Sir Isaac Newton (1642-1727)

It is impossible to pin down any particular Newtonian contribution to Fluidmechanics; for almost all the fundamental fluidmechanic concepts are built upon Newton's basic laws.

His second law introduced a new conception of force: as that which changes momentum, the latter being defined as the mass of a body multiplied by its speed of motion; and mass was defined by him as 'the volume of a body multiplied by its density'. Let me say right away that there are in nature no Newtonian (concentrated) forces; but Newton's definitions met, and continue to meet, all the basic needs of classical mechanics, therefore we shall 'tolerate' them for the time being without comment (see the last section of this book).

The second point of major historic importance is this. We have seen that problems of gravitation, inertia and motion of bodies were formulated before Newton. But none of them emerged in that simple, beautifully finalized, form as Newton's Laws of Motion. His *Philosophiæ Naturalis Principia Mathematica* became the hinge of all the levers of Mechanics—'the greatest production of the human mind' (Lagrange). As Laplace put it, Newton's contribution to scientific knowledge surpassed everything the entire history of science had achieved before, and *Principia* will always shine as the star of human genius.

If we condense Newton's Mechanics, the following laws emerge as its essential roots:

A material body does not alter its motion in any way, except under the action of a force applied to it; a material body at rest remains at rest, or in uniform motion – it continues to move in the same direction, with the same speed, unless a force is impressed upon it; the time-rate of change of momentum is proportional to the force which causes it; to every action of a force there exists a counter action, or a reaction. In a still more condensed, and popular, form these are reduced to the following Three Laws of Motion: (1) every material particle continues in its state of rest or of uniform motion in a straight line except insofar as it is compelled by force to change that state; (2) the time-rate of change of momentum ($m\bar{v}$) is proportional to the motive force and takes place in the direction of the straight line in which the force acts; and (3) the interaction between two particles is represented by two forces equal in magnitude but oppositely directed along the line joining the particles.

We shall see in due course that these laws, in spite of their profound importance, are open to philosophical criticism. But even these criticisms require the knowledge of . . . the same laws!, especially of the second law, which has the mathematical form

$$\boxed{\bar{F} = \frac{d}{dt}(m\bar{v}) = m\frac{d\bar{v}}{dt} = m.\bar{a} = \frac{W}{g}\bar{a}},$$

where $\bar{F}$ = force, m = mass, $\bar{v}$ = velocity, $\bar{a}$ = acceleration, W = weight, t = time, d = symbol of differentiation.

Sir Isaac Newton postulated that every fluid in nature consists of perfectly spherical elastic particles at equal distances from each other. Thus, according to Newton's first theory, a solid body moving in such a medium imparts momentum to all particles it meets on its way. The particles do not communicate their motion to neighbouring particles.

Starting from here, Newton tried to develop a theory of fluidmechanic resistance and produced interesting results. For a cylinder, he found that the resistance was equal to the weight of a cylinder of fluid of the same base as the solid, and whose height was twice that from which a heavy body would have to fall in order to acquire the velocity with which the solid moved. The resistance of a sphere appeared to be one-half the resistance of the cylinder, under the same conditions.

But it was Newton himself who soon disproved these conclusions. Assuming now that a fluid stream is not a stream of isolated balls lined up in perfect order, but a continuous chain of particles, he studied the effect of such a stream upon a curved surface and arrived at the new conclusion that the resistance experienced by a cylinder in translational motion was equal to the weight of a cylinder of fluid whose base is the same as that of the solid and whose weight is half that from which a heavy body would have to fall to acquire the velocity with which the solid moves in the fluid, i.e. four times less than by the first theory. He also found that the length of the cylinder did not affect the resistance – we shall return to this conclusion later.

The essence of Newton's theory is that molecules, or particles, are assumed to move in straight lines until they strike the body surface. When this happens, they lose the component of their momentum normal to the body. If we take, for example, an inclined flat plate (Figure 39) we shall have

$$\boxed{\begin{aligned} R &= (mv)_n = (\rho.A.v_\infty v)_n = \rho A_n v_\infty v_n = \rho A \sin\alpha\, v_\infty v_\infty \sin\alpha = \\ &= \rho A v_\infty^2 \sin^2\alpha \end{aligned}},$$

where ρ is the mass density of the fluid, A the area of the plate, and α its angle of incidence. This is, then, the so-called Newton's Sine-Square Law of air resistance. But, in fact, it cannot be found in Newton's work; it was deduced by other investigators.†

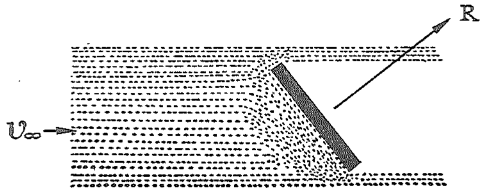

Fig. 39. Newton's model of flow past an inclined plate

We have so far been surveying the main milestones in the history of mechanics of ideal, inviscid, fluids. But this does not mean that the existence of friction in real fluids was unknown to the developers of Fluidmechanics.

Indeed, Michel Angelo (1475–1564), more widely known as Michelangelo (Buonarroti), the great Italian sculptor, painter, architect and poet of the High Renaissance, who was the author of many monumental hydraulic and other engineering projects in Rome, observed that the velocity of a water flow is greater in the centre of the flow than near the banks. This suggests that he knew of the existence of fluid friction. Leonardo da Vinci, in turn, went so far as to define the main parameters affecting the resistance met by a solid body moving in a real fluid. René Descartes (1596–1650), or Renatus Cartesius in Latin (hence the name Cartesian Coordinates), a French scientist and philosopher, who in his *La Géometrie*‡ established the Cartesian system of mathematical certitude, also studied the problem of friction between two liquid layers. Then there are Torricelli and Viviani who even tried to establish experimentally a relationship between the kinematic and frictional characteristics of water jets, while d'Alembert came to the conclusion that it was impossible to obtain reliable results without the help of experiments. 'I must confess', he wrote, 'that I do not know how the resistance of fluids can be explained theoretically, because theory gives zero resistance. . . .'

Nevertheless, the determination of the force of fluid resistance had to wait until the end of the XVII century, when Newton in England and Guillelmini in Italy, independently of each other, published their famous works. In his *Della natura de fiumi* (Roma, 1697), Guillelmini made a

† *Aerodynamics*. Theodore von Karman, Cornell University Press, Ithaca: 1954.
‡ See, for instance, *Philosophical Writings*. Descartes (selected and translated into English by Norman Kemp Smith), Random House, New York: 1958.

definite attempt to analyse the physical nature, and to establish the mathematical structure, of friction between fluids and solid surfaces.† But his effort was superseded, both in time and essence, by Sir Isaac Newton.‡

Newton, incidentally, wrote that a fluid is a body whose particles can move relative to one another under the action of *any* force, which suggested the absence of friction. In the same book, however, he not only accepts its existence, but determines its force: even Newton was not free of contradictions!

If portions of a mass of fluid are caused to move relative to one another, says Newton's theory, the motion gradually subsides unless sustained by external forces. Conversely, if a portion of a mass of fluid is kept moving, the motion gradually communicates itself to the rest of the fluid. These effects, observed generally long before he was born, were ascribed by him to a *defectus lubricitatis*, that is, to a lack of slipperiness, or internal friction, or viscosity in modern terminology. The corresponding foreign terms are: *frottement interieur* and *viscosite* in French; *innere Reibung* and *Viskosität* or *Zähigkeit*, in German; *vnutrenneye treniye* and *vyazkost*, in Russian.

If A and B (Figure 40) are two particles of a viscous flow sliding one

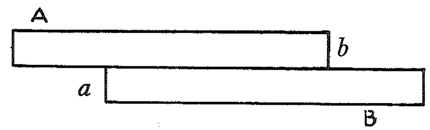

Fig. 40. Two fluid layers sliding over each other

over another, then there exists friction, or viscous resistance, along the surface ab. The force of this resistance is known as the *shear force*, while the shear force per unit area of friction is known as the *shear stress*.

The magnitude of the shear stress depends on the speed with which the two layers slide one over another:

$$\boxed{\tau = \frac{F}{A} = \mu \frac{dv}{dy}},$$

where μ is the absolute or dynamic viscosity coefficient, such that

$$\boxed{\mu = \rho\nu},$$

† *Traite d'hydrodinamique*. Bossu, t. 2.
‡ *Philosophiae naturalis principia, Mathematica*, Book II. Newton, 1687.

ν being the so-called 'kinematic viscosity', and ρ the mass density of the fluid.

With reference to Figure 41, Newton's actual description is as follows: '*If a solid cylinder infinitely long, in an uniform and infinite fluid, revolves with an uniform motion about an axis given in position, and the fluid be forced round by only this impulse of the cylinder, and every part of the fluid continues uniformly in its motion: I say, that the periodic times of the parts of the fluid are as their distances from the axis of the cylinder.*

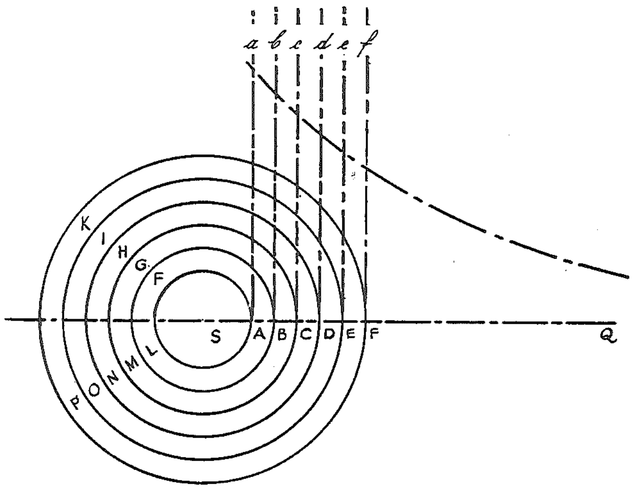

Fig. 41. Newton's scheme of fluid friction

Let *AFL* be a cylinder turning uniformly about the axis *S*, and let the concentric circles *BGM*, *CHN*, *DIO*, *EKP*, etc., divide the fluid into innumerable concentric cylindric solid orbs of the same thickness. Then, because the fluid is homogeneous, the impressions which the contiguous orbs make upon each other will be (by the Hypothesis) as their translations from each other, and as the contiguous surfaces upon which the impressions are made. If the impression made upon any orb be greater or less on its concave than on its convex side, the stronger impression will prevail, and will either accelerate or retard the motion of the orb, according as it agrees with, or is contrary to, the motion of the same. Therefore, that every orb may continue uniformly in its motion, the impressions made on both sides must be equal and their directions contrary. Therefore since the impressions are as the contiguous surfaces, and as their translations from one another, the translations will be inversely as the surfaces, that is, inversely as the distances of the surfaces

from the axis. But the differences of the angular motions about the axis are as those translations applied to the distances, or directly as the translations and inversely as the distances; that is, joining these ratios together, inversely as the squares of the distances. Therefore, if there be erected the lines *Aa*, *Bb*, *Cc*, *Dd*, *Ee*, etc., perpendicular to the several parts of the infinite right line *SABCDEQ*, and inversely proportional to the squares of *SA*, *SB*, *SC*, *SD*, *SE*, etc., and through the extremities of those perpendiculars there be supposed to pass a hyperbolic curve, the sums of the differences, that is, the whole angular motions, will be as the correspondent sums of the lines *Aa*, *Bb*, *Cc*, *Dd*, *Ee*, that is (if to constitute a medium uniformly fluid the number of the orbs be increased and their breadth diminished *in infinitum*) as the hyperbolic areas *AaQ*, *BbQ*, *CcQ*, *DdQ*, *EeQ*, etc., analogous to the sums; and the times, inversely proportional to the angular motions, will be also inversely proportional to those areas. Therefore, the periodic time of any particle, as *D*, is inversely as the area *DdQ*, that is (as appears from the known methods of quadratures of curves), directly as the distance *SD*.

Q.E.D.

COR. (corollary) I. Hence the angular motions of the particles of the fluid are inversely as their distances from the axis of the cylinder, and the absolute velocities are equal.

COR. II. If a fluid be contained in a cylindric vessel of an infinite length, and contain another cylinder within, and both the cylinders revolve about one common axis, and the times of their revolutions be as their semidiameters, and every part of the fluid continues in its motion, the periodic times of the several parts will be as the distances from the axis of the cylinders.

COR. III. If there be added or taken away any common quality of angular motion from the cylinder and fluid moving in this manner, yet because this new motion will not alter the mutual attrition of the parts of the fluid, the motion of the parts among themselves will not be changed; for the translations of the parts from one another depend upon the attrition. Any part will continue in that motion, which, by the attrition made on both sides with contrary directions, is no more accelerated than it is retarded.

COR. IV. Therefore, if there be taken away from this whole system of the cylinders and the fluid all the angular motion of the outward cylinder, we shall have the motion of the fluid in a quiescent cylinder.

COR. V. Therefore, if the fluid and outward cylinder are at rest, and the inward cylinder revolve uniformly, there will be communicated a circular motion to the fluid, which will be propagated by degrees through the whole fluid; and will go on continually increasing, till such time as the several parts of the fluid acquire the motion determined in Cor. IV.

COR. VI. And because the fluid endeavours to propagate its motion still further, its impulse will carry the outmost cylinder also about with it, unless the cylinder be forcibly held back; and accelerate its motion till the period times of both cylinders become equal with each other. But if the outward cylinder be forcibly held fast, it will make an effort to retard the motion of the fluid; and unless the inward cylinder preserve that motion by means of some external force impressed thereon, it will make it cease by degrees. All these things will be found true by making the experiment in deep standing water.

The role of temperature in fluid friction – viscosity – was neglected by Newton, but studied subsequently by Du Buat, Girard and others, and especially by Poiseuille. They showed that the force of friction decreases with the increase of temperature, and this, in turn, brought forward yet another question: what is the physical nature of viscosity?

Navier, first (150 years after Newton!) introduced μ into the general equations of motion (see p. 88), and Poisson ascribed fluid friction to the change of the 'repulsive forces' of fluid particles, a concept unacceptable to modern Physics. Saint-Venant, Kleitz, Helmholtz, Meyer, Kirchhoff and others advanced yet another theory, according to which rapid displacements of particles in a flowing fluid create in it forces which are proportional to the relative velocities of the layers of the fluid.

Daniel Bernoulli (1700-82)

Just as a great river is fed by small streams, some even barely noticeable at its source and along its banks, so science and technology proceeds from small individual contributions until it becomes an ever-increasing flow of knowledge and techniques. This big river of Fluidmechanics is closely associated with Daniel Bernoulli, the author of the first textbook in the field.† Reading and re-reading it, I entirely agree with its author that his theory was 'novel, because it considers both the pressure and the motion of fluids'. Since the book is not readily available to the average reader, I should like to give here some details of Bernoulli's methods and techniques.

Chapter 13 of the book is called *Hydrodynamica*. Let there be a large vessel *AFEB*, (Figure 42), begins Bernoulli, which must constantly be

† *Hydrodynamica, sive de viribus et motibus fluidorum commentarii.* Berlin (written in St. Petersburg): 1738.

kept full of water. Let the vessel have a horizontal cylindrical pipe *BD*, with a hole *O* at its end, through which the water is ejected with a steady velocity. It is required to determine the pressure of the water on the walls of the pipe *BD*.

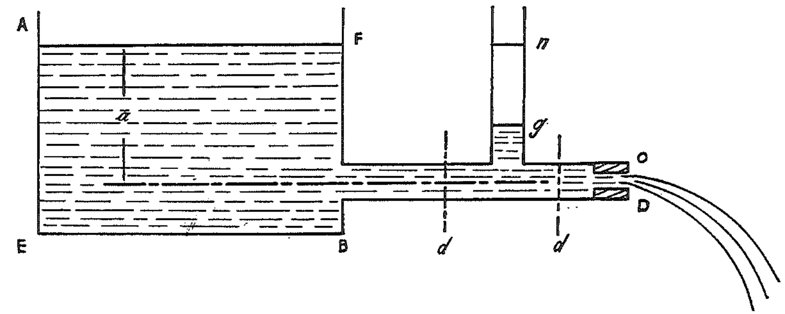

Fig. 42. Bernoulli's historic scheme of discharge of water, from which he derived his famous 'Bernoulli's Law'

Solution: let the height of the surface of the water above the hole *O* be equal to a; then the velocity of the water flowing out at *O*, if the first instants of the outflow are excluded, should be considered as steady and equal to $\sqrt{a}$, for we accept that the vessel is constantly kept full. And if one assumes that the ratio of the cross-section of the pipe and of its hole is equal to $n/1$, then the velocity of the water in the pipe will be equal to $\sqrt{a}/n$. But if orifice *OD* were missing entirely, then the limiting velocity of water in the same pipe would be equal to $\sqrt{a}$, which is greater than $\sqrt{a}/n$. Thus, water in the pipe tends to accelerate, but meets the resistance offered by the end *OD*. These obstacles and counteractions compress the water; the compression is transmitted to the walls of the pipe which experience, therefore, the same increased pressure. Thus, it is clear that the pressure on the walls is proportional to the acceleration or to the increase of velocity, which would be gained by water, if the obstacle were to disappear instantaneously and the water to flow right into the air.

So the problem is to determine the acceleration which would be gained by the particle *dd*, if the pipe *BD* were at some instant cut at *d*, the water continuing to flow through *O*; this is precisely the pressure of flowing water upon particle *dd* taken on the wall of the pipe. To find its value, consider the whole vessel *ABddD* and find for it the acceleration in the close neighbourhood of the outflowing water particle, which has velocity $\sqrt{a}/n$.

. . . Let v be a variable velocity in the pipe *Bd*. The cross-section of the pipe, as before, is n, and $Bd = c$, $dd = x$. At B, there is an equal particle ready to enter the pipe at the instant when particle dd leaves it. But when the particle at B, whose mass is equal to $n.dx$, enters the pipe, it acquires the velocity v and also the living force (i.e. kinetic energy) nv^2dx. Since vessel AB is infinitely large, the particle at B has been in a state of rest before entering the pipe, therefore nv^2dx is an entirely new force. We must add to this living force the increment of the living force gained by water at Bb, while particle dd flows out, namely $2ncvdv$. This sum corresponds to the real descent of the particle ndx from the height BE, i.e. a. Thus, we have $nv^2dx + 2ncvdv = nadx$, or

$$\boxed{\frac{vdv}{dx} = \frac{a - v^2}{2c}}$$

But, in any flow, the velocity increment dv is proportional to pressure multiplied by time, which in this case is equal to dx/v. Therefore, in our case, pressure experienced by particle dd is proportional to vdv/dx, that is, to $(a - v^2)/2c$.

At the instant when the pipe is cut, $v = \sqrt{a}/n$ or $v^2 = a/n^2$; this expression should be substituted in the right-hand side of the equation, which then becomes $(n^2 - 1)a/2n^2c$. And the latter represents a quantity proportional to the pressure of the water on the portion ac of the tube, whatever the cross-sections of the tube and of the orifice at its end.

So, if the pressure of the water is determined for one case, it is determined also for all the other cases, and this is so when the orifice is infinitely small or when n is infinitely large compared with unity; because it is self-evident that, in such a case, the water exerts its entire pressure corresponding to a; this pressure we call a. But when n is infinitely large, then unity is vanishingly small compared with n^2, and the quantity, which is proportional to pressure, appears to be equal to $a/2c$. Thus, if we wish to know what the pressure will be at any n, we should ask this question: if a corresponds to quantity $a/2c$, what pressure will correspond to quantity $(n^2 - 1)a/2n^2c$? In this way, one establishes that the answer is $(n^2 - 1)a/n^2$.

It follows from the disappearance of c from the calculation that all parts of the tube, those near the vessel AB as well as the more distant ones, experience the same pressure from the flowing water, which is less than that on the bottom EB. The greater the orifice O, the greater is this pressure difference. And the walls experience no pressure at all, when obstacle OD is missing, because, in this case, the water flows out through a full opening.

If we make somewhere in the wall of the tube a hole very small compared with O, water will shoot out through it with a velocity sufficient to elevate it (the water) to the height $(n^2a - a)/n^2$, if there are no obstacles in its way. . . .

The next problem Bernoulli analyses and solves is this: determine the pressure of water flowing with a constant velocity in a tube of any form. The solution obtained is

$$\boxed{\frac{vdv}{dx} = \frac{a - v^2}{2\alpha}},$$

where α is a constant number whose value depends on the geometry of the tube. If the actual pressure is $(a - b)$, b being the height corresponding to the actual v, and if $(a - b) < O$, then pressure becomes suction, i.e. the walls of the tube experience pressure from outside.

The rest of the Chapter (24 more pages) is given in the same typically Bernoulli style. The two enframed formulae are the two slightly different forms of the actual Bernoulli Equation. They appear on a number of pages of the textbook in modified forms. And so we find ourselves face to face with one of the most fascinating questions in the history of Fluidmechanics: who was, then, the author of the famous 'Bernoulli Equation' which appears in every textbook of Hydrodynamics,

$$\boxed{\frac{\rho v^2}{2} + p = \text{const}}?$$

Indeed, this formula has no direct resemblance to the above two equations by Bernoulli. It is true that he was the first to undertake a fundamental study of the $p = p(v)$ or $v = v(p)$ interdependence, and the first to conclude that an increase in v leads to a definite decrease in p, and vice-versa, but certainly not in the form of the formula which is ascribed to him. This 'mystery', however, is dispelled by a close examination of Euler's equations of motion.

Leonhard Euler (1707-83)

Leonhard Euler was not a contributor to, but the founder of, Fluidmechanics, its mathematical architect, its great river.

Let us recall that geometry is a branch of mathematics which treats the shape and size of things; while Fluidmechanics is the science of motion (and equilibrium) of bodies of deformable (and variable) shapes, under the action of forces. When one analyses these two definitions, it becomes clear that *some* theorems and axioms of geometry do not meet the philosophical and physical needs of mechanics generally, and of Fluidmechanics in particular; and it is difficult to imagine that Euler's genius could have been unaware of this.

For example, a point is usually defined as an element of geometry which has position but no extension; a line is defined as a path traced out by a point in motion; and motion is defined as a change of position in space. But motion and matter cannot be divorced. A point that has no extension lacks volume and, consequently, mass, therefore is nothing; and nothing can have neither path nor momentum, or motion.

Euler was, perhaps, the first to overcome this fundamental contradiction, by means of the introduction of his historic 'fluid particle', thus giving Fluidmechanics a powerful instrument of physical and mathematical analysis. A fluid particle is imagined as an infinitesimal body, small enough to be treated mathematically as a point, but large enough to possess such physical properties as volume, mass, density, inertia, etc. Like Newton, Euler defined mass as the product of volume by the *mass density* of the fluid which occupied the volume. Hence, the now classical definition of mass density: it is the amount, or quantity, of fluid per unit volume. Mathematically,

$$\boxed{\rho = \frac{dm}{dV} = \frac{M}{V}} \quad (\text{kg. sec}^2/\text{m}^4)$$

But since, by Newton's Second Law, the weight W of any physical body $W = m.g$, it follows that

$$\rho = \frac{W}{gV} = \frac{\gamma}{g}, \text{ or } \boxed{\gamma = \rho g} \ (\text{kg/m}^3).$$

So, the fluid particle concept became meaningful, logical, powerful. From then on, everyone knew that a fluid particle was not a mathematical, but a *physical* point possessing volume, weight, mass, densities, specific heats, etc. But in what shape should it be imagined? and how should its motion be determined?

Again, Euler produced the most beautiful answers, directly or by implication. Let us imagine a continuous curved line l, within a fluid flow, at any given instance of time tangential to velocity vectors of all fluid particles through which it passes, the so-called stream-line. The word *tangential* implies that anywhere along the stream-line the velocity vector is parallel to the portion of l where it acts. Euler exploited this

fact in a somewhat complicated way; but if we apply to it the theorem (of vectorial mechanics) that the vector product of two parallel vectors is zero, we have:

$$\bar{v} \times dl = \begin{vmatrix} \bar{i}, & \bar{j}, & \bar{k} \\ v_x, & v_y, & v_z \\ dl_x, & dl_y, & dl_z \end{vmatrix} = 0$$

or, since the unit vectors $\bar{i} \neq 0, \bar{j} \neq 0$ and $\bar{k} \neq 0$, and denoting $dl_x = dx$, $dl_y = dy$, $dl_z = dz$,

$$\boxed{\frac{dx}{v_x} = \frac{dy}{v_y} = \frac{dz}{v_z}}$$

This is, then, Euler's classical differential equation of the stream-line, one of the golden keys to the mysteries of Fluidmechanics. It answers the second of the above two questions and, moreover, leads to a number of further ideas and concepts. For example, we can imagine, within the flow, a stream-tube composed of stream-lines (Figure 43). Now, since

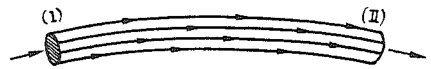

Fig. 43. The imagery stream tube

the velocity vectors are parallel to the stream-lines at the points of their action, and since the walls of the stream-tube are composed of stream-lines, it follows that no amount of fluid enters or leaves the imaginary tube through the walls. Therefore, the amount (mass) of fluid entering through cross-section (I) per unit time must be exactly equal to the amount (mass) of fluid leaving the tube through cross-section (II) per unit time. That is, there can be in the stream-tube neither an accumulation nor a loss of mass, i.e. $m_1 - m_2 = dm = 0$. But $dm = \rho dV$.

What should the configuration of dV be? The general answer is: any configuration. Euler chose, however, an infinitesimal parallelopiped with dx, dy, dz as its sides (Figure 44) and, consequently, with $dV = dxdydz$ as its volume. Thus, the condition of 'no accumulation and no loss of mass' in the stream-tube assumed the form $dm = \rho dV = \rho dxdydz = 0$. And the integration of this expression produced yet another typically Eulerian historic formula,

$$\boxed{m = \underset{(V)}{\iiint} \rho dxdydz + \text{const}},$$

which represents, in fact, the law of conservation of mass in fluid flows, and remains fundamental in every branch of Fluidmechanics.

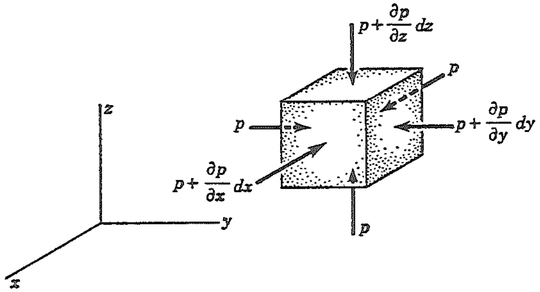

Fig. 44. Euler's infinitesimal fluid parallelopiped

Any magnitude which has size, in the ordinary algebraic sense of the word, as well as direction in space, is a *vector*; velocity, acceleration and force are examples of vectors. The common algebraic magnitudes, which have nothing to do with direction in space and have no directional properties, but are each determined completely by a single (real) number, are *scalars*; mass and temperature are typical examples of *scalars*. Leonhard Euler used in his analysis both vectors and scalars, but without calling them so. All his mathematical operations appeared in the Cartesian (rectangular) components. But they were, of course, the components of vectors. For if R_x, R_y, R_z be generalized rectangular components, then $\bar{\imath}R_x + \bar{\jmath}R_y + \bar{k}R_z = \bar{R}$, the latter being the generalized vector, and $\bar{\imath}, \bar{\jmath}, \bar{k}$ being its unit vectors.

Now, since Euler's hydrodynamic theory was based on the mass conservation and continuity concepts (actually, these are two modes of one and the same thing), it follows that R_x, R_y, and R_z (therefore also $\bar{R}$), whatever they may represent, must be continuous functions of space (co-ordinates) and time. Which leads us to important definitions. First, fluid particles in motion have weights, accelerations, velocities, masses, and so on. The space occupied by a flow – by particles in motion, that is – is, therefore, full of vectors and scalars, endowed with and permeated by them. Secondly, when to every point of the space occupied by a continuous flow there corresponds a vector $\bar{R}$, of definite direction and tensor, generally varying from point to point, the space is said to be a *vector field*, i.e. the seat of the vectors $\bar{R}$. When $\bar{R}$ stands for $\bar{v}$, we have a velocity field, i.e. the field of the velocity vector $\bar{v}$ representing at each point the direction and the absolute value of the flow. When $\bar{R}$ stands for $\bar{a}$, we have a field of accelerations. And so on.

A vector field is analytically continuous, if $\bar{R}$, both in value and direction, varies in a continuous manner from point to point and in time; in which case $\bar{R}$ – and its components (whatever they represent) – admit everywhere definite differential coefficients, at least of the first and second orders, with respect to space (co-ordinates), and time.

Having established these basic foundation stones, directly or by implication, Leonhard Euler embarked on the building of the edifice of mathematical operations itself. And here he displayed the prodigious ability of an absolutely towering master. As Lagrange put it, Euler 'did not contribute to Fluidmechanics but created it'. It is a matter of regret that the purpose of this book prevents us from reproducing here Euler's garden of roses of mathematical virtuosity. When I look at these brilliant differential equations, with their mirror-like partial derivatives, I cannot help feeling that Euler was for Fluidmechanics what Leonardo da Vinci or Rembrandt was for the arts.

It would, however, be unfair to our reader not to give here at least one or two examples of Euler's mathematical forms. They are:

the differential equation of continuity

$$\boxed{\frac{d\rho}{dt} + \rho\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right) = 0},$$

and the differential equations of fluid motion

$$\boxed{\begin{aligned}
\frac{dv_x}{dt} &= a_x = g_x - \frac{1}{\rho}\frac{\partial p}{\partial x} = \frac{\partial v_x}{\partial t} + v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y} + v_z\frac{\partial v_x}{\partial z}\\
\frac{dv_y}{dt} &= a_y = g_y - \frac{1}{\rho}\frac{\partial p}{\partial y} = \frac{\partial v_y}{\partial t} + v_x\frac{\partial v_y}{\partial x} + v_y\frac{\partial v_y}{\partial y} + v_z\frac{\partial v_y}{\partial z}\\
\frac{dv_z}{dt} &= a_z = g_z - \frac{1}{\rho}\frac{\partial p}{\partial z} = \frac{\partial v_z}{\partial t} + v_x\frac{\partial v_z}{\partial x} + v_y\frac{\partial v_z}{\partial y} + v_z\frac{\partial v_z}{\partial z}
\end{aligned}}$$

These are the blood, the flesh and the bones of Fluidmechanics. It is remarkable that they have not changed at all since the day Euler derived them. By their discovery by Euler, wrote Lagrange, the whole mechanics of fluids was reduced to a matter of analysis alone, and if the equations ever prove to be integrable, the characteristics of the flow, and the behaviour of a fluid under the action of forces, will be determined for all circumstances.

Louis de Lagrange (1736-1813)

Lagrange's 'if' was overcome by Lagrange himself, and in a most masterly manner. Anyone who has ever tried to study his famous *Mechanique analytique* first published in France in 1788, will agree that he distinguished himself in the history of Fluidmechanics on such a scale that his name can easily be put side by side with Euler's; moreover, in some respects he surpassed Euler. Aristotle, Archimedes, Galileo, Torricelli, Stevinus, Pascal, Huyghens, Bernoulli, Clairaut, Descartes, d'Alembert, and many others contributed to the formation of the discipline; but Newton was the first to cement the foundation of the edifice, Euler to erect its walls and floor, and Lagrange to add all, or almost all, those other major parts which make an edifice a safe and rather enjoyable house. He became preoccupied with the organization and perfection of mechanics, of its mathematical language and methods.

'My objectives were', wrote Lagrange, 'to reduce the theory of mechanics and the art of solving the associated problems to general formulae and to unite the different principles in mechanics'. He succeeded in this with the glitter of a genius. However, I am mainly interested in his direct contributions to Fluidmechanics.

Euler considered the motion of individual fluid particles along their trajectories. But Lagrange felt that, since the number of such particles is infinitely large, the motion of each individual particle required to be specified in some way. To achieve this, he suggested choosing the initial (or starting) co-ordinates of a particle, at $t = 0$, as the characteristics of its motion.

Namely, let (a,b,c) be the co-ordinates of a fluid particle at $t = 0$. Then the trajectory of this particle (among the trajectories of an infinitely large number of particles) will be that one which will pass through point (a,b,c). Thus, the co-ordinates (x,y,z) of the point under consideration along its trajectory will be $x = x(a,b,c,t)$, $y = y(a,b,c,t)$ and $z = z(a,b,c,t)$, where (a,b,c,t) are known today as Lagrange's Variables. These equations represent a family of trajectories, which fill the whole region of flow, (a,b,c) being their parameters.

Thus, in the Euler method the velocity components of the fluid particle are functions of the co-ordinates (space) and time, while in the Lagrange method they appear to be given by the above equations. Mathematically speaking, $v_x = v_x(x,y,z,t)$, $v_y = v_y(x,y,z,t)$, and

$v_z = v_z(x,y,z,t)$ – in Euler's method, and $v_x = \partial x/\partial t$, $v_y = \partial y/\partial t$, and $v_z = \partial z/\partial t$ – in Lagrange's method. And the latter himself acknowledged that Euler's method was and remains more convenient and logical.

The trouble was, however, that the above three Euler differential equations of fluid motion contained five unknowns (v_x, v_y, v_z, p, ρ), therefore two additional equations were needed. Euler hinted and Lagrange showed that the differential equation of continuity and the equation of physical state could serve as such equations. So, according to the formal logic of mathematics, the equations must have been integrable. 'In the circumstances, Lagrange a man of great academic principles and pride, could not accept defeat, he had to develop a solution' (D. P. Riabouchinsky). And this is where we must return to our earlier question: who was the real author of the so-called Bernoulli Equation?

The notion 'total differential' was known already to Euler and to A. C. Clairaut (1713–65); the latter applied it to the solution of problems in fluidstatics.† But Lagrange was the first to develop it into a powerful tool of fluidmechanics. He came to the conclusion that Euler's equations could be solved only for two specific conditions: (1) for potential (irrotational) flows, and (2) for non-potential (rotational) but steady flows.

The first of these cases required the introduction of the so-called 'Velocity Potential' $\varphi = \varphi(x,y,z)$, such that $v_x = \partial\varphi/\partial x$, $v_y = \partial\varphi/\partial y$, $v_z = \partial\varphi/\partial z$. This was yet another revolutionary development in the formation of Fluidmechanics, which remains vital up to these days. The introduction of the velocity potential made it possible to carry out extremely interesting mathematical operations, and thus to reduce Euler's equations of motion to a single total differential equation,

$$d\left(\frac{v^2}{2} + \int\frac{dp}{\rho} + \frac{\partial\varphi}{\partial t} - d\Phi\right) = 0$$

The integral of this equation is

$$\boxed{\frac{v^2}{2} + \int\frac{dp}{\rho} + \frac{\partial\varphi}{\partial t} - d\Phi = C(t)}$$

This is, then, Lagrange's integral of Euler's equations of motion of an irrotational (potential) compressible fluid. The integral for the second case (steady flow) is similar. For a steady flow, $\partial\varphi/\partial t = 0$ and the time-dependent constant $C(t)$ becomes simply C. If, in addition, the flow is incompressible, $\int dp/\rho = p/\rho$. Thus,

† *Theorie de la figure de la Terre tiree des principes de l'hydrodynamique.* Durand, Paris, 1743.

$$\boxed{\frac{v^2}{2} + \frac{p}{\rho} - \Phi = C = \text{const}}, \text{ or } \boxed{\frac{\rho v^2}{2} + p - \rho\Phi = \text{const}}$$

Here $\Phi = \Phi(x,y,z)$ is the gravitational potential, such that $\partial\Phi/\partial x = g_x$, $\partial\Phi/\partial y = g_y$, $\partial\Phi/\partial z = g_z$, where g is the gravitational acceleration. If we choose the oy-axis vertically upward, then $g_x = g_z = 0$ and $\partial\Phi/\partial y = -g_y = -g$, therefore $\Phi = -gy$ and $-\rho\Phi = \rho gy$. Let us recall now the equation of state: $p = \rho RT$, or $\rho = p/RT$. Then $\rho gy = pgy/RT$, or $\rho gy/p = gy/RT$. Substituting here the well-known values ($T = 288°$ and $g = 9{\cdot}81$ m/sec^2), we find that the third term of the left-hand side of Lagrange's integral is negligibly small compared with the first and second terms, so that

$$\boxed{\frac{\rho v^2}{2} + p = \text{const}}$$

To sum up, we have shown that the so-called 'Bernoulli Equation', probably the most fundamental equation in the entire Fluidmechanics, ascribed to D. Bernoulli in every textbook known to us, is, in fact, not Bernoulli's at all, but LAGRANGE'S INTEGRAL OF EULER'S EQUATIONS OF MOTION.

In due course, however, other methods of development of the equation emerged.

Jean le Rond d'Alembert (1717-83)

Lagrange's integral of Euler's equations shows, as, indeed, was shown also by Daniel Bernoulli, that the greater the velocity at a given point of a flow, the less the pressure at that point, and vice versa. But, by Leonardo da Vinci's law, where the velocity is greater, the cross-sectional area is smaller, and vice versa.

Many interesting conclusions follow from these fluidmechanic laws. For example, our ancestors knew that rivers flow faster where they are narrower; but they were unaware of the fact that pressure p was lower in the narrower places. Then, experienced captains know that it is dangerous to keep two moving ships close to each other, because the cross-sectional area of water between them becomes small, narrow, therefore the water speed increases and, consequently, the water pressure decreases, which creates a danger of collision, of drawing them towards each other. Another example: almost every chimney has been

built with a contraction at the exit; this contraction, this reduction of the cross-sectional area, results in an increase of the smoke velocity and, consequently, in a corresponding decrease of the pressure, which, in turn, creates a sucking effect, thus preventing smoke from escaping into the room.

Jean le Rond d'Alembert was the first to apply these phenomena to the study of the resistance offered by an ideal fluid to a body moving in it.† But the results of his almost heroic efforts were so complex mathematically, and so far above the grasp of the average reader, that they had little impact on the further development of Fluidmechanics. They produced, however, some valuable new ideas and theorems, which deserve attention.

The widely known and important notions 'Stagnation Point' (i.e. point on the surface of a body exposed to fluid flow where the flow velocity is zero) and 'Stagnation Region' were introduced by d'Alembert. This is how he described them: the particles moving along the central streamline towards O do not travel as far as O, they stop just before reaching the point, therefore at O and immediately in front of O the fluid is necessarily stagnant.‡

He was disturbed by his result, because he simply refused to understand 'the role' of a stagnant fluid, however small its quantity. In order to avoid this difficulty, he proposed a body configuration with an 'infinitely sharp leading edge' (Figure 45), so that 'there will no longer be

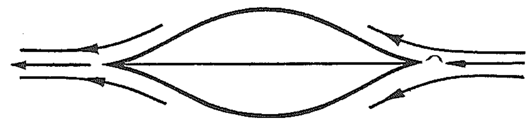

Fig. 45. d'Alembert's sharp-edged body configuration

stagnant fluid, and the whole fluid will run past the forward surface' without disturbance. Whether we should accept this as a prevision of the modern high-speed (sharp leading edge) aerodynamic configurations, is a matter of speculation and I leave it at that.

But what about the rear-side of the body? It may seem at first sight, writes d'Alembert, that the motion must be different here. But my theory of motion of fluids, he writes, shows that the velocity components in front and behind the body are exactly the same, therefore all the other conditions will also be the same. If so, he continues, providing mathematical proofs in his peculiar manner, 'the pressure of the fluid on the forward

† *Traite de l'Equilibre et du Mouvement des Fluides*, Paris: 1744.
‡ *Essai d'une nouvelle theorie de la resistance des fluides*. Paris: 1752.

surface is equal and opposite to the pressure on the backward surface, therefore the resultant pressure will be absolutely nothing'.

This is, then, the 'd'Alembert Paradox', which can be formulated more precisely as follows: the fluidmechanic resistance of a body moving steadily in an ideal fluid is zero. Its modern proof can be found in standard textbooks of Fluidmechanics, therefore we do not reproduce it.

D'Alembert was also the first to introduce something like the laminar flow concept, i.e. the concept of a flow composed of parallel slices of fluid. In general, let the velocities of the different slices of the fluid, at one and the same instant, be represented by the variable v. Then imagine that dv is the increment of the velocity in the next instant, the quantities of dv being different for the different slices, positive for some and negative for others. Or, briefly, imagine that $v \mp dv$ expresses the velocity of each layer when it takes the place of that which is immediately below. 'I say', wrote d'Alembert, 'that if each layer is supposed to tend to move with an infinitely small velocity $\pm\, dv$ (in relation to its neighbouring layers), the fluid remains in equilibrium.†

For since the velocity of each slide is supposed not to change in direction, each layer can be regarded, at the instant that v changes to $v \mp dv$, as if it had both the velocity $v \mp dv$ and $v \pm dv$. Now, since it only retains the first of these velocities, it follows that the velocity $\pm\, dv$ must be such that it does not affect the first and is reduced to nothing. Therefore, if each slide were actuated by the velocity $\pm\, dv$, the fluid would remain at rest.'‡

The conclusions drawn by d'Alembert from this analysis are of great interest to students of Fluidmechanics. In addition, he also proved analytically that the principle of 'living force' can be applied to fluids. He then analysed the now classical problem of the velocity of a fluid leaving a vessel which is kept filled to a constant height. And we should not forget, of course, the d'Alembert Principle, which states that Newton's third law holds for forces acting upon bodies entirely free to move as well as upon fixed bodies in stationary equilibrium.

Chevalier de Borda (1733-99) and others

We are moving closer and closer to modern Fluidmechanics, which emerged at long last, unfortunately, as a much more complex system

† *Essai d'une Nouvelle Théorie de la Résistance des Fluides.* Paris, David: 1752.
‡ *Traité de l'Equilibre et du Mouvement des Fluides.* Paris: 1744.

than has been described so far. As Chevalier de Borda, a French mathematician and nautical astronomer, remarked, real fluid flows are 'more sophisticated than the most sophisticated lady's character'. He sounded a warning, for example, that not all flows are 'in harmony' with Daniel Bernoulli's, Leonardo da Vinci's and Lagrange's laws, i.e. not always does an increase in the cross-sectional area lead to the proportional decrease in the flow velocity and increase in flow pressure. When (he says) a perfectly normal flow experiences a sudden expansion, which may happen in a pipe as well as on the surface of a body, it gets disturbed to such an extent that it loses a part of its kinetic energy, or living force'.†

This became known as the Borda Theorem (see Plate 1). Its implications for applied Fluidmechanics proved to be far reaching. For instance, the concept of *flow separation* became an organic part of the study of real flows, and it encouraged the emergence of a new branch of fluid flows – flows through valves and orifices of all kinds. It would be difficult to call them all Borda Flows, nor can we say that they were all studied during Borda's lifetime. But it may be relevant to describe them in connection with his contributions.

An orifice is an opening having a closed perimeter through which a fluid may discharge. It may be open to the atmosphere, which is the case of free discharge, or it may be partially or entirely submerged in the discharged fluid. Typical examples of orifices are shown in Figure 46 (*a,b,c*).

An orifice may be very small, as in the case of those used for leak

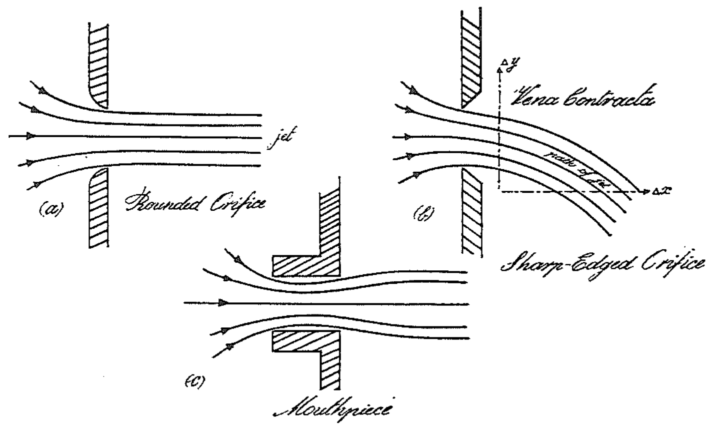

Fig. 46. Typical schemes of orifices

† *Mémoirs de l'Académie des Sciences.* Paris: 1766.