a} \cdot \delta\mathbf{r} = \sum_m \mathbf{F}_a \cdot \delta\mathbf{r}. \tag{12.16}$$

Eq. (12.16) is a statement of the generalized principle of virtual work and is the starting point for Lagrange's theory of mechanics.

Beginning with eq. (12.16), Lagrange derived a system of differential equations to describe the motion of the system. Using the constraints, one reduces the description of the system to the specification of n 'generalized' variables $q_1, q_2, \ldots, q_m$. (If the system consists of n bodies moving freely then $m = 3n$ and the q_i would be the $3n$ Cartesian coordinates of the bodies.) Each of these variables is independent of the others, and each is a function of time. A system with suitably smooth constraints is then described by the m differential equations

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i} + \frac{\partial Q}{\partial q_i} = 0, \tag{12.17}$$

Here $\dot{q}_i$ denotes the time derivative of q_i, the function T is what later would become known as the kinetic energy of the system and Q_i is what later would be called the potential, although these terms were not used by Lagrange. Eq. (12.17) became known in later dynamics as the Lagrangian equations of motion.

12.5 Statics of Elastic Bodies

The elastic behaviour of rods and beams emerged as a subject of interest in the early eighteenth century in two related problem of statics. In the problem of fracture one attempted to determine the maximum load that a beam of given material and dimensions can sustain without breaking. Typically it was assumed that the beam was cantilevered to a wall and that the rupture took place close to the wall. In the problem of elastic bending one was concerned with determining the shape assumed by a rod or lamina in equilibrium when subject to external forces. In the example of the elastica, where the elastic rod was treated mathematically as a line, the forces were assumed to act at the ends of the rod and to cause the rod to bend into a curve. The first problem had been considered by Galileo and had attracted the attention of Varignon and Antoine Parent, among others. Jakob Bernoulli initiated the study of the second problem, and his work became the basis for further researches by Euler. (For histories of elasticity in the eighteenth century see (Truesdell 1960) and (Szabo 1977). The subject of strength of materials is explored by Timoshenko (1953).)

It is important to note that research on elasticity was carried out without the general theoretical perspective that is provided today by the concept of elastic stress. This concept, which underlies such basic modern formulas as the stress–strain relation and the flexure formula, only emerged explicitly in the 1820s in the writings of Claude Navier and Augustin Cauchy. Although one can discern in the earlier work some of the elements that enter into the modern concept of stress, the essential idea—that of cutting a body by an arbitrary plane and considering forces per unit area acting across this plane—was absent.

The divide that separates the modern theory and that of the eighteenth century is illustrated by the problem of elastic bending. Consider the derivation today of the formula for the bending moment of a beam. One begins by assuming that there is a neutral axis running through the beam that neither stretches nor contracts in bending. We apply elementary stress analysis and consider at an arbitrary point of the beam a cross-sectional plane cutting transversely the neutral axis. Elastic stresses distributed over the section are assumed to act across it. Calculation of their moment about the line that lies in the section, is perpendicular to the plane of bending, and passes through the neutral axis leads to the flexure formula, $M = SI/c$, where M is the bending moment, I is the moment of area of the section about the line, c is the distance of the outermost unit of area of the section from the line, and S is the stress at this outermost area.

In the problem of fracture eighteenth-century researchers obtained results that can be readily interpreted in terms of modern formulas and theory. Typically they assumed that the beam was joined transversely to a wall and that the rupture occurred at the joining with the wall. Here the physical situation directly concentrated attention on the plane of fracture—something of concrete significance and no mere analytical abstraction. The conception then current of the loaded beam as comprised of longitudinal fibres in tension is readily understood today in terms of stresses acting across this plane.

By contrast, in the problem of elastic bending researchers were much slower to develop an analysis that connected the phenomenon in question to the internal structure of the beam. Here there was nothing in the physical situation that identified for immediate study any particular cross-sectional plane. In all of Jakob Bernoulli's seminal writings on the elastica the central idea of stress fails to receive clear identification and development.

Although a general theory did not emerge in the eighteenth century, there were many partial results and successful analyses of particular problems. We will consider one such result in some detail: the derivation by Euler of the buckling formula for a loaded column. (An account of Euler's results is given by Fraser (1990).) Euler obtained this result as a corollary to his analysis of the elastica. We are given an elastic lamina oriented vertically, in which the ends A and B are pinned and forces P and $-P$