§ 10. *D'Alembert's Principle; Introduction of Inertial Forces*

As we have seen, all bodies have the tendency to remain in a state of rest or of uniform rectilinear motion. We can think of this tendency as a resistance to changes in the motion, an inertial resistance, or, for brevity, as an *inertial force*. The definition of inertial force $\mathbf{F}^*$ for the single mass point is therefore

$$\mathbf{F}^* \equiv -\dot{\mathbf{p}} \tag{1}$$

and the fundamental law $\dot{\mathbf{p}} = \mathbf{F}$ takes on the form

$$\mathbf{F}^* + \mathbf{F} = 0. \tag{2}$$

The inertial force is in vectorial equilibrium with the applied force.

While $\mathbf{F}$ is a force given by the physical situation, $\mathbf{F}^*$ is a fictitious force. We introduce it in order to reduce problems of motion to problems involving equilibrium, a procedure that is often convenient.

Inertial forces are familiar to us from everyday life. When we set the heavy revolving door of an hotel in motion, it is not the force of gravity or friction, but the inertia of the door that has to be overcome. A similar example is that of the sliding doors of street cars and trolleys.[3] On the forward platform the door opens in the direction of travel. When the car brakes, the door tends to move forward and can therefore be opened easily. When the car accelerates after a stop, the open door seeks to retain its position of rest; it therefore tends to move to the rear and can be closed without effort. It is easier to get on and off at the front platform than at the rear, where the door opens in the reverse manner.

The best-known form of an inertial force is the *centrifugal force*, which is noticeable in any curved motion. It, too, is a fictitious force. It corresponds to the acceleration $\dot{v}_n$ normal to the curve, which is a *centripetal* acceleration, i.e., directed toward the center of curvature. According to (5.9) the centrifugal force is given by

$$\mathbf{C} = -m\dot{\mathbf{v}}_n, \quad |\mathbf{C}| = m|-\dot{\mathbf{v}}_n| = m\frac{v^2}{\rho}, \tag{3}$$

where the minus sign refers to the outward direction.

The *Coriolis force* (cf. § 28) and the various gyroscopic effects (cf. § 27) also come under the heading of inertial forces.

Incidentally the operation of railroads furnishes a very vivid example of the fact that the "fictitious" centrifugal force has a very real existence.

[3] The translator does not guarantee that the following is applicable to trolleys in the United States. It applies at least in part to the streetcars of San Francisco, which belong, however, to a breed rapidly approaching extinction.

On a curve the rail bed is banked in such a way that the outer rail is higher than the inner. The difference in height is always such that for some mean velocity of the train the resultant of gravity and centrifugal force is perpendicular to the rail bed. This procedure eliminates not only the danger of overturning about the outer rail, but also a harmful unequal loading of the rails.

Strangely enough, the great Heinrich Hertz raises objections to the introduction of the centrifugal force in the unusually beautiful and beautifully written introduction to his " Mechanics " (Collected Works, Vol. III, p. 6):

" We swing a stone attached to a string in a circle; we thereby consciously exert a force on the stone; this force constantly deviates the stone from a straight path, and if we alter this force, the mass of the stone or the length of the string, we discover that indeed the motion of the stone occurs at all times in agreement with Newton's second law. Now the third law demands a force opposing that which is exerted by our hand on the stone. If we ask for this force, we obtain the answer familiar to everybody, that the stone reacts on the hand by virtue of the centrifugal force, and that this centrifugal force is indeed equal and opposite to the force exerted by us on the stone. Is this mode of expression admissible ? Is that which we now call centrifugal force anything but the inertia of the stone ? "

We answer this question with a flat no; indeed the centrifugal force, by virtue of our definition (3), is identical to the inertia of the stone. But the force opposing that which we exert on the stone, i.e., really on the string, is the pull which the string exerts on our hand. Hertz further remarks that " we are forced to the conclusion that the classification of the centrifugal force as a force is not suitable; its name, just like that of live force, is to be regarded as a heritage passed down from former times; and from the point of view of usefulness the retention of this name is easier to excuse than to justify." In regard to this we would like to say that the name centrifugal force needs no justification, for it rests, like the more general term, inertial force, on a clear definition.

Incidentally, it is precisely this alleged lack of clarity of the force concept which induced Hertz, in an interesting but not very fruitful attempt, to construct his mechanics entirely without the notion of force (cf. § 1, p. 5).

We now come to the achievement of d'Alembert (mathematician, philosopher, astronomer, physicist, encyclopedist; " Traité de Dynamique," 1758).

If a mass point k, part of an arbitrary mechanical system, is acted on by an applied force $\mathbf{F}$, Eq. (2) must be changed to read

$$\mathbf{F}_k{}^* + \mathbf{F}_k + \sum_i \mathbf{R}_{ik} = 0. \tag{4}$$

Here $\mathbf{R}_{ik}$ is the reaction which the mass point i connected with k exerts on k. According to our general postulate of p. 52, the $\mathbf{R}_{ik}$, taken together, do no work in an arbitrary virtual displacement compatible with the (here internal) constraints. It follows that the virtual work of the sum of all the $\mathbf{F}^* + \mathbf{F}$ is zero as well,

$$\sum_k (\mathbf{F}_k{}^* + \mathbf{F}_k) \cdot \delta \mathbf{s}_k = 0. \tag{5}$$

Recalling now the principle of virtual work, we can express Eq. (5) by saying that *the inertial forces of a system are in equilibrium with the forces applied to the system.* A knowledge of the reactions is not required.

This is *d'Alembert's principle* in its simplest and most natural form. In order to obtain another interesting formulation of the principle, let us look at the quantity

$$\mathbf{F}_k + \mathbf{F}_k{}^* = \mathbf{F}_k - \dot{\mathbf{p}}_k.$$

It is that part of the force $\mathbf{F}_k$ that cannot be converted into motion of the point k. We can call this part the "lost force" and can therefore re-frame (5) by stating that *the lost forces of a system are in equilibrium.*

A formulation of d'Alembert's principle widely used in textbooks is that expressed in Cartesian coordinates. We call the components of $\mathbf{F}_k$, X_k, Y_k, Z_k and those of $\delta \mathbf{s}_k$, δx_k, δy_k, δz_k. Furthermore, we stipulate that the masses m_k involved are constant; for a system consisting of n mass points we can then replace (5) by

$$\sum_{k=1}^{n} \{(X_k - m_k \ddot{x}_k)\delta x_k + (Y_k - m_k \ddot{y}_k)\delta y_k + (Z_k - m_k \ddot{z}_k)\delta z_k\} = 0. \tag{6}$$

It is here required that the δx_k, δy_k, δz_k be compatible with the constraints of the system. Let us at once consider the general case of non-holonomic constraints. There relations of type (7.4) exist; if we replace the general coordinates q of (7.4) by Cartesian coordinates, these relations become

$$\sum_{\mu=1}^{n} [F_\mu(x_1 \ldots z_n)\delta x_\mu + G_\mu(x_1 \ldots z_n)\delta y_\mu + H_\mu(x_1 \ldots z_n)\delta z_\mu] = 0. \tag{6a}$$

If f is the number of degrees of freedom for infinitesimal motion, there must be $3n - f$ such relations for the δx, δy, δz (cf. p. 50). In the case of holonomic constraints the F_μ, G_μ, H_μ are derivatives of one and the same function with respect to x_μ, y_μ, z_μ.

Let the reader be warned emphatically not to look for the true content of d'Alembert's principle in the clumsy formulation (6), (6a). Equation (5) or the statement of equilibrium equivalent to it is not only more readily useful, but also, by virtue of its invariant form, more natural.

§ 11. *Application of d'Alembert's Principle to the Simplest Problems*

(1) Rotation of a Rigid Body About a Fixed Axis

Here we are dealing with a single degree of freedom, viz., the angle of rotation ϕ. We let $\dot{\phi}=\omega$ be the angular velocity, $\ddot{\phi}=\dot{\omega}$ the angular acceleration. For the present we are not interested in the axle bearings.

We suppose that arbitrary applied forces **F** act on the body. According to § 9, Eq. (7), their virtual work is given by the sum of their moments about the axis of rotation, i.e., by

$$\delta W=\mathbf{L}\cdot\delta\boldsymbol{\phi}=L_a\delta\phi \tag{1}$$

where L_a is the sum of the moments of the **F** about the axis of rotation a. We also wish to know the work done by the inertial forces $\mathbf{F}^*$. For this purpose we subdivide the body into mass elements dm. In view of (10.3) the inertial force acting on dm directed normal to the path is the centrifugal force $dm\frac{v^2}{r}=dm\omega v$. (In circular motion the radius of curvature ρ is of course equal to the distance r from the rotation axis, the velocity v of each element of mass therefore becomes $r\omega$, and its acceleration $\dot{v}$ along the path is $r\cdot\dot{\omega}$). But the centrifugal force does no work. Along the path direction, on the other hand, the inertial force is

$$-dm\dot{v}=-dm\,r\dot{\omega}.$$

The total virtual work of the inertial forces is therefore

$$\sum(-dm\dot{v})\delta s=\sum -dm\,r\dot{\omega}r\,\delta\phi=-\delta\phi\dot{\omega}\int r^2\,dm=-\delta\phi\dot{\omega}I, \tag{2}$$

where

$$I=\int r^2 dm \tag{3}$$

is the *moment of inertia* of the body. The dimensions of I are ML^2, therefore g cm² in the absolute system, g cm sec² in the gravitational system.

By virtue of (1) and (2) d'Alembert's principle takes the form

$$\delta\phi(L_a-I\dot{\omega})=0$$