Principles of the motion of fluids*

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Here are treated the elements of the theory of the motion of fluids in general, the whole matter being reduced to this: given a mass of fluid, either free or confined in vessels, upon which an arbitrary motion is impressed, and which in turn is acted upon by arbitrary forces, to determine the motion carrying forward each particle, and at the same time to ascertain the pressure exerted by each part, acting on it as well as on the sides of the vessel. At first in this memoir, before undertaking the investigation of these effects of the forces, the Most Famous Author[1] carefully evaluates all the possible motions which can actually take place in the fluid. Indeed, even if the individual particles of the fluid are free from each other, motions in which the particles interpenetrate are nevertheless excluded, since we are dealing with fluids that do not permit any compression into a narrower volume. Thus it is clear that an arbitrary small portion of fluid cannot receive a motion other than the one which constantly conserves the same volume; even though meanwhile the shape is changed in any way. It would hold indeed, as long as no elementary portion would be compressed at any time into a smaller volume; furthermore[2] if the portion expanded into a larger volume, the continuity of the particles was violated, these were dispersed and no longer clinged together, such a motion would no longer pertain to the science of the motion of fluids; but individual droplets would separately perform their motion. Therefore, this case being excluded, the motion of the fluids must be restricted by this rule that each small portion must retain for ever the same volume; and this principle restricts the general expressions of motion for elements of the fluid. Plainly, considering an arbitrary small portion of the fluid, its individual points have to be carried by such a motion that, when at a moment of time they arrive at the next location, till then they occupy a volume equal to the previous one; thus if, as usual, the motion of a point is decomposed parallel to fixed orthogonal directions, it is necessary that a certain established relation hold between these three velocities, which the Author has determined in the first part.

In the second part the author proceeds to the determination of the motion of a fluid produced by arbitrary forces, in which matter the whole investigation reduces to this that the pressure with which the parts of the fluid at each point act upon one another shall be ascertained; which pressure is denoted most conveniently, as customary for water, by a certain height; this is to be understood thus, that the each element of the fluid sustains a pressure the same as if were pressed by a heavy column of the same fluid, whose height is equal to that amount. Thus, in such way in each point of the fluid the height referring to the state of the pressure will be given; since it is not equal to the one in the neighbourhood, it will perturb the motion of the elements. But this pressure depends as well on the forces acting on each element of the fluid, as on those, acting in the whole mass; thus, by the given forces, the pressure in each point and thereupon the acceleration of each element – or its retardation – can be assigned for the motion, all which determinations are expressed by the author through differential formulae. But, in fact, the full development of these formulas mostly involves the greatest difficulties. But nevertheless this whole theory has been reduced to pure analysis, and what remains to be completed in it depends solely upon subsequent progress in Analysis. Thus it is far from true that purely analytic researches are of no use in applied mathematics; rather, important additions in pure analysis are now required.

I. FIRST PART

1. Since liquid substances differ from solid ones by the fact that their particles are mutually independent of each other, they can also receive most diverse motions; the motion performed by an arbitrary particle of the fluid is not determined by the motion of the remaining particles to the point that it cannot move in any other way. The matter is very different in solid bodies, which, if they were inflexible, would not undergo any change in their shape; in whatsoever way they be moved, each of their particles would constantly keep the same location and distance with respect to other particles; it thus follows that, the motion of two or, if necessary, three of all the particles being known, the motion of any other particle can be defined; furthermore the motion of two or three particles of such a body cannot be chosen at will, but must be constrained in such a way that these particles preserve constantly their positions with respect to each other.[1]

2. But if, moreover, solid bodies are flexible, the motion of each particle is less constrained: because of the bending,

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*This is an English adaptation by Walter Pauls of Euler's memoir 'Principia motus fluidorum' (Euler, 1756–1757). Updated versions of the translation may become available at `www.oca.eu/etc7/EE250/texts/euler1761eng.pdf`. For a detailed presentation of Euler's fluid dynamics papers, cf. Truesdell, 1954, which has also been helpful for this translation. Euler's work is discussed in the perspective of eighteenth century fluid dynamics research by Darrigol and Frisch, 2008. The help of O. Darrigol, U. Frisch, G. Grimberg and G. Mikhailov is also acknowledged.
Explanatory footnotes and references have been supplied where necessary; Euler's memoir had neither footnotes nor a list of references.

[1]Summaries, which at that time were not placed at the beginning of the corresponding paper, were published under the responsibility of the Academy; the presence of the words "Most Famous Author", rather common at the time, cannot be taken as evidence that Euler usually referred to himself in this way.

[2]In the original, we find "verum quoniam"; the litteral translation "since indeed" does not seem logically consistent.

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[1] Here Euler refers to the motion of rigid solid bodies treated previously in Euler, 1750.

the distance as well as the relative position of each particle can be subject to changes. However, the manner itself of the bending constitutes a certain law which various particles of such a body have to obey in their motion: certainly what has to be taken care of is that the parts that experience in their neighborhood such a strong bending with respect to each other are neither torn apart from the inside nor penetrate into each other. Indeed, as we shall see, impenetrability is demanded for all bodies.

3. In fluid bodies, whose particles are united among themselves by no bond, the motion of each particle is much less restricted: the motion of the remaining particles is not determined from the motion of any number of particles. Even knowing the motion of one hundred particles, the future motion permitted to the remaining particles still can vary in infinitely many ways. From which it is seen that the motion of these fluid particles plainly does not depend on the motion of the remaining ones, unless it be enclosed by these so that it is constrained to follow them.

4. However, it cannot happen that the motion of all particles of the fluid suffers no restrictions at all. Furthermore, one cannot at will invent a motion that is conceived to occur for each particle. Since, indeed, the particles are impenetrable, it is immediately clear that a motion cannot be maintained in which some particles go through other particles and, accordingly, penetrate each other: also, because of this reason such motion certainly cannot be conceived to occur in the fluid. Therefore, infinitely many motions must be excluded; after their determination the remaining ones are grouped together. It is seen worthwhile to define them more accurately regarding the property which distinguishes them from the previous ones.

5. But before the motion by which the fluid is agitated at any place can be defined, it is necessary to see how every motion, which can definitely be maintained in this fluid, be recognized: these motions, here, I will call possible, which I will distinguish from impossible motions which certainly cannot take place. We must then find what characteristic is appropriate to possible motions, separating them from impossible ones. When this is done, we shall have to determine which one of all possible motions in a certain case ought actually to occur. Plainly we must then turn to the forces which act upon the water, so that the motion appropriate to them may be determined from the principles of mechanics.

6. Thus, I decided to inquire into the character of the possible motions, such that no violation of the impenetrability can occur in the fluid. I shall assume the fluid to be such as never to permit itself to be forced into a lesser space, nor should its continuity be interrupted. Once the theory of fluids has been adjusted to fluids of this nature, it will not be difficult to extend it also to those fluids whose density is variable and which do not necessarily require continuity.[2]

7. If, thus, we consider an arbitrary portion in such a fluid, the motion, by which each of its particles is carried has to be set up so that at each time they occupy an equal volume. When this occurs in separate portions, any expansion into a larger volume, or compression into a smaller volume is prohibited. And, if we turn attention to this only property, we can have only such motion that the fluid is not permitted to expand or compress. Furthermore, what is said here about arbitrary portions of the fluid, has to be understood for each of its elements; so that the volume of its elements must constantly preserve its value.

8. Thus, assuming that this condition holds, let an arbitrary motion be considered to occur at each point of the fluid; moreover, given any element of the fluid, consider the brief translations of each of its boundaries. In this manner the volume, in which the element is contained after a very short time, becomes known. From there on, this volume is posed to be equal to the one occupied previously, and this equation will prescribe the calculation of the motion, in so far as it will be possible. Since all elements occupy the same volumes during all periods of time, no compression of the fluid, nor expansion can occur; and the motion is arranged in such a way that this becomes possible.

9. Since we consider not only the velocity[3] of the motion occuring at every points of the fluid but also its direction, both aspects are most conveniently handled, if the motion of each point is decomposed along fixed directions. Moreover, this decomposition is usually carried out with respect to two or three directions:[4] the former is appropriate for the decomposition, if the motion of all points is completed in the same plane; but if their motion is not contained in the same plane, it is appropriate to decompose the motion following three fixed axes. Because the latter case is more difficult to treat, it is more convenient to begin the investigation of possible motions with the former case; once this has been done, the latter case will be easily completed.

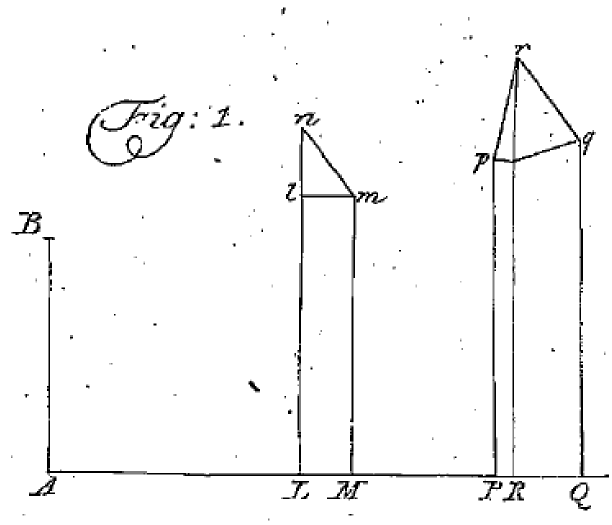
10. First I will assign to the fluid two dimensions in such a way that all of its particles are now not only found with certainty in the same plane, but also their motion is performed in it. Let this plane be represented in the plane of the table (Fig. 1), let an arbitrary point l of the fluid be considered, its position being denoted by orthogonal coordinates $\mathrm{AL} = x$ and $\mathrm{L}l = y$. The motion is decomposed following these directions, giving a velocity $lm = u$ parallel to the axis AL and $ln = v$ parallel to the other axis AB: so that the true future velocity of this point is $= \sqrt{(uu + vv)}$, and its direction with respect to the axis AL is inclined by an angle with the tangent $\frac{v}{u}$.

11. Since the state of the motion, presented in a way which suits the each point of the fluid, is supposed to evolve, the velocities u and v will depend on the position l of the point and will therefore be functions of the coordinates x and y.

[2] See the English translation of "General laws of the motion of fluids" in these Proceedings.

[3] Meaning here the absolute value of the velocity.

[4] Depending on the dimension: Euler treats both the two- and the three-dimensional cases.



Thus, we put upon a differentiation

$$du = \mathrm{L}dx + ldy \quad \text{and} \quad dv = \mathrm{M}dx + mdy,$$

which differential formulas, since they are complete,[5] satisfy furthermore $\frac{d\mathrm{L}}{dy} = \frac{dl}{dx}$ and $\frac{d\mathrm{M}}{dy} = \frac{dm}{dx}$. Here it is to note that in such expression $\frac{d\mathrm{L}}{dy}$, the differential of L itself or dL, is understood to be obtained from the variability with respect to y; in similar manner in the expression dl/dx, for dl the differential of l itself has to be taken, which arises if we take x to vary.

12. Thus, it is in order to be cautious and not to take in such fractional expressions $\frac{d\mathrm{L}}{dy}$, $\frac{dl}{dx}$, $\frac{d\mathrm{M}}{dy}$, and $\frac{dm}{dx}$ the numerators dL, dl, dM, and dm as denoting the complete differentials of the functions L, l, M and m; but constantly they designate such differentials that are obtained from the variation of only one coordinate, obviously the one, whose differential is represented in the denominator; thus, such expressions will always represent finite and well defined quantities. Furthermore, in the same way are understood L$= \frac{du}{dx}$, $l = \frac{du}{dy}$, M$= \frac{dv}{dx}$ and $m = \frac{dv}{dy}$; which notation of ratios has been used for the first time by the most enlightened *Fontaine*,[6] and I will also apply it here, since it gives a non negligible advantage of calculation.

13. Since $du = \mathrm{L}dx + ldy$ and $dv = \mathrm{M}dx + mdy$, here it is appropriate to assign a pair of velocities to the point which is at an infinitely small distance from the point l; if the distance of such a point from the point l parallel to the axis AL is dx, and parallel to the axis AB is dy, then the velocity of this point parallel to the axis AL will be $u + \mathrm{L}dx + ldy$; furthermore, the velocity parallel to the other axis AB is $v + \mathrm{M}dx + mdy$. Thus, during the infinitely short time dt this point will be carried parallel to the direction of the axis AL the distance $dt(u + \mathrm{L}dx + ldy)$ and parallel to the direction of the other axis AB the distance $dt(v + \mathrm{M}dx + mdy)$.

14. Having noted these things, let us consider a triangular element lmn of water, and let us seek the location into which it is carried by the motion during the time dt. Let lm be the side parallel to the axis AL and let ln be the side parallel to the axis AB: let us also put $lm = dx$ and $ln = dy$; or let the coordinates of the point m be $x + dx$ and y; the coordinates of the point n be x and $y + dy$. It is plain, since we do not define the relation between the differentials dx and dy, which can be taken negative as well as positive, that in thought the whole mass of fluid may be divided into elements of this sort, so that what we determine for one in general will extend equally to all.

15. To find out how far the element lmn is carried during the time dt due to the local motion, we search for the points p, q and r, to which its vertices, or the points l, m and n are transferred during the time dt. Since

	of point l	of point m	of point n
Velocity w.r.t. AL=	u	$u + \mathrm{L}dx$	$u + ldy$
Velocity w.r.t. AB=	v	$v + \mathrm{M}dx$	$v + mdy$

in the time dt the point l reaches the point p, chosen such that:

$$\mathrm{AP} - \mathrm{AL} = udt \quad \text{and} \quad \mathrm{P}p - \mathrm{L}l = vdt.$$

Furthermore, the point m reaches the point q, such that

$$\mathrm{AQ} - \mathrm{AM} = (u + \mathrm{L}dx)dt \quad \text{and} \quad \mathrm{Q}q - \mathrm{M}m = (v + \mathrm{M}dx)dt.$$

Also, the point n is carried to r, chosen such that

$$\mathrm{AR} - \mathrm{AL} = (u + ldy)dt \quad \text{and} \quad \mathrm{R}r - \mathrm{L}n = (v + mdy)dt.$$

16. Since the points l, m and n are carried to the points p, q and r, the triangle lmn made of the joined straight lines pq, pr and qr, is thought to be arriving at the location defined by the triangle pqr. Because the triangle lmn is infinitely small, its sides cannot receive any curvature from the motion, and therefore, after having performed the translation of the element of water lmn in the time dt, it will conserve the straight and triangular form. Since this element lmn must not be either extended to a larger volume, nor compressed into a smaller one, the motion should be arranged so that the volume of the triangle pqr is rendered to be equal to the area of the triangle lmn.

17. The area of the triangle lmn, being rectangular at l, is $= \frac{1}{2}dxdy$, value to which the area of the triangle pqr should be put equal. To find this area, the pair of coordinates of the points p, q and r must be examined, which are:

$$\mathrm{AP} = x + udt; \quad \mathrm{AQ} = x + dx + (u + \mathrm{L}dx)dt;$$
$$\mathrm{AR} = x + (u + ldy)dt; \quad \mathrm{P}p = y + vdt$$
$$\mathrm{Q}q = y + (v + \mathrm{M}dx)dt, \quad \mathrm{R}r = y + dy + (v + mdy)dt$$

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[5] Exact differentials.

[6] A paper "Sur le calcul intégral" containing the notation $\frac{df}{dx}$ for the partial derivative of f with respect to x was presented by Alexis Fontaine des Bertins to the Paris Academy of Sciences in 1738, but it was published only a quarter of a century later (Fontaine, 1764). Nevertheless, Fontaine's paper was widely known among mathematicians from the beginning of the 1740s, and, particularly, was discussed in the correspondence between Euler, Daniel Bernoulli and Clairaut; cf. Euler, 1980: 65–246.

Then, indeed, the area of the triangle pqr is found from the area of the succeeding trapezoids, so that

$$pqr = \mathrm{P}pr\mathrm{R} + \mathrm{R}rq\mathrm{Q} - \mathrm{P}pq\mathrm{Q}.$$

Since these trapezoids have a pair of sides parallel and perpendicular to the base AQ, their areas are easily found.

18. Plainly, these areas are given by the expressions

$$\begin{aligned}\mathrm{P}pr\mathrm{R} &= \frac{1}{2}\mathrm{PR}(\mathrm{P}p + \mathrm{R}r)\\ \mathrm{R}rq\mathrm{Q} &= \frac{1}{2}\mathrm{RQ}(\mathrm{R}r + \mathrm{Q}q)\\ \mathrm{P}pq\mathrm{Q} &= \frac{1}{2}\mathrm{PQ}(\mathrm{P}p + \mathrm{Q}q)\end{aligned}$$

By putting these together we find:

$$\Delta pqr = \frac{1}{2}\mathrm{PQ}.\mathrm{R}r - \frac{1}{2}\mathrm{RQ}.\mathrm{P}p - \frac{1}{2}\mathrm{PR}.\mathrm{Q}q$$

Let us set for brevity

$\mathrm{AQ} = \mathrm{AP} + \mathrm{Q}$; $\mathrm{AR} = \mathrm{AP} + \mathrm{R}$; $\mathrm{Q}q = \mathrm{P}p + q$; and $\mathrm{R}r = \mathrm{P}p + r$,

so that $\mathrm{PQ} = \mathrm{Q}$, $\mathrm{PR} = \mathrm{R}$, and $\mathrm{RQ} = \mathrm{Q} - \mathrm{R}$, and we have $\Delta pqr = \frac{1}{2}\mathrm{Q}(\mathrm{P}p + r) - \frac{1}{2}(\mathrm{Q} - \mathrm{R})\mathrm{P}p - \frac{1}{2}\mathrm{R}(\mathrm{P}p + q)$ or $\Delta pqr = \frac{1}{2}\mathrm{Q}.r - \frac{1}{2}\mathrm{R}.q$.

19. Truly, from the values of the coordinates represented before it follows that

$$\begin{aligned}&\mathrm{Q} = dx + \mathrm{L}dxdt; \quad q = \mathrm{M}dxdt\\ &\mathrm{R} = ldydt; \quad r = dy + mdydt,\end{aligned}$$

upon the substitution of these values, the area of the triangle is obtained

$$\begin{aligned}pqr &= \frac{1}{2}dxdy(1 + \mathrm{L}dt)(1 + mdt) - \frac{1}{2}\mathrm{M}l\, dxdydt^2, \quad \text{or}\\ pqr &= \frac{1}{2}dxdy(1 + \mathrm{L}dt + mdt + \mathrm{L}mdt^2 - \mathrm{M}ldt^2).\end{aligned}$$

This should be equal to the area of the triangle lmn, that is $= \frac{1}{2}dxdy$; hence we obtain the following equation

$$\begin{aligned}&\mathrm{L}dt + mdt + \mathrm{L}mdt^2 - \mathrm{M}ldt^2 = 0 \quad \text{or}\\ &\mathrm{L} + m + \mathrm{L}mdt - \mathrm{M}ldt = 0.\end{aligned}$$

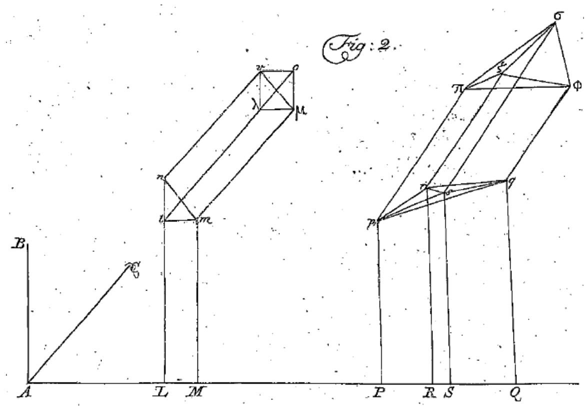
20. Since the terms $\mathrm{L}mdt$ and $\mathrm{M}ldt$ vanish for finite L and m, we will have the equation $\mathrm{L} + m = 0$. Hence, for the motion to be possible, the velocities u and v of any point l have to be arranged such that after calculating their differentials

$$du = \mathrm{L}dx + ldy, \quad \text{and} \quad dv = \mathrm{M}dx + mdy,$$

one has $\mathrm{L} + m = 0$. Or, since $\mathrm{L} = \frac{du}{dx}$ and $m = \frac{dv}{dy}$, the velocities u and v, which are considered to occur at the point l parallel to the axes AL and AB, must be functions of the coordinates x and y such that $\frac{du}{dx} + \frac{dv}{dy} = 0$, and thus, the criterion of possible motions consists in this that $\frac{du}{dx} + \frac{dv}{dy} = 0$;[7] and unless this condition holds, the motion of the fluid cannot take place.

21. We shall proceed identically when the motion of the fluid is not confined to the same plane. Let us assume, to investigate this question in the broadest sense, that all particles of the fluid are agitated among themselves by an arbitrary motion, with the only law to be respected that neither condensation nor expansion of the parts occurs anywhere: in the same way, we seek which condition should apply to the velocities that are considered to occur at every point, so that the motion is possible: or, which amounts to the same, all motions that are opposed to these conditions should be eliminated from the possible ones, this being the criterion of possible motions.

22. Let us consider an arbitrary point of the fluid λ. To represent its location we use three fixed axes AL, AB and AC orthogonal to each other (Fig. 2). Let the triple coordinates parallel to these axes be AL$= x$, L$l = y$ and $l\lambda = z$; which are obtained if firstly a perpendicular λl is dropped from the point λ to the plane determined by the two axes AL and AB; and then a perpendicular lL is drawn from the point l to the axis AL. In this manner the location of the point λ is expressed through three such coordinates in the most general way and can be adapted to all points of the fluid.



23. Whatever the later motion of the point λ, it can be resolved following the three directions $\lambda\mu$, $\lambda\nu$, λo, parallel to the axes AL, AB and AC. For the motion of the point λ we set

$$\begin{aligned}&\text{the velocity parallel to the direction} \quad \lambda\mu = u,\\ &\text{the velocity parallel to the direction} \quad \lambda\nu = v,\\ &\text{the velocity parallel to the direction} \quad \lambda o = w.\end{aligned}$$

Since these velocities can vary in an arbitrary manner for different locations of the point λ, they will have to be considered as functions of the three coordinates x, y and z. After differ-

[7] This is the two-dimensional incompressibility condition, which in a slightly different form has already been established by d'Alembert, 1752; cf. also Darrigol and Frisch, 2008:§ III.

entiating them, let us put to proceed

$$du = \mathrm{L}dx + ldy + \lambda dz$$
$$dv = \mathrm{M}dx + mdy + \mu dz$$
$$dw = \mathrm{N}dx + ndy + \nu dz.$$

Henceforth the quantities L, l, λ, M, m, μ, N, n, ν will be functions of the coordinates x, y and z.

24. Because these formulas are complete differentials, we obtain as above

$$\frac{d\mathrm{L}}{dy} = \frac{dl}{dx}; \quad \frac{d\mathrm{L}}{dz} = \frac{d\lambda}{dx}; \quad \frac{dl}{dz} = \frac{d\lambda}{dy}$$
$$\frac{d\mathrm{M}}{dy} = \frac{dm}{dx}; \quad \frac{d\mathrm{M}}{dz} = \frac{d\mu}{dx}; \quad \frac{dm}{dz} = \frac{d\mu}{dy}$$
$$\frac{d\mathrm{N}}{dy} = \frac{dn}{dx}; \quad \frac{d\mathrm{N}}{dz} = \frac{d\nu}{dx}; \quad \frac{dn}{dz} = \frac{d\nu}{dy},$$

where it is assumed that the only varying coordinate is that whose differential appears in the denominator varies.[8]

25. Thus, this point λ will be moved in the time dt by this threefold motion, which is considered to take place at the point X; hence it moves

$$\text{parallel to the axis AL the distance} = udt$$
$$\text{parallel to the axis AB the distance} = vdt$$
$$\text{parallel to the axis AC the distance} = wdt$$

The true velocity of the point λ, denoted by $= V$, which clearly arises from the composition of this triple motion, is given in view of orthogonality of the three directions by $V = \sqrt{(uu + vv + ww)}$ and the elementary distance, which is traveled in the time dt through its motion, will be Vdt.

26. Let us consider an arbitrary solid element of the fluid to see whereto it is carried during the time dt; since it amounts to the same, let us assign a quite arbitrary shape to that element, but of a kind such that the entire mass of the fluid can be divided into such elements; to investigate by calculation, let the shape be a right triangular pyramid, bounded by four solid angles λ, μ, ν and o, so that for each one there are three coordinates

	of point λ	of point μ	of point ν	of point o
w.r.t. AL	x	$x + dx$	x	x
w.r.t. AB	y	y	$y + dy$	y
w.r.t. AC	z	z	z	$z + dz$

Since the base of this pyramid is $\lambda\mu\nu = lmn = \frac{1}{2}dxdy$ and the hight $\lambda o = dz$, its volume will be $= \frac{1}{6}dxdydz$.

27. Let us investigate, whereto these vertices λ, μ, ν and o are carried during the time dt: for which purpose their three velocities parallel to the directions of the three axes must be considered. The differential values of the velocities u, v and w are given by

Velocity	of point λ	of point μ	of point ν	of point o
w.r.t. AL	u	$u + \mathrm{L}dx$	$u + ldy$	$u + \lambda dz$
w.r.t. AB	v	$v + \mathrm{M}dx$	$v + mdy$	$v + \mu dz$
w.r.t. AC	w	$w + \mathrm{N}dx$	$w + ndy$	$w + odz$

28. If we let the points λ, μ, ν and o be transferred to the points π, Φ, ρ and σ in the time dt, and establish the three coordinates of these points parallel to the axes, the small displacement parallel to these axes will be

$$\begin{array}{ll}
\mathrm{AP} - \mathrm{A\,L} & = u\,dt \\
\mathrm{AQ} - \mathrm{A\,M} & = (u + \mathrm{L}\,dx)\,dt \\
\mathrm{AR} - \mathrm{A\,L} & = (u + l\ \ dy)\,dt \\
\mathrm{AS} - \mathrm{A\,L} & = (u + \lambda\ dz)\,dt \\
\hline
\mathrm{P}p\ - \mathrm{L}l & = v\,dt \\
\mathrm{Q}q\ - \mathrm{M}m & = (v + \mathrm{M}dx)\,dt \\
\mathrm{R}r\ - \mathrm{L}n & = (v + mdy)\,dt \\
\mathrm{S}\,s\ - \mathrm{L}l & = (v + \mu\ dz)\,dt \\
\hline
p\,\pi\ - l\ \lambda & = w\,dt \\
q\,\Phi - m\mu & = (w + \mathrm{N}\,dx)\,dt \\
r\,\rho\ - n\nu & = (w + n\ dy)\,dt \\
s\,\sigma\ - l\ o & = (w + \nu\ dz)\,dt
\end{array}$$

Thus the three coordinates for these four points π, Φ, ρ and σ will be

$$\begin{array}{rl}
\mathrm{AP} = x + udt; & \mathrm{P}p = y + vdt; \\
p\pi = z + wdt & \\
\mathrm{RQ} = x + dx + (u + \mathrm{L}dx)dt; & \mathrm{Q}q = y + (v + \mathrm{M}dx)dt; \\
q\Phi = z + (w + \mathrm{N}dx)dt & \\
\mathrm{AR} = x + (u + ldy)dt; & \mathrm{R}r = y + dy + (v + mdy)dt; \\
r\rho = z + (w + ndy)dt & \\
\mathrm{AS} = x + (u + \lambda dz)dt; & \mathrm{S}s = y + (v + \mu dz)dt; \\
s\sigma = z + dz + (w + \nu dz)dt &
\end{array}$$

29. Since after the time dt has elapsed the vertices λ, μ, ν and o of the pyramid are transferred to the points π, Φ, ρ and σ, $\pi\Phi\rho\sigma$ defines a similar triangular pyramid. Due to the nature of the fluid the volume of the pyramid $\pi\Phi\rho\sigma$ should be equal to the volume of the pyramid $\lambda\mu\nu o$ put forward, that is $= \frac{1}{6}dxdydz$. Thus, the whole matter is reduced to determining the volume of the pyramid $\pi\Phi\rho\sigma$. Clearly, it remains a pyramid, if the solid $pqr\pi\Phi\rho\sigma$ is removed from the solid $pqr\pi\Phi\rho\sigma$; the latter solid is a prism orthogonally incident to the triangular basis pqr, and cut by the upper oblique section $\pi\rho\Phi$.

30. The other solid $pqr\pi\Phi\rho\sigma$ can be divided by similarly into three prisms truncated in this manner, namely

$$\text{I. } pqrs\pi\Phi\sigma; \quad \text{II. } prs\pi\rho\sigma; \quad \text{III. } qrs\Phi\rho\sigma$$

This has to be accomplished in such a way that

$$\frac{1}{6}dxdydz = pqs\pi\Phi\sigma + prs\pi\rho\sigma + qrs\Phi\rho\sigma - pqr\pi\Phi\rho.$$

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[8] The partial differential notation was so new that Euler had to remind the reader of its definition.

Since such a prism is orthogonally incident to its lower base, and furthermore has three unequal heights, its volume is found by multiplying the base by one third of the sum of these heights.

31. Thus, the volumes of these truncated prisms will be

$$\begin{aligned} p\,q\,s\,\pi\Phi\sigma &= \frac{1}{3}p\,q\,s\,(p\,\pi + q\,\Phi + s\,\sigma) \\ p\,r\,s\,\pi\rho\,\sigma &= \frac{1}{3}p\,r\,s\,(p\,\pi + r\,\rho + s\,\sigma) \\ q\,r\,s\,\Phi\rho\,\sigma &= \frac{1}{3}q\,r\,s\,(q\,\Phi + r\,\rho + s\,\sigma) \\ p\,q\,r\,\pi\Phi\rho &= \frac{1}{3}p\,q\,r\,(p\,\pi + q\,\Phi + r\,\rho). \end{aligned}$$

Since $pqr = pqs + prs + qrs$, the sum of the first three prisms will definitely be small, or

$$\frac{1}{6}dxdydz = -\frac{1}{3}p\pi.qrs - \frac{1}{3}q\Phi.prs - \frac{1}{3}r\rho.pqs + \frac{1}{3}s\sigma.pqr,$$

or

$$dxdydz = 2pqr.s\sigma - 2pqs.r\rho - 2prs.q\Phi - 2qrs.p\pi.$$

32. Thus, it remains to define the bases of these prisms: but before we do this, let us put

$$\begin{aligned} &\mathrm{AQ} = \mathrm{AP} + \mathrm{Q}; \quad \mathrm{Q}q = \mathrm{P}p + q\,; \quad q\Phi = p\pi + \Phi; \\ &\mathrm{AR} = \mathrm{AP} + \mathrm{R}; \quad \mathrm{R}r = \mathrm{P}p + r\,; \quad r\rho = p\pi + \rho\,; \\ &\mathrm{AS} = \mathrm{AP} + \mathrm{S}\,; \quad \mathrm{S}s = \mathrm{P}p + s\,; \quad s\sigma = p\pi + \sigma\,, \end{aligned}$$

in order to shorten the following calculations. After the substitution of these values, the terms containing $p\pi$ will annihilate each other, and we shall have

$$dxdydz = 2pqr.\sigma - 2pqs.\rho - 2prs.\Phi$$

so that the value of the bases to be investigated is smaller.

33. Furthermore the triangle pqr is obtained by removing the trapezoid PpqQ from the figure PprqQ, the latter being the sum of the trapezoids PprR and RrqQ; from which it follows that

$$\Delta pqr = \frac{1}{2}\mathrm{PR}(\mathrm{P}p + \mathrm{R}r) + \frac{1}{2}\mathrm{RQ}(\mathrm{R}r + \mathrm{Q}q) - \frac{1}{2}\mathrm{PQ}(\mathrm{P}p + \mathrm{Q}q);$$

or, because of PR = R; RQ = Q − R; and PQ = Q we shall have

$$\Delta pqr = \frac{1}{2}\mathrm{R}(\mathrm{P}p - \mathrm{Q}q) + \frac{1}{2}\mathrm{Q}(\mathrm{R}r - \mathrm{P}p) = \frac{1}{2}\mathrm{Q}r - \frac{1}{2}\mathrm{R}q.$$

In the same manner we have

$$\Delta pqs = \frac{1}{2}\mathrm{PS}(\mathrm{P}p + \mathrm{S}s) + \frac{1}{2}\mathrm{SQ}(\mathrm{S}s + \mathrm{Q}q) - \frac{1}{2}\mathrm{PQ}(\mathrm{P}p + \mathrm{Q}q),$$

or

$$\Delta pqs = \frac{1}{2}\mathrm{S}(\mathrm{P}p + \mathrm{S}s) + \frac{1}{2}(\mathrm{Q} - \mathrm{S})(\mathrm{S}s + \mathrm{Q}q) - \frac{1}{2}\mathrm{Q}(\mathrm{P}p + \mathrm{Q}q),$$

from where it follows that:

$$\Delta pqs = \frac{1}{2}\mathrm{S}(\mathrm{P}p - \mathrm{Q}q) + \frac{1}{2}\mathrm{Q}(\mathrm{S}s - \mathrm{P}p) = \frac{1}{2}\mathrm{Q}s - \frac{1}{2}\mathrm{S}q.$$

And finally

$$\Delta prs = \frac{1}{2}\mathrm{PR}(\mathrm{P}p + \mathrm{R}r) + \frac{1}{2}\mathrm{RS}(\mathrm{R}r + \mathrm{S}s) - \frac{1}{2}\mathrm{PS}(\mathrm{P}p + \mathrm{S}s),$$

or

$$\Delta prs = \frac{1}{2}\mathrm{R}(\mathrm{P}p + \mathrm{R}r) + \frac{1}{2}(\mathrm{S} - \mathrm{R})(\mathrm{R}r + \mathrm{S}s) - \frac{1}{2}\mathrm{S}(\mathrm{P}p + \mathrm{S}s)$$

from where it follows that

$$\Delta prs = \frac{1}{2}\mathrm{R}(\mathrm{P}p - \mathrm{S}s) + \frac{1}{2}\mathrm{S}(\mathrm{R}r - \mathrm{P}p) = \frac{1}{2}\mathrm{S}r - \frac{1}{2}\mathrm{R}s.$$

34. After the substitution of these values we will obtain

$$dxdydz = (\mathrm{Q}r - \mathrm{R}q)\sigma + (\mathrm{S}q - \mathrm{Q}s)\rho + (\mathrm{R}s - \mathrm{S}r)\Phi;$$

thus the volume of the pyramid $\pi\Phi\rho\sigma$ will be

$$\frac{1}{6}(\mathrm{Q}r - \mathrm{R}q)\sigma + \frac{1}{6}(\mathrm{S}q - \mathrm{Q}s)\rho + \frac{1}{6}(\mathrm{R}s - \mathrm{S}r)\Phi.$$

From the values of the coordinates presented above in §. 28 follows

$$\begin{aligned} &\mathrm{Q} = dx + \mathrm{L}dxdt \quad && q = \mathrm{M}dxdt \quad && \Phi = \mathrm{N}dxdt \\ &\mathrm{R} = ldydt && r = dy + mdydt && \rho = ndydt \\ &\mathrm{S} = \lambda dzdt && s = \mu dzdt && \sigma = dz + \nu dzdt. \end{aligned}$$

35. Since here we have

$$\begin{aligned} \mathrm{Q}r - \mathrm{R}q &= dxdy(1 + \mathrm{L}dt + mdt + \mathrm{L}mdt^2 - \mathrm{M}ldt^2) \\ \mathrm{S}q - \mathrm{Q}s &= dxdz(-\mu dt - \mathrm{L}\mu dt^2 + \mathrm{M}\lambda dt^2) \\ \mathrm{R}s - \mathrm{S}r &= dydz(-\lambda dt - m\lambda dt^2 + l\mu dt^2) \end{aligned}$$

the volume of the pyramid $\pi\Phi\rho\sigma$ is found to be expressed as

$$\frac{1}{6}dxdydz\left\{\begin{array}{llll} 1 & +\mathrm{L}\,dt & +\mathrm{L}m\,dt^2 & +\mathrm{L}m\nu\,dt^3 \\ & +mdt & -\mathrm{M}l\,dt^2 & -\mathrm{M}l\nu\,dt^3 \\ & +\nu\,dt & +\mathrm{L}\nu\,dt^2 & -\mathrm{L}n\mu\,dt^3 \\ & & +m\nu\,dt^2 & +\mathrm{M}n\lambda\,dt^3 \\ & & -n\mu\,dt^2 & -\mathrm{N}m\lambda\,dt^3 \\ & & -\mathrm{N}\lambda\,dt^2 & +\mathrm{N}l\mu\,dt^3 \end{array}\right\},$$

which (volume), since it must be equal to that of the pyramid $\lambda\mu\nu o = \frac{1}{6}dxdydz$, will satisfy, after performing a division by dt the following equation[9]

$$\begin{aligned} 0 = &\mathrm{L} + m + \nu + dt(\mathrm{L}m + \mathrm{L}\nu + m\nu - \mathrm{M}l - \mathrm{N}\lambda - n\mu) \\ &+ dt^2(\mathrm{L}m\nu + \mathrm{M}n\lambda + \mathrm{N}l\mu - \mathrm{L}n\mu - \mathrm{M}l\nu - \mathrm{N}l\mu). \end{aligned}$$

36. Discarding the infinitely small terms, we get this equation:[10] $\mathrm{L} + m + \nu = 0$, through which is determined the relation between u, v and w, so that the motion of the fluid be

[9] This is the calculation to which Euler refers in his later French memoir Euler, 1755.

[10] This is the three-dimensional incompressibility condition.

possible. Since L$=\frac{du}{dx}$, $m=\frac{dv}{dy}$ and $\nu=\frac{dw}{dz}$, at an arbitrary point of the fluid λ, whose position is defined by the three coordinates x, y and z, and the velocities u, v and w are assigned in the same manner to be directed along these same coordinates, the criterion of possible motions is such that

$$\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}=0.$$

This condition expresses that through the motion no part of the fluid is carried into a greater or or lesser space, but perpetually the continuity of the fluid as well as the identical density is conserved.

37. This property is to be interpreted further so that at the same instant it is extended to all points of the fluid: of course, the three velocities of all the points must be such functions of the three coordinates x, y and z that we have $\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}=0$: in this way the nature of those functions defines the motion of every point of the fluid at a given instant. At another time the motion of the same points may be howsoever different, provided that at an arbitrary point of time the property holds for the whole fluid. Up to now I have considered the time simply as a constant quantity.

38. If however, we also wish to consider the time as variable so that the motion of the point λ whose position is given by the three coordinates AL$=x$, L$l=y$ and $l\lambda=z$ has to be defined after the elapsed time t, it is certain that the three velocities u, v and w depend not only on the coordinates x, y and z but additionally on the time t, that is they will be functions of these four quantities x, y, z and t; furthermore, their differentials are going to have the following form

$$\begin{aligned}du &= \mathrm{L}dx+ldy+\lambda dz+\mathfrak{L}dt;\\ dv &= \mathrm{M}dx+mdy+\mu dz+\mathfrak{M}dt;\\ dw &= \mathrm{N}dx+ndy+\nu dz+\mathfrak{N}dt;\end{aligned}$$

Meanwhile we shall always have $\mathrm{L}+m+\nu=0$, therefore at every arbitrary instant the time t is considered to be constant, or $dt=0$. Howsoever the functions u, v and w vary with time t, it is necessary that at every moment of time the following holds:

$$\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}=0.$$

Since the condition expresses that any arbitrary portion of the fluid is carried in a time dt into a volume equal to itself, the same will have to happen, due to the same condition, in the next time interval, and therefore in all the following time intervals.

II. SECOND PART

39. Having exposed what pertains to all possible motions, let us now investigate the nature of the motion which can really occur in the fluid. Here, besides the continuity of the fluid and the constancy of its density, we will also have to consider the forces which act on every element of the fluid. When the motion of any element is either non-uniform or varying in its direction, the change of the motion must be in accordance with the forces acting on this element. The change of the motion becomes known from the known forces, and the preceding formulas contain this change; we will now deduce new conditions[11] which single out the actual motion among all those possible up to this point.

40. Let us arrange this investigation in two parts as well; at first let us consider all motions being performed in the same plane. Let AL $=x$, L$l=y$ be, as before, the defining coordinates of the position of an arbitrary point l; now, after the elapsed time t, the two velocities of the point l parallel to the axes AL and AB are u and v: since the variability of time has to be taken into account, u and v will be functions of x, y and t themselves. in respect of which we put

$$du=\mathrm{L}dx+ldy+\mathfrak{L}dt\quad\text{and}dv=\mathrm{M}dx+mdy+\mathfrak{M}dt$$

and we have established above that because of the former condition encoutered above, we have $\mathrm{L}+m=0$.

41. After an elapsed small time interval dt the point l is carried to p, and it has travelled a distance udt parallel to the axis AL, a distance vdt parallel to the other axis AB. Hence, to obtain the increments in the velocities u and v of the point l which are induced during the time dt, for dx and dy we must write the distance udt and vdt, from which will arise these true increments of the velocities

$$du=\mathrm{L}udt+lvdt+\mathfrak{L}dt\quad\text{and}\quad dv=\mathrm{M}udt+mvdt+\mathfrak{M}dt.$$

Therefore the accelerating forces, which produce these accelerations are

$$\begin{aligned}&\text{Accel. force w.r.t. AL}=2(\mathrm{L}u+lv+\mathfrak{L})\\ &\text{Accel. force w.r.t. AB}=2(\mathrm{M}u+mv+\mathfrak{M})\end{aligned}$$

to which therefore the forces acting upon the particle of water ought to be equal.[12]

42. Among the forces which in fact act upon the particles of water, the first to be considered is gravity; its effect, however, if the plane of motion is horizontal, amounts to nothing. Yet if the plane is inclined, the axis AL following the inclination, the other being horizontal, gravity generates a constant accelerating force parallel to the axis AL, let it be α. Next we must not neglect friction, which often hinders the motion of water, and not a little. Although its laws have not yet been explored sufficiently, nevertheless, following the law of friction for solid bodies, probably we shall not wander too far astray if we set the friction everywhere proportional to the pressure with which the particles of water press upon one another.[13]

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[11] Here Euler probably has in mind the condition of potentiality, which he will obtain in §§. 47 and 54 for the two-dimensional case and in §. 60 for the three-dimensional case.

[12] The unusual factors of 2 in the previous equations have to do with a choice of units which soon became obsolete; cf. Truesdell, 1954; Mikahailov, 1999.

[13] It is actually not clear why Euler takes the friction force proportional to the pressure.