Hydraulico-statics

When Daniel Bernoulli brought up the subject of hydrostatics (to which he made no important new contributions), he made the following declaration concerning its relationship to hydrodynamics:

> The pressure of water at rest must be clearly distinguished from the pressure of flowing waters, although no one, as far I know, has been aware of this. Hence it is that the rules presented by other authors are only valid for water at rest, although they employ terms that might lead us to believe that such rules refer to flowing water. [II.§.17]

Certainly, no previous author had noticed this fact, one of such overriding importance that it constitutes the base of all hydrodynamics. What is more, hydrostatics and hydraulics had hitherto been two separate disciplines, the only nexus of union being the fluid. From now on the separation will disappear. We have seen that in the chapters devoted to movement in pipes and to discharges, his sole aim was to obtain velocities, but he said nothing about pressures, which constitute an element of hydrostatics. From the moment at which they proceed side-by-side, a new discipline appears which he baptises as 'hydraulico-statics', as both participate in it. He says that 'in this *hydraulico-statics* it is surprising that the pressures of waters cannot be defined without previously having grasped their motions' [XII.§.2], and he devotes himself to this, arriving at the first definition of what is now known as 'Bernoulli's theorem', which is the first theorem that any student of fluid mechanics encounters.[29]

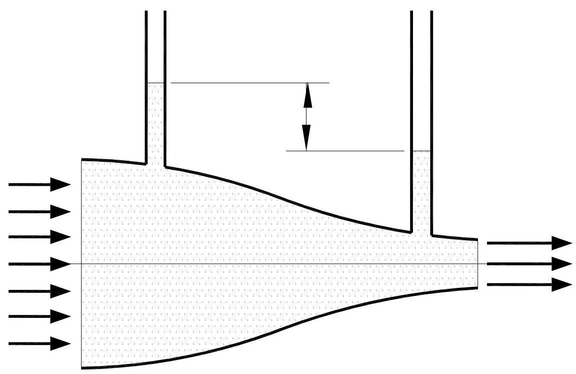

Fig. 7-10. Water manometer

[29] From my own experience, I can state that it is difficult to understand something of the abstruse science of fluids if this theorem has not been fully understood.

But Daniel's contributions are not limited to the theoretical field. We have already said that he was a great experimenter, in this respect we must acknowledge that he used the column water manometer in order to measure pressures[30] (Fig. 7-10). This consists of a narrow pipe placed vertically in a small lateral orifice of a pipe through which water circulates, and through which the water will ascend until it reaches the height proportional to the pressure.[31] It is not surprising that Daniel was the first to use this instrument, as there is actually a close relation between the phenomenon of 'hydraulico-statics' and this measuring instrument. So much so, that we believe these pipes and measurements existed before the theoretical formulation of the theorem, as we shall go on to show.

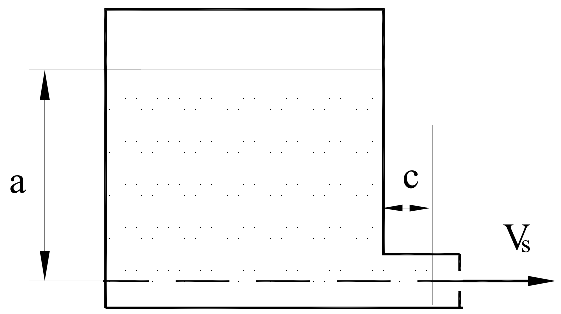

Fig. 7-11. Discharge of a reservoir through a tube

Employing a methodological procedure standard in his study of the relation between pressure and velocity, he proposes a study of motion based on an apparatus that he uses as a model. It is a reservoir that discharges through a horizontal tube closed in turn by a perforated seal, as shown in Fig. 7-11. In principle the discharge is like those already studied in the previous chapters. The difference in level between the free surface of the water and the outlet orifice is designated by him as *a*, which will remain constant as he considers the surface of the

[30] We must remember that Daniel was trained as a doctor, and first he developed this manometer to measure the pressure of arterial flow. Perhaps following Varignon's invention of the manometer in 1705, Bernoulli also experimented by puncturing the wall of a pipe with a small open-ended straw and noted that the height to which the fluid rose in the straw was related to fluid's pressure in the pipe. Based on this observation, doctors began to measure blood pressure by sticking sharpened glass tubes directly into their patients' arteries. The less-painful sphygmomanometer (blood-pressure cuff) was not invented until the close of the nineteenth century (This comment is due to Larrie Ferreiro).

[31] The pressure at the base of column of water is expressed by the well known formula $p = \rho g h$, i.e., if h is known, p will also be know. These apparatus have continued to be used up to the present-day, be it with water or another liquid, and so much so that in experimental aerodynamics the pressures are frequently quoted as heights of water or mercury.

reservoir to be very large. According to Torricelli's Law, the speed of the liquid flowing out through the orifice will be $v_s = \sqrt{2ga}$,[32] and consequently the velocity in the tube, obviously less, must be v_s/r, where r is the relation between the sections of the tube and the orifice. From what has already been said, it follows that the velocity inside the reservoir is zero, given its large size.

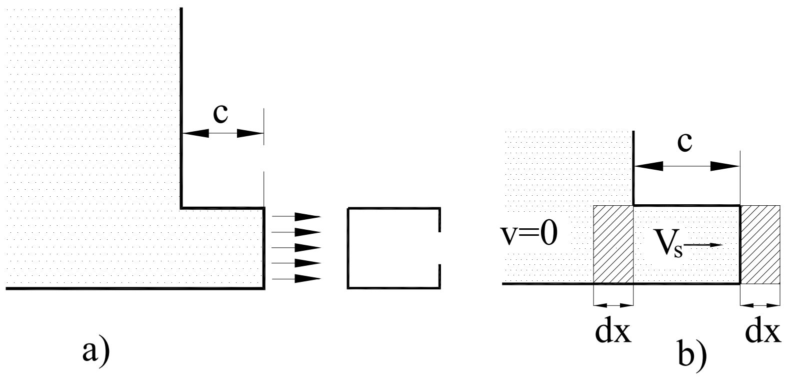

Fig. 7-12. Separation of the spout

Given this, let us imagine that the right part of the horizontal tube (Fig. 7-12a) was to disappear suddenly. It is clear that the liquid in the tube would accelerate from its previous velocity v_s/r to v_s, which would be the velocity accorded it by Torricelli's Law. Therefore, according to Daniel, the effect of the perforated plug can be interpreted as if its presence were compressing and retaining the water, pressing it against the walls of the reservoir and preventing it from expanding. This retention pressure will be greater as the velocity of the water circulating through the tube is slower, because the water will have greater acceleration capability upon the disappearance of the plug, which is the obstacle preventing free movement.

The result of this compression and retention (*nisus et renisus*) is that the water is compressed along the axial hub of the tube, and this pressure is transmitted to the lateral walls. We see that there is a likeness between this containing pressure and accelerating force that appears when we remove the plug. He writes:

[32] From this point on of the *Hydrodynamica*, Daniel changes the conceptual sense of the velocity, which goes from being represented by the kinetic height to its intrinsic sense of a space travelled in a unit of time.

> It seems that the pressure of the lateral walls is proportional to the acceleration or the increment of velocity which the water would receive if the entire obstacle to motion were to vanish in an instant, so that [the water] might pour out directly into the air. [XII.§.5]

In order to reduce this reasoning to equations he makes use of the principle of live forces, not as a potential ascent however but in the form $\Sigma mv^2 = 2g\Sigma ma$.[33] After the disappearance of the plug in an intermediate moment of acceleration process, the velocity of the liquid in the tube will be v, and this will be increased in dv in a time dt. In this interval the mass of water $m_s = \rho Sdx$ will egress the tube through the cut section (Fig. 7-12b), and will be substituted for an equal mass coming from the reservoir. As this entering mass has no velocity, it will pass from repose to the velocity $v + dv$. The increase of live force of the entire set will be the sum of that acquired by the mass of water entering, which is ρSv^2dx, plus the increase corresponding to the mass that was inside the tube that passes through v to $v + dv$, and which will be $2\rho Scvdv$. On the other hand, the real descent will be that corresponding to the fall of the mass m_s from the height a, which is $2g\rho Sadx$. Equalling will result in:

$$v^2dx + 2cvdv = 2gadx \qquad [7.26]$$

which yields:

$$\frac{2ga - v^2}{2a} = \frac{vdv}{dx} \qquad [7.27]$$

He says that in all motion the increase of velocity is proportional to the pressure multiplied by the increase in time, which would be $dv = kpdt$, or rather $dv = kpdx/v$. At this point we note that he uses pressure instead of force,[34] which would be justified if he were talking about the internal pressure on the bases of the cylinder of fluid that he has isolated, as this would act in the same direction as the variation of the velocity. However, he indicates that the pressure on the walls is what he is looking for, and in some intuitive way he likens these.[35] On

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[33] If one follows the original text, one observes that the factor $2g$ does not appear, and that the formulas are not non-dimensional. It is also appreciated that the relation between the velocity and the height is simply $v = \sqrt{a}$, as the same factor is also missing. This is explained because in his system of units it is verified that $g = 1/2$ as he warned in Note no. 20.

[34] This will be later criticised by d'Alembert, in the *Traité de l'équilibre et mouvement des fluides*.

[35] The concept of internal pressure would be introduced by his father later on, but Daniel here has an inkling of this.

the other hand, the equation he gives Newton's second law,[36] which he introduces as a differential equality between the impulse and the variation of the quantity of motion. With these clarifications the previous equation changes to:

$$\frac{2ga - v^2}{2a} = kp \qquad [7.28]$$

This equation refers to an intermediate moment between the separation of the tube and the final condition. In the initial instant the velocity was $v_s / r = \sqrt{2ga} / r$, and the pressure was that which existed before the separation of the tube, and which he likens to a height of z as ρgz. Introduced into the last formula, the result is:

$$\frac{2ga}{2c}\left(1 - \frac{1}{r^2}\right) = kp = \rho\, gz \qquad [7.29]$$

In order to eliminate all the unknown parameters that still remain in this formula, he imagines the case in which the outlet orifice is infinitesimal, i.e., $r \rightarrow \infty$. In this condition the outlet flow will be practically null, and therefore the pressure would be ρga, then $c = 1/\rho$, which introduced into the last equation leads to:

$$z = \frac{r^2 - 1}{r^2} a \qquad [7.30]$$

This is the final formula presented by Daniel, in which a relation is established between the velocities, determined by the magnitude r and the height of the pressure z. If we make a small transformation with the aim of updating of formula[37] we arrive at:

$$p = p_0 - \frac{1}{2} \rho\, v^2 \qquad [7.31]$$

an equation which is much more familiar to the present-day student.

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[36] We recall that Newton formulated his law in an integral manner, not differential. Therefore, the approximation made by Daniel Bernoulli in this point should no surprise us. See above in previous Chapter 2, 'Resistance in aeriform fluids'.

[37] On one hand $p = \rho gz$, and $p_0 = \rho ga$, which is the pressure that would exist without egress or repose pressure. On other hand $r = S_T/S_s = v_s/v_T = v_s/\sqrt{(2ga)}$. Substituting, $p = p_0 - p_0/r^2 = p_0 - \frac{1}{2}\rho v_T^2$.

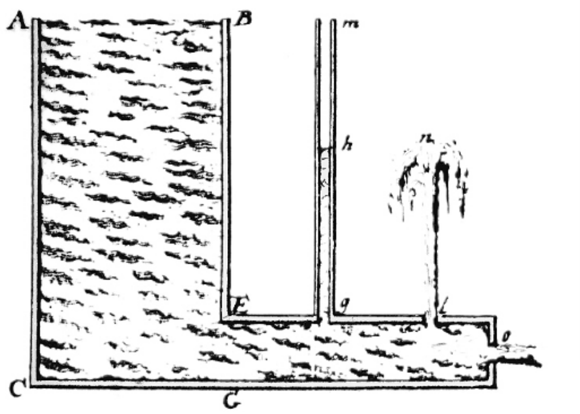

Fig. 7-13. Manometer and jet

Daniel suggests several experiments and corroborations for this formula, and here we reproduce one of them (Fig. 7-13). We can see the manometer tube and a small jet, and in both the water reaches a height *z*, which is a function of the velocity *v*. This apparatus clearly testifies to the intimate relation existing between the height of the water in the manometers and the formula, which was the reason why we venture to suggest that perhaps the instrument existed prior to the law. It is common to try to pinpoint an exact date for an important factor. However, this desire is frequently impossible to satisfy, not through ignorance or lack of documentary proof, but because the thought follows a line of evolutionary maturity, and there is no definite crystallisation point. Something like this happens with Bernoulli's formula. What is more, there are a couple of indications that the idea was mulling around in Daniel's head quite a long time. We know of a letter that he sent to Golbach, dated 17 July 1730, in which he writes that:

> In these past days I have made a new discovery which can be of great use for the design of ducts for water, but which above all will bring in a new day in physiology: it is to have found the statics of running water, which no one before me has considered, so far as I know ….[38]

He encloses a drawing of an apparatus similar to that of the *Hydrodynamica*, and the final formula without any type of demonstration.

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[38] Comment by Truesdell. Cf. 'Rat. Fluid Mech-12', p. XXX, that specifies that the mention of physiology is due to the fact that Daniel was a medical practitioner. See also previous note 31.

Another note on this is found in the article ‘Experimenta coram societate instituta in confirmationem theoriae pressionum quas lateera canalis ab aqua transfluente sustinent’ (‘Experiments made before the Academy in confirmation of the theory of pressures exercised by the waters circulating in the laterals of chanels’) published in the *Commentarii petropolitanæ* (1729), in which there is a drawing of an experimental apparatus, that we reproduce in Fig. 7-14, and also the formula, as well as some paragraphs that are later repeated almost word for word in the *Hydrodynamica*. The volume is from the year 1729, but it was published in 1735. The drawing shows the tube he used as a manometer, and the removable plug with the orifice used in the experiments. It is clearly the precursor of the model he will use to demonstrate his theory.

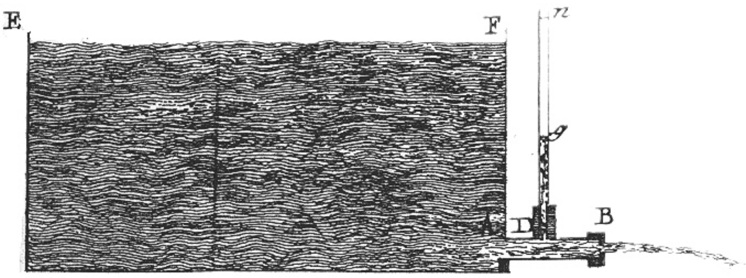

Fig. 7-14. The *Commentarii* apparatus

It is curious that in both cases he gives the formula of the height of the pressure as $(1 - 1/r^2)a$ without proof. This seems to indicate that he obtained this formula from experimental data, which is not unusual, as it is very simple and easy to conjecture, especially for an excellent experimenter like Daniel was. If this is so, then his work consisted in looking for a basis for this equation, which would constitute another of the many cases in which theory has to justify experiments. This could also explain his father’s anger, as he very probably had the empirical formula, perhaps even before the publication of the *Commentarii* in 1735 and he could have been very close to its theoretical reduction. Nevertheless, this is a simple conjecture, interesting and even likely, but not verified.

The discharge of elastic fluids

In the *Hydrodynamica* there is a chapter subtitled ‘Concerning Properties and Motions of Elastic Fluids, but especially of Air’, the only chapter that does not deal with liquids. This section begins with a kinetic theory of gases, one of the