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Four— The Paradigm Constructed: *On Motion* , Theorems 1, 2, and 3

Newton's thoughts on dynamics were awakened late in 1679 by a series of letters from Robert Hooke, who had recently become secretary of the Royal Society and was attempting to revive Newton's interest in contributing to its proceedings. The Royal Society had been founded in London in 1661, and its meetings and publications served to inform the intellectual community of progress in natural philosophy. Newton had been elected to membership in 1672 but had threatened to resign the following year because of criticism of his paper on the theory of colors. Hooke's letter of conciliation of November 1679 opened with the observation that some individuals would misrepresent him to Newton and Newton to him but that differences of opinion, "especially on philosophical issues," should not be the basis of enmity.[1] Hooke offered an olive branch in the form of a request for Newton's opinion on Hooke's hypothesis concerning planetary motion. In contrast to Newton's earlier Cartesian view of curvilinear motion with a tangential endeavor and an outward endeavor counterbalanced by an inward force, Hooke proposed that planetary motion was a dynamic compound of only two motions: a motion along the tangent line due to inertia and a motion toward a central body due to an inward attractive force.[2] The Cartesian outward endeavor is not mentioned.

This revision of the dynamical principle appears not to have been fully appreciated either by Hooke (who never did solve the problem) or by Newton (who did not take up the problem at once). In his immediate reply to Hooke's letter, Newton said that he had not had time to "entertain philosophical meditations" and in fact had "long grudged the time spent in the study."[3] Hooke's initial letter had spoken both of the dynamics of planetary motion and of the astronomer John Flamsteed's claims concerning parallax. Newton had chosen not to respond to the topic of planetary

motion but rather spoke to the question of falling bodies on a moving earth, a question that had been raised indirectly by Hooke's report of Flamsteed's measurements of parallax. The correspondence that followed was concerned with that topic. Newton did produce a response to Hooke's planetary challenge, however, at least if one is to trust Newton's recall of his activities some thirty-five years later. In the autumn of 1714, Newton was in the midst of a controversy with the German philosopher and mathematician Wilhelm Gottfried Leibniz over who could claim the invention of calculus, and in the course of that debate, Newton recalled his activities of the winter of 1679.

> In the end of the year 1679 in answer to a Letter from Dr Hook then Secretary of the R.S. I wrote that whereas it had been objected against the diurnal motion of the earth that it would cause bodies to fall to the west, the contrary was true. . . . He had made some experiments thereof and found that they would not fall down to the center of the earth but rise up again and describe an Oval as the Planets do in their orbs. Whereupon I computed what would be the Orb described by the Planets. For I had found before by the sesquialterate proportion of the tempora periodica of the Planets with respect to their distances from the sun, that the forces which kept them in their Orbs about the sun were as the squares of their mean distances from the sun reciprocally: and I found now that whatsoever was the law of the forces which kept the planets in their Orbs, the area described by a Radius drawn from them to the sun would be proportional to the times in which they were described. And by the help of these two Propositions I found that their Orbs would be such Ellipses as Kepler had described.[4]

Newton's statement produces as much confusion as it provides clarification. If taken to be literally true, then he claimed to have solved the inverse problem (given the force, find the orbit) and not the direct problem (given the orbit, find the force). Newton stated that he obtained the knowledge of the inverse square nature of the force from Kepler's third law, in which the cube of the planetary period is proportional to the square of the mean planetary radius. Independent of that result, he determined that the planetary radius sweeps out equal areas in equal times for any central force. Then, using these two propositions, Newton claimed that he found "their Orbs would be such Ellipses as Kepler had described." The exact form of this solution of 1679 must remain a matter of conjecture, however, for no copy of it has ever been found.[5]

Newton may have already settled the direct question for himself in 1679, but the problem of the planets still remained one of the great unanswered questions for the rest of the academic establishment. Halley admitted at a meeting of the Royal Society early in 1684 that he had tried to solve it and failed. Hooke claimed that he had a solution but that he would not reveal it until others tried and realized the difficulty involved. Sir Christopher

Wren offered a small prize of a book to the person who could give him a demonstration within two months, but none was forthcoming within the time period (not even from Hooke). In August of that same year, Halley had occasion to visit Cambridge, perhaps on family business, and took the opportunity to call on Newton, with whom he had become acquainted during discussions of the comet of 1680.[6] In the course of their meeting, the subject of the planetary problem arose. Newton's report of the discussion comes secondhand from the French mathematician Abraham Demoivre who, after Newton's death in 1727, told of a conversation that he had with Newton:

> In 1684 Dr Halley came to visit him [Newton] at Cambridge. . . . The Dr asked him what he thought the Curve would be that would be described by the Planets supposing the force of attraction towards the Sun to be reciprocal to the square of the distance from it. Sir Isaac replied immediately that it would be an Ellipsis. . . . Dr Halley asked him for his calculation without any further delay. Sir Isaac looked among his papers but could not find it, but he promised him to renew it, and then to send it [to] him.[7]

Demoivre recalled that Newton recalled that Halley requested the solution to the inverse problem (i.e., given the force, find the path). In contrast to these recollections, however, stand the copies of the solutions Newton actually produced in fulfillment of his promise to Halley "to renew it [the lost solution of 1679] and then send it to him." The solution that Halley received from Newton after the visit to Cambridge demonstrated that the force, given an elliptical orbit with a force center at a focus, was inversely proportional to the square of the distance (i.e., the direct problem).[8]

True to his promise to Edmund Halley in August of 1684, Newton reproduced a version of the lost solution to the direct Kepler problem that he had generated in 1679 following his correspondence with Robert Hooke. In November of 1684, he sent it to Halley in London. If Halley's initial request in August had been specifically for a solution to the inverse problem, then by December he must have changed his mind, for he received the solution to the direct problem with nothing but exclamations of great admiration.[9] The text that Halley received from Newton, however, contained much more than a determination of the force required to maintain elliptical planetary motion about a focal sun center of force. Newton sent him a tract in which the direct Kepler problem appears as "Problem 3." The tract begins with a set of three definitions, four hypotheses, and two lemmas, each consisting of one or two lines of text. Newton then gave three theorems (with demonstrations), to establish a general procedure for solving direct problems.

> Theorem 1. *All orbiting bodies describe, by radii having been constructed to their center, areas proportional to the times* .

> Theorem 2. *For bodies orbiting uniformly on the circumferences of circles, the centripetal forces are as the squares of the arcs described in the same time divided by the radii of the circles* .

> Theorem 3. *If a body, by orbiting around the center* S, *should describe any curved line* APQ, *and if . . . [then] I assert that the centripetal force would be reciprocally as the solid* $SP^2 \times QT^2 / QR$, *provided that the quantity of that solid that ultimately occurs when the points* P *and* Q *coalesce is always taken* .

Newton then provided the solutions to three problems as examples of how the general dynamical algorithm of Theorem 3 could be applied to a set of specific direct problems.

> Problem 1. *A body orbits on the circumference of a circle; there is required the law of centripetal force being directed to some point on the circumference* .

> Problem 2. *A body orbits on a classical ellipse; there is required the law of centripetal force being directed to the center of the ellipse* .

> Problem 3. *A body orbits on an ellipse; there is required the law of centripetal force directed to a focus of the ellipse* .

Problem 1 derives the force function necessary to describe uniform circular motion with a force center on the circumference of a circle, and Problem 2 derives the force function necessary to describe elliptical motion with the force center at the center of an ellipse. The solution to Problem 1 is much simpler than the solution to Problem 2, which in turn is simpler than the solution to Problem 3. Further, since neither Problem 1 nor 2 has any obvious physical application, it appears that Newton introduced them as exemplars to demonstrate how Theorem 3 could be used to solve direct problems.**[10]** The solution to Problem 3, however, does have a physical application. It is the solution to the distinguished direct Kepler problem of the planets (i.e., the force function necessary to describe elliptical motion about a focal force center is an inverse square function of the distance). Although others had anticipated this solution, Newton was the first to produce its demonstration.

The solution to the direct Kepler problem, however, did not exhaust the list of items Newton sent to Halley. Theorem 4 demonstrates Kepler's law relating the periods of planets to their mean radii, and Problem 4 is a version of an inverse problem in which the magnitude of an inverse square force is given and the nature of the resulting orbit is sought. Newton assumes that the body is initially moving in an ellipse with the center of the inverse square force located at a focus of the ellipse, a condition that the solution in Problem 3 of the direct Kepler problem demonstrates is possible (a sufficient condition but not yet demonstrated to be a necessary condition). He then demonstrates that for any other given position and

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velocity the path will remain an ellipse or will become a parabola or a hyperbola.

> Theorem 4. *Supposing that the centripetal force is reciprocally proportional to the square of the distance from the center, the squares of the periodic times in ellipses are as the cubes of their transverse axes* .

> Problem 4. *Supposing that the centripetal force be made reciprocally proportional to the square of the distance from its center, and that the absolute quantity of that force is known; there is required the ellipse which a body will describe when released from a given position with a given speed along a given straight line* .

Newton then turned from his analysis of celestial motion in a nonresistive medium to a comparison of terrestrial motion in nonresistive and resistive media. Problem 5 derives the distance a body would fall freely in a void under an inverse square force, and Problems 6 and 7 combine to extend the analysis to the motion of projectiles in a uniformly resisting medium.

> Problem 5. *Supposing that the centripetal force is reciprocally proportional to the square of the distance from the center, [it is required] to define the spaces which a body falling in a straight line describes in given times* .

> Problem 6. *To define the motion of a body carried by its innate force alone through a uniformly resisting medium* .

> Problem 7. *Supposing a uniform centripetal force, [it is required] to define the motion of a body ascending and descending straight up and down in a homogeneous medium* .

Newton did not explain why he included these problems of terrestrial motion in a tract that was motivated by questions of celestial motion. It may well be that he wished to call attention to the difference between calculated ideal terrestrial motion in a void and observed terrestrial motion in a resistive medium, such as projectiles fired in the atmosphere of the earth. In contrast to the motion of terrestrial projectiles, the motion of celestial bodies does not display a marked difference between calculated ideal motion in a void and observed celestial motion in the ether. The law of equal areas in equal times is derived for ideal motion in a void, and it appears to correspond to the observed motion in the ether. Thus, any assumption of a mechanical ether, such as that made by Descartes, becomes suspect. If the ether is so fine as to offer no resistance to the motion of celestial bodies, then it surely is too fine to provide the necessary mechanical collisions to account for the gravitational force. The implication of such a difference concerning the role of the ether in producing the gravitational force has been discussed in chapter 2.**[11]**

The tract that Halley received from Newton in November of 1684 contained only a fraction of what would shortly be demonstrated in the

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1687 edition of the *Principia* , but it was much more than anyone had thus far been able to accomplish, and much more than Halley had requested. Shortly after receiving it, Halley made yet another journey to Cambridge in order to consult Newton once again. He found Newton already at work revising and enlarging the tract into what would ultimately become the 1687 edition of the *Principia* . Halley returned to London and early in December 1684 he communicated the initial copy of the tract to the Royal Society in order to secure Newton's rights of authorship, even as work on the extended version continued. What follows includes a translation of the Latin copy of the tract that appears in the *Register Book* of the Royal Society and now bears the title, *On Motion of Bodies in Orbit* or, simply, *On Motion* . The version in the *Register Book* contains some obvious mistakes of transcription, particularly in the diagrams. Corrections were made to the transcript used in this translation by comparing it to other versions of this tract. Although the authenticity of the text is not in question, the original manuscript that Newton sent to Halley has never been found.[12]

Introduction: Definitions, Hypotheses, and Lemmas

Definitions

Newton began *On Motion* with three definitions: centripetal force, innate force, and resistance. The list of definitions and the details of their descriptions were greatly enlarged in subsequent versions of the work, but in this first text they were set out very simply and compactly.

> Definition 1. *I call centripetal the force by which a body is impelled or attracted toward some point which is regarded as the center* .

Definition 1 contains the first use of the term "centripetal" (center-seeking), which Newton coined as a complement to the term "centrifugal" (center-fleeing) that Huygens had employed in his writings.[13] The term signaled a major clarification of Newton's analysis of dynamics as found in the demonstrations of circular motion in the *Waste Book* (1665) and *On Circular Motion* (pre-1669) that were discussed in chapter 3. In those works, Newton, consistent with Descartes, refers to an outward endeavor, and in *On Motion* Newton, consistent with Hooke, does not mention an outward endeavor. Newton's rejection of both Cartesian perspective and terminology is not a change in Newton's method of demonstration, however. In both the early work and this later tract Newton employs the parabolic approximation in which the force is directly proportional to the radial displacement and inversely proportional to the square of the time. Moreover, Newton continued to employ the term "centrifugal" in other contexts, well after the writing of this work.[14]

> Definition 2. *Moreover,* [*I call*] *the force of a body,* [*the force*] *innate in a body, that by which it endeavors to persevere in its own motion along a straight line* .

In Newton's analysis, the use of the "force innate in a body" most closely conforms to the contemporary use of the "magnitude of the linear momentum." Therefore, Newton's use of the word "force" in this context is at variance with modern usage, which reserves the term "force" for the "time rate of change of the linear momentum." Regardless of his choice of term, he uses "innate force" in a manner consistent with modern analysis.

> Definition 3. *And* [*I call*] *resistance the force which comes from a regularly impeding medium* .

The topic of motion in a resistive medium will not be included in the material covered in this study.

Hypotheses

Newton provided four hypotheses: the first sets out assumptions concerning resistance to motion; the second describes force-free motion; the third states the parallelogram rule for the addition of displacements produced by separate forces; and the fourth expresses a version of Galileo's time-squared dependence of linear displacement under a constant force.

> Hypothesis 1. *In the next nine propositions the resistance is zero; in those propositions following, the resistance is conjointly as the speed of the body and the density of the medium* .

As noted, the first nine propositions consist of four theorems and five problems, all of which are concerned with ideal motion in the absence of resistance. The final two propositions (Problems 5 and 6) treat motion in a resistive medium.

Hypothesis 2. *Every body by its innate force alone progresses uniformly along a straight line to infinity unless something impedes it from outside* .

Following Descartes, Newton states that motion free from an external force (i.e., motion subject only to "innate force") takes place at a uniform rate along an infinite straight line. An enlarged version of this statement appears as the first law of motion in the *Principia* .[15]

Hypothesis 3. *A body, in a given time, with forces having been conjoined, is carried to the place where it is carried by separated forces in successively equal times* .

This rule for the combination of displacements as a measure of forces was implicit in Newton's pre-1665 analysis of circular motion but is made explicit here. Although Newton gives it as a hypothesis in *On Motion* , in the first revision of the tract Newton adds a demonstration and promotes the hypothesis to the status of a lemma.[16]

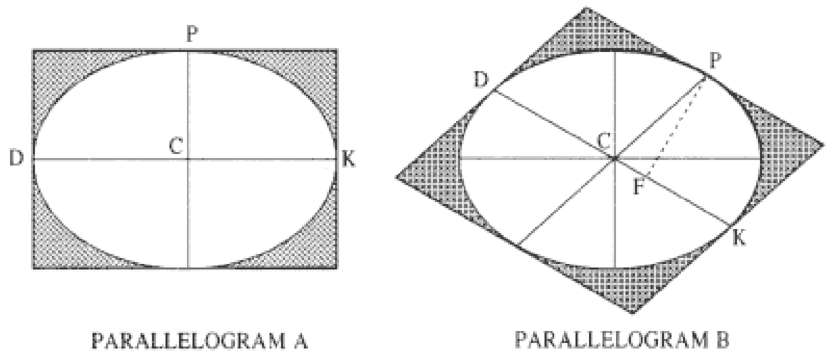

Figure 4.1
The area of parallelogram A is equal to the area of parallelogram B (Proposition 31, Book 7, of the *Conics* of Apollonius of Perga).

Hypothesis 4. *The space which a body, with some centripetal force impelling it, describes at the very beginning of its motion, is in the doubled ratio of the time* .

This relationship is critical to all of Newton's analysis of action under a continuous centripetal force. It is the very core of his analysis; yet it is given here very simply and with little explanation. This hypothesis will be revised by the addition of a demonstration, and it also will be promoted to the status of a lemma.

Lemmas

Lemma 1. *All parallelograms described around a given ellipse are equal to each other. This is established from the* Conics.

This lemma is demonstrated in Book 7, Proposition 31 in the *Conics* of Apollonius of Perga (c. 262–c. 200 B.C.); see figure 4.1.[17] The area of the circumscribed parallelogram *A* is 2*PC* × *DK* , and it is equal to the area of the circumscribed parallelogram *B* , which is 2*PF* × *DK* . It is important to note in parallelogram *B* that *PF* is the normal to *DK* , while in parallelogram *A* that *PC* is the normal to *DK* . The sides of parallelogram *B* are tangent to the ellipse at points *P* and *D* , where *DK* is constructed parallel to the line tangent at point *P* . This relationship will appear as Lemma 12 in the *Principia* , where it is employed in the solution of the direct Kepler problem.

Lemma 2. *Quantities proportional to their differences are continuously proportional. Set* A: (A – B) = B: (B – C) = C: (C – D) = . . . *and by dividing there will be produced* A:B = B:C = C:D = . . .

This lemma has application only to motion in a resistive medium, a topic which does not appear in the first three sections of Book One of the *Principia* .

Following the completion of the draft of *On Motion of Bodies in Orbit* (or simply *On Motion*) that was sent to Halley, and hence to the Royal Society, Newton produced a slightly enlarged version of the tract, entitled *On the Motion of Spherical Bodies in Fluids* .[18] The body of the tract, which supplied the method and solutions to the direct problem in nonresisting media, was essentially unchanged from the first draft (except for the addition of a paragraph at the end of the scholium to Theorem 4). Newton did expand, however, the rather sparse statement of the fundamental hypotheses that was just discussed.

Hypothesis 3 Becomes Lemma 1

> Hypothesis 3. *A body, in a given time, with forces having been conjoined, is carried to the place where it is carried by separated forces in successively equal times* .

Hypothesis 3 of the first draft of *On Motion* now appears as Lemma 1 in the second draft. The initial statement is slightly revised, and Newton appends a detailed demonstration. An earlier version of this parallelogram rule was discussed in chapter 3 during the analysis of uniform circular motion. It is important to note that the measure of a force is the displacement it produces in a given time, and it is the displacements that are combined when the "forces are conjoined."

> Lemma 1. *A body, with forces having been conjoined, describes the diagonal of a parallelogram in the same time as it describes the sides, with* [*forces*] *having been separated* .
>
> *If a body in a given time were to be carried from* A *to* B *by the action of the force* M *alone and from* A *to* C *by the force* N *alone* , [*then*] *complete the parallelogram* ABDC, *and it will be carried in the same time from* A *to* D *by both forces* . [See fig. 4.2.]
>
> *For since force* N *acts along the line* AC *parallel to* BD, *by Law 2 this force* [N] *will do nothing to change the speed of* [*the body's*] *approaching the line* BD, *impressed by the other force* [M]. *The body will therefore approach the line* BD *in the same time whether the force* AC [N] *is impressed or not; and so at the end of that time it will be found somewhere on the line* BD. *By the same reasoning it will at the end of the same time be found somewhere on the line* CD, *and consequently must be found at the meeting* D *of both lines* .

Newton applied the parallelogram rule implicitly in all his dynamics. He does not, however, give an explicit formal defense of the application of this lemma to the polygonal and parabolic approximations, either in this tract or in the first edition of the *Principia* . Only in the revised editions of the *Principia* does he offer an explicit demonstration of its application.[19]

Figure 4.3 displays the situation to which the parallelogram rule is applied in the format of the polygonal approximation in Theorem 1 (to follow). A body moves with uniform rectilinear motion from point *A* to point

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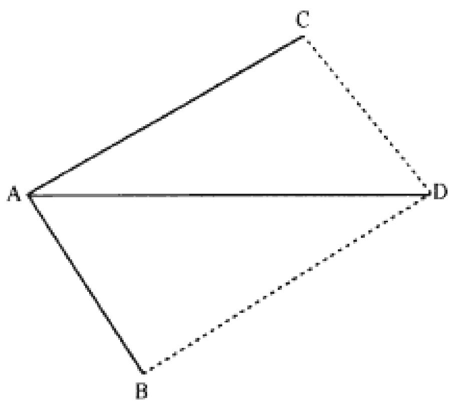

Figure 4.2
Based on Newton's drawing for the demonstration of Lemma 1.

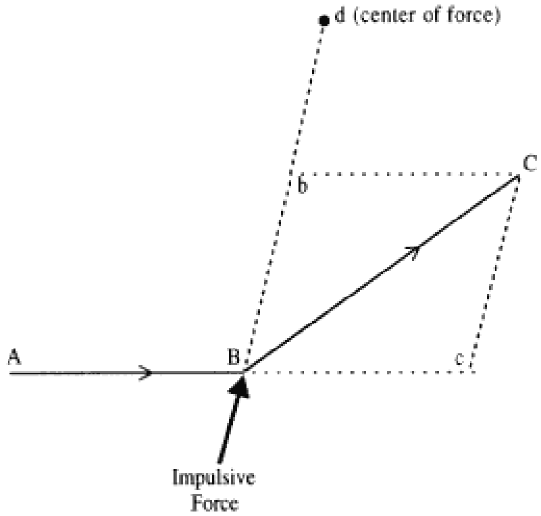

Figure 4.3
The parallelogram rule as applied to the polygonal approximation. In a given time, *Bc* is the displacement due to the initial "innate force," *Bb* is the displacement due to the impulsive force, and *BC* is the resultant displacement.

B in a given time *DT* . At point *B* an impulsive centripetal force, directed toward the center of force at point *d* , acts for a vanishingly small time *dt* on the body. The body then moves with uniform rectilinear motion from point *B* to point *C* in the given time *DT* . If the impulsive force had not acted at point *B* , then the body would have moved with uniform rectilinear motion from point *B* to point *c* in the time *DT* under the action of only its initial motion. If the body had been at rest at point *B* , then it would have moved with uniform rectilinear motion from point *B* to point *b* in the time *DT* under the action of only the motion produced by the impulsive force. (Note that the displacement *Bb* must be in the direction of the line of action of the impulsive centripetal force.) The composite uniform motion from point *B* to point *C* is along the diagonal of the parallelogram formed by the initial motion *Bc* (or *bC*) and the added motion *Bb* (or *cC*). Thus, a body is carried in a given time [*DT*] by combined forces [the initial "force of the body's motion" plus the "change in the body's motion" due to the impulse] to the place [*B* ®*C*] where it is carried by separated forces [*B*®*c* by the initial motion and *B*®*b* by the impulsive motion] in successively equal times [*DT*].

If the force is continuous rather than impulsive, then the curve is also continuous rather than polygonal. For a continuous force the body moves along the curved path between the points *B* and *C* , and Newton considers a situation in which the point *C* approaches very closely to the point *B* . Thus, the interval of time is extremely small, and the force can be assumed to be constant over that interval. Galileo has demonstrated that ideal projectile motion under a constant force is parabolic, and hence the element of arc *BC* is approximated by a parabolic element (i.e., the parabolic approximation discussed in chapters 1 and 2). The displacement *BC* of any future point *C* on the elemental parabolic arc can be found by using the parallelogram rule to combine the displacement *BC* due to the initial tangential velocity with the deviation *Bb* due to the constant force. Thus, this rule is applied to both impulsive and continuous forces in a consistent manner.

Theorem 1—
The Law of Equal Areas in Equal Times

In Newton's theorems that follow Theorem 1, the path of the particle appears as a continuous curve and the force changes continuously as the particle traverses the path. In Theorem 1, however, the particle travels with a given uniform rectilinear motion between points *A* and *B* . At point *B* it experiences an impulsive force and then travels with a different uniform rectilinear motion between points *B* and *C* . At point *C* it experiences

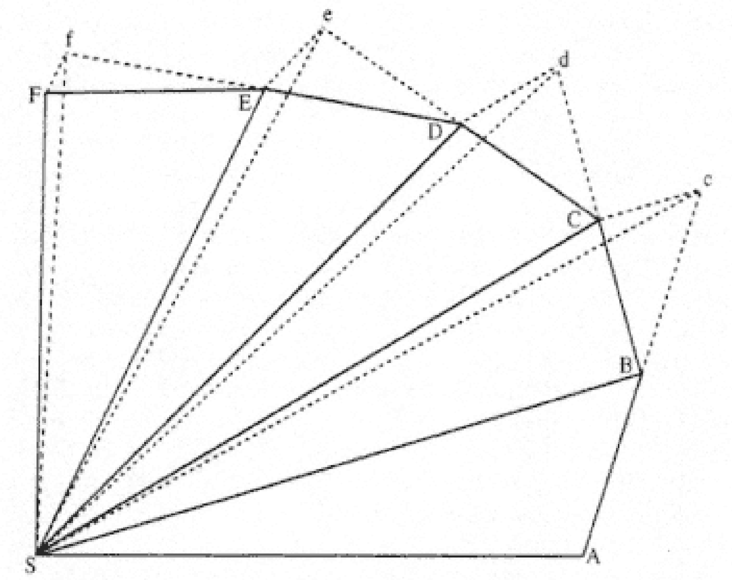

Figure 4.4
Based on Newton's drawing for Theorem 1.

yet another impulsive force. The process is repeated again and again. Ultimately, Newton required that the distance between points become infinitely small, and thus he required the polygonal path of the particle to become a smooth curve. This type of approximation is similar to the polygonal approximation that Newton used in his early work of 1665 on uniform circular motion and is particularly well suited to the demonstration of this theorem. Figure 4.4 is based on Newton's diagram that accompanies Theorem 1 in *On Motion* ; it is instructive to note how the diagram is constructed.

Line *AB* . One starts at point *A* and draws a line of arbitrary length to point *B* . The line *AB* represents the displacement of the body for a given time with a given velocity. At point *B* an impulsive force that is directed toward the center *S* acts on the body.

Line *Bc* . Then the line is extended from point *B* to point *c* , where line *Bc* is equal in length to line *AB* . The line *Bc* represents the displacement that would have taken place in the given interval of time if no force had acted at point *B* .

Line *BC* . Because an impulsive force does act at point *B* , however, one must construct a line *BC* that represents the actual displacement that does take place in that given time.

Line *Cc* . Then the actual displacement *BC* is connected to the hypothet-

ical force-free displacement *Bc* by a line *Cc* . The line *Cc* represents the deviation of the body from *Bc* due to the action of the impulsive force at *B* . Therefore, *Cc* is constructed parallel to the line of force *SB* because the displacement *Cc* must be in the same direction as the line of action of the impulsive force. The rest of the construction is a repetition of the preceding procedure.

It simply remains to demonstrate (1) that the area *ASB* is equal to the area *BSc* and (2) that area *BSc* is equal to the area *BSC* . Then, in a given time, (3) area *ASB* is equal to the area *BSC* , which in turn is equal to area *CSD* , and so on. Thus, for a series of discrete impulsive forces, equal areas will be swept out in equal times. The correspondence with a force that acts continuously is achieved by passing to the limit by letting "the triangles be infinite in number and infinitely small, so that the individual triangles correspond to the individual moments of time."

What follows is Newton's statement of this important theorem, one line at a time, each line followed by a detailed discussion.

Demonstration

Theorem 1. *All orbiting bodies describe, by radii having been constructed to their center, areas proportional to the times* .

The only restriction on the force is that it be directed to a given point *S* . For such a central force, Newton demonstrates that the radius linking the point *S* and the body *P* sweeps out equal areas in equal times. For the special example of uniform circular motion about a point at the center of the circle, the radius also sweeps out equal angles in equal times. In his early analysis of such motion, Newton employed this more restricted angular version of the area law to relate the period and circumference to arc and deviation, and thus to express the results of the time / distance relationship in the desired form (see chapter 2). In Theorem 1, however, Newton demonstrates that the law holds true for any centripetal force acting in a medium devoid of resistance, and thus he can express the time, and hence the force, in terms of the geometric elements of the figure for nonuniform motion.

[A] *Let the time be divided into equal parts, and in the first part of the time let a body by its innate force describe the straight line* AB.

The line *AB* is the displacement due to the uniform rectilinear motion of the body between points *A* and *B* . The body leaves point *A* with a given speed and travels to point *B* in the absence of any external force.

[B] *The same body would then, if nothing impeded it, proceed directly to* c *in the second part of the time, describing the line* Bc *equal to itself* AB,

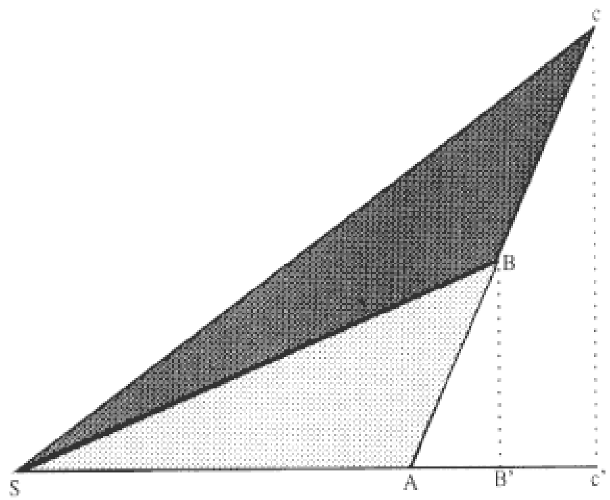

Figure 4.5
The area of triangle *ASc* is equal to twice the area of triangle *ASB* and thus area *ASB* equals area *BSc* .

In this theorem rectilinear motion along the tangent occurs when nothing impedes the tangential motion (i.e., in the *absence of a centripetal* "center-seeking" force). In the pre-1669 tract, *On Circular Motion* , however, rectilinear motion along the tangent occurs when there is no impediment to the Cartesian outward endeavor (i.e., in the *presence of a centrifugal* "center-fleeing" endeavor). After 1679, however, Newton set aside the Cartesian outward endeavor and was concerned only with the two dynamical elements that Hooke stressed in his early paper, the tangential displacement and the center-seeking force.

[C] *so that, when the radii* AS, BS, *and* cS *were extended to the center, areas* ASB *and* BSc *would be made equal* .

See figure 4.5. The line segment *AB* is the displacement the body makes during the first time interval and the line segment *Bc* is the displacement, equal to *AB* , that the body would have made in an equal time if no impulsive force had acted upon it at point *B* . Thus, the displacement *Ac* is equal to twice the displacement *AB* and thus the height *cc* ' is equal to twice the height *BB* '. The area of a triangle is equal to one-half the product of the base and the height. Triangles *ASB* and *ASc* have a common base *SA* and the height of triangle *ASc* is twice the area of triangle *ASB* . Thus, the area of triangle *ASc* is twice the area of triangle *ASB* . From figure 4.5 one can express the area of triangle *BSc* as the difference between the area of triangle *ASc* and the area of triangle *ASB* . Thus, area *BSc* = area *ASc* – area *ASB* and, because area *ASc* = 2 area *ASB* , then area *BSc* = area *ASB* .

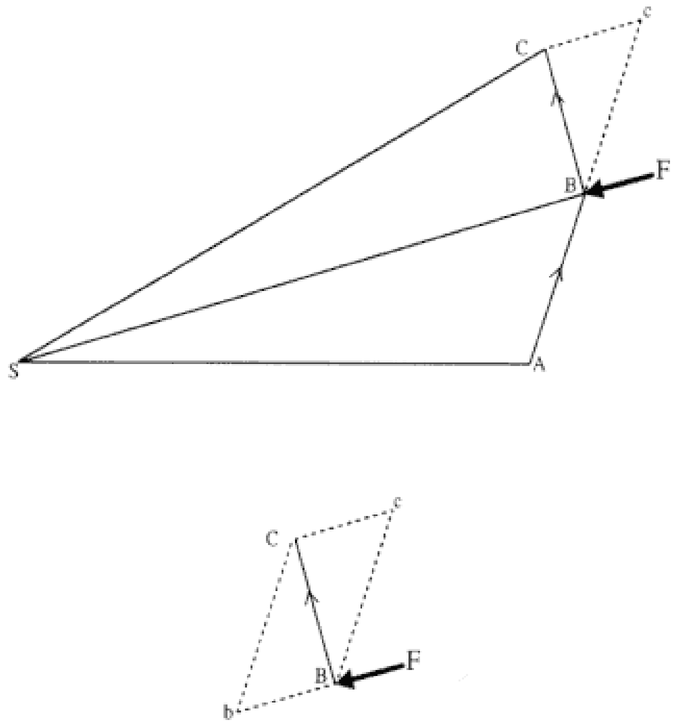

Figure 4.6
The parallelogram rule as applied to Theorem 1.

[D] *Now when the body comes to* B, *let the centripetal force act with one great impulse, and let it make the body deflect from the straight line* Bc *and proceed along the straight line* BC.

In the early discussion (1665) of circular motion in the *Waste Book* , the "force or pression" came from the "collision" or "reflection" of the ball from the curve and was the impediment to the "outward endeavor." In this more mature work (1684), the impulse delivered to the ball comes from an external unbalanced "centripetal force" and there is no mention of the Cartesian "outward endeavor."

[E] *Parallel to the same* BS, *let* cC *be extended, meeting* BC *at* C, *and when the second interval of time is finished, the body will be found at* C.

See figure 4.6. If no force had acted at point *B* , then the particle would have made the hypothetical displacement *Bc* equal in length to the initial displacement *AB* . Because a force *F* does act at point *B* , however, the particle is diverted and instead makes the actual displacement *BC* . The deviation or difference between the hypothetical displacement *Bc* and the

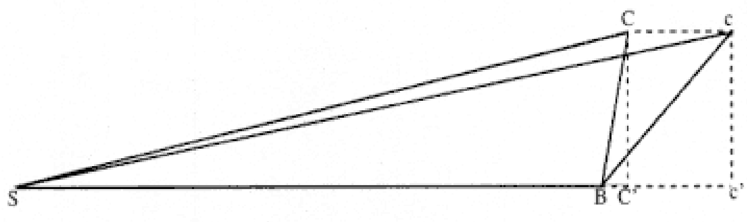

Figure 4.7
The area of triangle *SBC* equals the area of triangle *SBc* .

actual displacement *BC* is given by the dashed line *Cc* . The small construction in figure 4.6 is an application of the parallelogram rule given in Hypothesis 3. The two uniform rectilinear motions to be combined at point *B* are (1) the uniform motion along *Bc* , which was retained from the body's initial motion at point *B* , and (2) the uniform motion along *Bb* , which was generated by the impulsive force *F* at point *b* . The resultant uniform rectilinear motion is along *BC* . In the given time, the particle will make the displacement *BC* , where *BC* is the diagonal of the parallelogram *BbCc* formed with sides *Bc* and *Bb* , where *Bc* and *Bb* are the displacements the body would have made separately in the given time. Consistent with the parallelogram rule, the deviation *Cc* is constructed parallel to the line *Bb* and hence parallel to the line of force *SB* .

[F] *Join* S *and* C *and because of the parallels* SB *and* Cc, *the triangle* SCB *will be equal to the triangle* SBc *and hence also to the triangle* SAB.

See figure 4.7. The triangles *SBC* and *SBc* have the same base *SB* and, because the deviation *Cc* is parallel to the line of force *SB* , they have the same perpendicular to that base through *C* and *c* (i.e., the height *C'C* equals the height *c'c*). Thus, the areas of triangles *SBC* and *SBc* are equal because they have a common base and equal heights.

[G] *By a similar argument, if the centripetal force should act successively at* C, D, E, *etc., making the body in separate moments of time describe the separate straight lines* CD, DE, EF, *etc., the triangle* SCD *will be equal to the triangle* SBC, SDE *to* SCD, SEF *to* SDE (*and so on*).

Simply repeat [F] for each of the successive blows at the points *C, D, E* , . . ., that are shown above in the original drawing, figure 4.4.

[H] *In equal times, therefore, equal areas are described* .

The conclusion is drawn from [G].

[I] *Now let these triangles be infinite in number and infinitely small, so that the individual triangles correspond to the individual moments of time, the centripetal force acting without interruption, and the proposition will be established* .

This limiting procedure is similar to that which Newton employed in the polygonal approximation solution to the problem of uniform circular motion in the early work in the *Waste Book* (discussed in chapter 3). Theorem 1 above becomes Proposition 1 in the 1687 edition of the *Principia* much in the form that it appears here, and Newton revises and extends it in the later editions.[20]

Newton's demonstration of Kepler's law of equal areas in equal times in Theorem 1 is a major step forward in his construction of a paradigm for the solution of direct problems. The challenge of the direct problem is to find the functional dependence of the force upon the distance between the body and the center of force necessary to describe the orbital motion. The only requirement imposed on the nature of the force by Theorem 1 is that it be directed toward a fixed center. Theorem 1 enables Newton to express the time of motion in terms of the area swept out and hence in terms of the dimensions of the orbit. He employs it and the parabolic approximation in Theorem 3 to develop a general measure of centripetal force.

Theorem 2—
Uniform Circular Motion

Theorem 2 on uniform circular motion employs the parabolic approximation as in the pre-1669 tract *On Circular Motion* rather than the polygonal approximation as found in the solution given in the *Waste Book* (both solutions are discussed in chapter 2). In Corollary 2 of the pre-1669 tract *On Circular Motion* , Newton wrote that "the endeavors from the centers in diverse circles are as the diameters divided by the squares of the times of revolution." Here in Corollary 2 of the post-1679 tract he wrote that "[the centripetal forces are] reciprocally as the squares of the periodic times divided by the radii." The change from diameters to radii is trivial, but the shift in emphasis from the "outward endeavor" to the center-seeking "centripetal force" is significant. For uniform circular motion, the arcs are proportional to the speeds. Newton elected to express the theorem in terms of the arc, however, and to reserve the statement in terms of the speeds for the first corollary, where the force is given as proportional to the square of the speeds divided by the radii.[21]

Figure 4.8 is based on Newton's diagram that accompanies this theorem in *On Motion* . There are two circles because the results are to be expressed as the ratio of the forces required to maintain uniform circular motion for two different radii, *SB* and *sb* . As in his earlier analysis, the line segments *BC* and *bc* represent the tangential displacements that would have taken place if no force had acted on the bodies at points *B* and *b* ; the arcs *DB* and *db* are the actual circular paths followed under the action of the centripetal forces; and the deviations that measure the forces are the

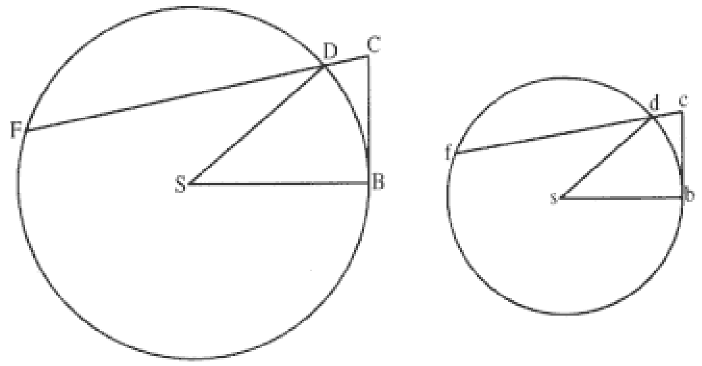

Figure 4.8
Based on Newton's drawing for Theorem 2.

line segments *DC* and *dc* . Newton's shift in emphasis from a Cartesian outward endeavor to an inward centripetal force does not affect his use of the parabolic approximation. In the limit, the force still is assumed to be approximately constant and its magnitude still is proportional to the deviation *DB* and inversely proportional to the square of the time. The following lines from the text are interspersed with detailed explanations and discussions.

Demonstration

[A] Theorem 2. *Let the bodies* B *and* b *orbiting on the circumferences of the circles* BD *and* bd *describe in the same time the arcs* BD *and* bd. *By their innate force alone they would describe the tangent lines* BC *and* bc *equal to these arcs* .

As in previous examples, the important elements shown in the diagram in figure 4.8 above are the inertial displacements *BC* and *bc* , which would have taken place along the tangent lines in the absence of a force, and the arc lengths *BD* and *bd* of the actual path. The arc lengths *BD* or *bd* in a given time are equal in length to the tangent lengths *BC* or *bc* . If the tangent *BC* were wrapped around the circle, then the end of the tangent *C* would travel on a curved arc *CD* (its involute) about *B* terminating on point *D* (see fig. 4.9).**[22]** Thus, the tangent length *BC* is exactly equal to the arc length *BD* .

[B] *The centripetal forces are those that perpetually draw bodies back from the tangents toward the circumferences* [*of the circles*], *and hence are to each other as the distances* CD *and* cd *surmounted by them* ,

Or, $F_C / f_c = CD / cd$, where F_C and f_c are the centripetal forces and *CD* and *cd* are the deviations from the tangential paths relative to the circular

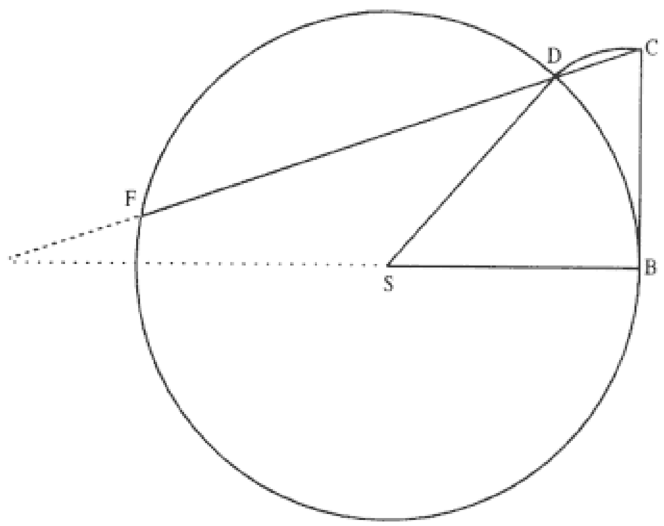

Figure 4.9
The length of the arc *BD* is equal to the length of the tangent segment

BC .

paths. The deviation *CD* is the chord of the arc *CD* about *B* (see fig. 4.9) and the deviation *cd* is the chord of the arc *cd* about *b* . In his pre-1669 analysis (see chapter 3) Newton spoke of the outward endeavor, and the deviation *CD* was directed incorrectly along a diameter of the circle, as suggested by the Cartesian terminology. Figure 4.10 is a comparison of the form of the diagram in his pre-1669 analysis of uniform circular motion with the form of the diagram in this post-1679 analysis. Note how in this later figure the deviation *CD* is not along a diameter but along the chord *DF* . This change in the slope and location of *CD* is now determined by the correct requirement in [A] that the tangent length *BC* must equal the arc length *BD* (see fig. 4.9). The deviation *CD* is no longer a potential but impeded outward displacement that acts in a radial direction, as suggested by Descartes. The revision of the diagram does not invalidate Newton's earlier solution because the Euclidean theorem to be employed in the next step is valid for both chords and diameters. The forces in either case are proportional to the deviations *DC* and *dc* , or $F_C / f_c = CD / cd$, as required.

[C] *that is, on producing* CD *to* F *and* cd *to* f, *as* BC^2 / CF *to* bc^2 / cf *or as* BD^2 / (1/2)CF *to* bd^2 / (1/2)cf.

Or $F_C / f_c = CD / cd = (BC^2 / CF) / (bc^2 / cf) = (BC^2 / (1/2)\,CF) / (bc^2 / (1/2)\,cf)$. In the earlier pre-1669 drawing (see fig. 4.10), Newton specifically called upon Proposition 36 of Book 3 of Euclid's *Elements* to relate the lines *CB* (the tangent), *CD* (the deviation), and *CE* (the diameter plus the

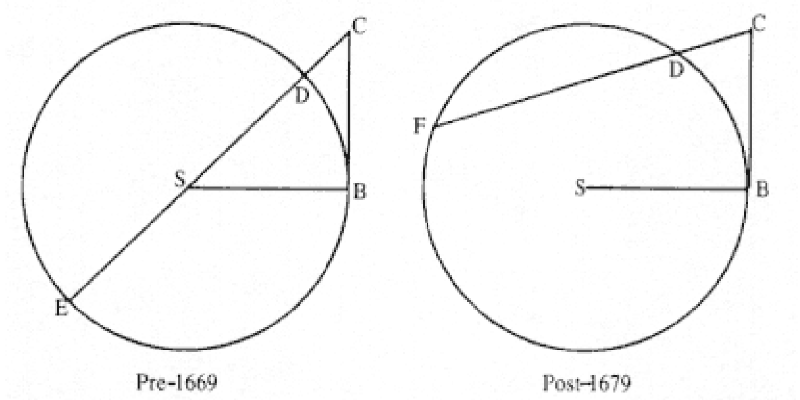

Figure 4.10
A comparison of the pre-1669 diagram with the post-1679 diagram in the analysis of circular motion.

deviation) as follows: $CE / CB = CB / CD$. In this later text he did not give a specific reference to Euclid but the same relationship holds for the lines *CB, CD* , and *CF* (the chord of the circle plus the deviation) in the revised post-1679 diagram. In figure 4.11A, Euclid's theorem is valid when *CF* contains *any* chord of the circle.[23] Thus, in figure 4.11B, the deviation *CD* (dashed line) is obtained from $CD / BC = BC / CF$ or $CD = BC^2 / CF$. Newton chose to express the results in terms of the radius (1/2) *CF* rather than the diameter *CF* . Thus, $F / f = CD / cd = (BC^2 / CF) / (bc^2 / cf) = (BC^2 / (1/2)\,CF) / (bc^2 / (1/2)cf)$.

[D] *I am speaking of the very minute distances* BD *and* bd, *to be diminished into infinity, so that in place of* (1/2)CF *and* (1/2)cf, *it would be allowed to write the radii* SB *and* sb *of the circles. This done, the Proposition will be established* .

Or, as point *D* approaches point *B* , then line *CF* approaches line 2*SB* (see fig. 4.12). And, (1/2) *CF* can be replaced by the radius *SB* . Thus, the proposition is demonstrated, and the forces are proportional to the square of their arcs divided by the radii of their circles (i.e., $F_C / f_c = (BC^2 / ((1/2)\,CF) / (bc^2 / ((1/2)\,cf) = (BC^2 / SB) / (bc^2 / (sb))$).

Corollary 1. [*Hence*] *the centripetal forces are as the squares of the speeds divided by the radii of the circles* .

For uniform circular motion, the arc is proportional to the tangential speed. From [D], the ratios of the forces $F_1 / F_2 = (arc_1^2 / r_1) / (arc_2^2 / r_2)$ and because $arc_1 / arc_2 = v_1 / v_2$ then $F_1 / F_2 = (v_1^2 / r_1) / (v_2^2 / r_2)$.

Corollary 2. *And reciprocally as the squares of the periodic times divided by the radii of the circles* .

Or, F_1 / F_2= (T_2^2/ r_2) / (T_1^2/ r_1). The tangential speed v = circumference / period = 2pr / T , thus, v^2 / r = 4p^2 (r / T)^2 / r . From Corollary 1, F_1 /

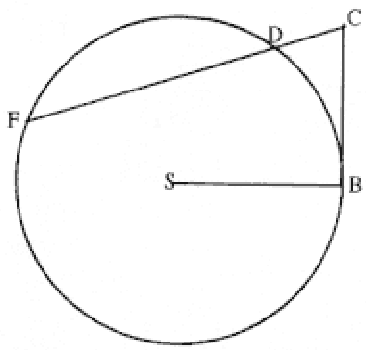

Figure 4.11A
From Proposition 36 of Book 3 of Euclid's
Elements , one has *CD / BC = BC / CF* .

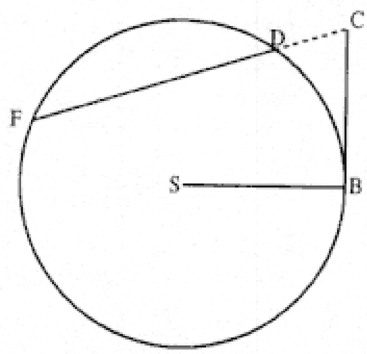

Figure 4.11B
In the diagram for Theorem 2, one has
CD / BC = BC / CF or the deviation
CD = BC^2 / CF.

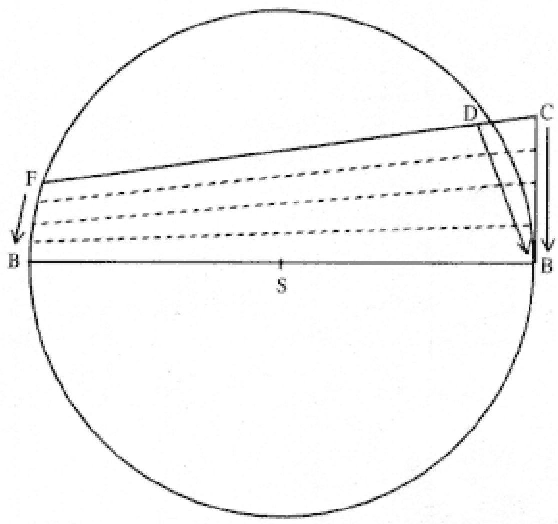

Figure 4.12
As the point *D* approaches the point *B* , the line *CF*
approaches the diameter 2*SB* .

F_2 = (v_1^2 / r_1) / (v_2^2 / r_2), which becomes (4p^2 (r_1 / T_1)^2 / r_1) / (4p^2 (r_2 /T_2)^2 /r_2) = (r_1 / T_1^2) / (r_2 /T_2^2) = (T_2^2 / r_2) / (T_1^2 /r_1), or the force is reciprocally proportional to the squares of the periods divided by the radii.

Corollary 3. *From this, if the squares of the periodic times are as the radii of the circles,* [*then*] *the centripetal forces are equal, and conversely* .

If T^2 is proportional to r , then $(T_2^2/ r_2) / (T_1^2/ r_1) = 1$. From Corollary 2, $F_1/ F_2 = (T_2^2/ r_2) / (T_1^2/ r_1) = 1$ and thus $F_1 = F_2$.

Corollary 4. *If the squares of the periodic times are as the squares of the radii,* [*then*] *the centripetal forces are reciprocally as the radii, and conversely* .

Or, $F_1/ F_2 = r_2/r_1$. If T^2 is proportional to r^2 , then from Corollary 2, $(T_2^2/ r_1) / (T_1^2/ r_2) = r_2/r_1$ and thus $F_1/ F_2 = r_2/r_1$.

Corollary 5. *If the squares of the periodic times are as the cubes of the radii, [then] the centripetal forces are reciprocally as the squares of the radii, and conversely* .

Or, $F_1/ F_2 = r_2^2/ r_1^2$. If T^2 is proportional to r^3 , then from Corollary 2, $(T_2^2/ r_2) / (T_1^2/ r_1) = r_2^2/ r_1^2$ and thus $F_1/ F_2 = r_2^2/ r_1^2$.

Scholium

The case of the fifth corollary holds true in the celestial bodies. The squares of the periodic times are as the cubes of the distances from the common center around which they revolve. Astronomers already agree that this holds true in the major planets orbiting around the sun and in the minor ones around Jupiter and Saturn .

Newton did not give credit here to Kepler for the relationship in this scholium, but the result is found in Book 5 of Kepler's *World Harmony* (1619).**[24]** In Theorem 4 of this tract, Newton demonstrates that for elliptical motion the square of the period is as the cube of the transverse (major) axis of the ellipse.

Theorem 3—
The Linear Dynamics Ratio

Theorem 3 is Newton's crowning achievement in dynamics. It sets forth the combination of geometric elements that provide the basic paradigm for his primary solution of direct problems. Figure 4.13 makes visual these basic geometric elements: the displacement *QR* is the deviation from the linear path *PR* produced by the centripetal force directed toward the center of force *S* , and the triangular area *PSQ* is given by one-half of the product $QT \times SP$. In the limit as the point *Q* approaches the point *P* , the force is assumed to be constant and its magnitude is proportional to the displacement *QR* divided by the square of the time (i.e., the parabolic approximation). From Theorem 1, the time is proportional to the area *PSQ* . Thus, combining the parabolic approximation and the area law, the force is shown to be proportional to the linear dynamics ratio $QR / (QT^2 \times SP^2)$, or as Newton preferred, "the centripetal force would be reciprocally as the solid $(SP^2 \times QT^2) / QR$."

The results of Theorem 3 are valid for any general curve and are not restricted to circles or ellipses (for example, Newton employed it to calculate the force necessary to produce a spiral motion). Nevertheless, New-

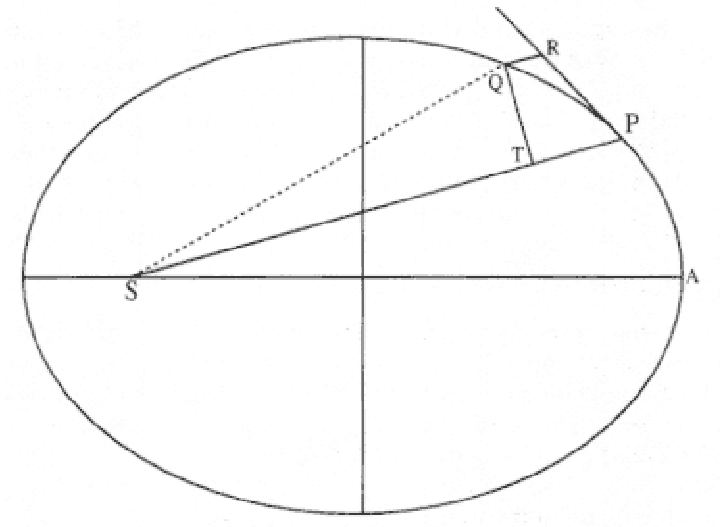

Figure 4.13
Based on Newton's diagram for Theorem 3.

ton's thoughts must have been on the distinguished Kepler problem when he drew the diagram for this theorem, because the general curve in figure 4.13 looks very much like an ellipse (in fact, it is an ellipse). He does not employ any of the specific properties of an ellipse, however, in obtaining the general relationship between the force and the dynamic elements of the general curve. In the following, the theorem is divided into a number of separate lines and detailed commentaries are given for each line.

Demonstration

[A] Theorem 3. *In the indefinitely small figure* QRPT *the line segment* QR *is, with the time given, as the centripetal force* ,

Or, *QRμF* for a given time. The applied force is assumed to be approximately constant in magnitude and direction in the limit as the point *Q* approaches the point *P* , or as Newton puts it, "in the indefinitely small figure *QRPT* ." Thus, the elemental arc of the general curve can be approximated by an elemental arc of a parabola. The tangential displacement *PR* is the inertial displacement the body would have made in a given time in the absence of a centripetal force under the sole action of the velocity it had at point *P* . The deviation *QR* is the linear displacement the body would have made in the same given time under the sole action of a constant applied force directed along the line *SP* from the initial point *P* to the center of

force *S* . The resultant displacement *PQ* is given from the parallelogram rule by the sum of the tangential displacement *PR* due to the initial "innate force" at point *P* and the linear displacement *QR* due to the continuous applied force. [In terms of modern notation, the displacement $QR = (1/2)\ (F / m)\ (t^2)$ for a constant force *F* and mass *m* , where F / m is the acceleration. Thus, for a given time, $QR = (t^2 / 2m)\ F = k_1 F$, (where k_1 is a constant equal to $t^2 / 2m$) or the displacement *QR* is directly proportional to the force *F* .]

[B] *and, with the force given, as the square of the time* ,

Or, *QRμ*t^2 for a given force. This condition follows from Hypothesis 4 (or in the revised Lemma 2: "The space with which a body, with some centripetal force impelling it, describes at the very beginning of its motion, is in the doubled ratio of the time"). [In modern notation, from section [A], $QR = (1/2)\ (F/m)(t^2) = (F/2m)\ (t^2) = k_2 t^2$, where k_2 is a constant equal to $F/2m$. Thus, for a given force *F* the displacement *QR* is directly proportional to the square of the time *t* .]

[C] *and hence, when neither is given, as the centripetal force and the square of the time conjointly* ;

Or, *QRμ* (*F*) (t^2). From the expressions [A] and [B], when neither the time nor the force is given, the line segment *QR* depends directly upon both the force and the square of the time. [In modern notation, $QR = (1/2m)(F)(t^2) = k_3(F)(t^2)$, where k_3 is a constant equal to $1/2m$. Thus, *QR* is directly proportional to the product of the force and the square of the time.]

[D] *that is, as the centripetal force taken once and the area* SQP *proportional to the time (or its double* , SP × QT) *taken twice* .

Or, $QR \mu (F)(t^2) \mu (F)(SP \times QT)^2$. From [C], the displacement is proportional to the first power of the force ("taken once") and the second power of the time ("taken twice"). From Theorem 1, the time is proportional to the triangular area *SQP* , which is equal to one-half the base *SP* times the height *QT* (see fig. 4.14), or the product *SP* × *QT* is the double of the area where ultimately the arc *QP* will approach the chord *QP* .

[E] *Let each part of this proportionality be divided by the line segment* QR *and there will result unity as the centripetal force and* $SP^2 \times QT^2 / QR$ *conjointly, that is the centripetal force reciprocally as* $SP^2 \times QT^2 / QR$. *Which was to be proven* .

Or, $1/F \mu (SP^2 \times QT^2)/(QR)$. Thus, "the centripetal force would be reciprocally as the solid $SP^2 \times QT^2 / QR$." [Divide $QR = k_3(F)(SP^2 \times QT^2)$ by *QR* , and one has $1 = k_3(F)(SP^2) \times QT^2)/QR = k_3(F)/(QR)/(SP^2 \times QT^2)$, or $1/F = k_3(SP^2 \times QT^2)/(QR)$.]

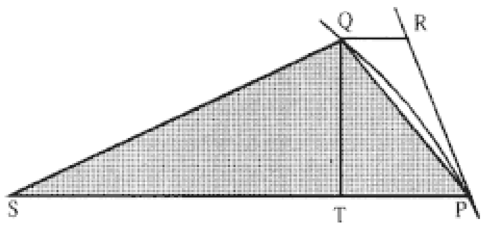

Figure 4.14
The shaded triangular area is equal to one-half of the product of the base *SP* and the height *QT* . As the point *Q* approaches the point *P* , the arc *QP* approaches the chord *QP* .

Corollary. *Hence if any figure is given, and on it a point toward which the centripetal force is directed* , [*then*] *it is possible for a law of centripetal force to be found which will make a body orbit on the perimeter of that figure* .

The result above represents a general theorem to be employed in obtaining solutions to direct problems: Given the orbit of the particle and given the center of force relative to that path, one seeks to find the law of force that will maintain the given orbit in terms of the distance *SP* from the particle *P* to the force center *S* .

[Corollary] *Specifically, the solid* $SP^2 \times QT^2 / QR$ *reciprocally proportional to this force must be computed. We shall give examples of this point in the following problems* .

Specifically, from the geometry of the given orbit and force center, one must express the discriminate ratio QR / QT^2 in terms of *SP* and constants of the orbits. The force will then be expressed in terms of the distance *SP* alone.

Conclusion

In what follows in the next chapter, Newton gives detailed solutions for three direct problems: Problem 1, find the force that generates a circular orbit with the center of force on the circumference of the circle; Problem 2, find the force that generates an elliptical orbit with the center of force at the center of the ellipse; and finally, Problem 3, find the force that generates a planetary elliptical orbit with the force center (the sun) at a focus of the ellipse. Theorem 3 provides the fundamental paradigm for solving direct problems in which one is given an orbit and a center of force fixed relative to that orbit. The theorem enables one to express the discriminate ratio QR / QT^2 in terms of *SP* and constants of the orbits and thus to determine the nature of the force required to maintain the given orbit about the given center of force. In Theorem 3, Newton fashioned the ratio that measures the force $QR / SP^2 \times QT^2$ from three elements: (1) the parabolic approximation: in the limit as the point *Q* approaches the point *P* , then the force is approximately constant in magnitude and direction; (2) Galileo's relationship (in Hypothesis 4): the displacement is proportional to the square of the time for a constant force; and (3) Kepler's relationship (in Theorem 1): the area swept out is proportional to the time for a centripetal force. The result is simplicity itself. From items (1) and (2) the force *F* is proportional to the deviation *QR* and inversely proportional to the square of the time *t* . From item (3) the time *t* is proportional to the area $SP \times QT$. Thus, the force *F* is proportional to the linear dynamics ratio $QR / SP^2 \times QT^2$. In the following chapter, this basic result is applied to three specific direct problems, of which the final one is the distinguished Kepler problem of planetary motion.

Four— The Paradigm Constructed:On Motion , Theorems 1, 2, and 3

Five—
The Paradigm Applied:
On Motion , Problems 1, 2, and 3

Theorem 3 in Newton's tract *On Motion* provides the basic paradigm for solutions to direct problems: Given the orbit and the location of the force center, find the force. As Newton put it, "specifically the solid $SP^2 \times QT^2 / QR$ must be computed." He concluded Theorem 3 with the statement, "We shall give examples of this point in the following problems." In this tract, he elected to solve three examples of direct problems. The most important example was Problem 3, the Kepler problem: find the centripetal force required to maintain planetary elliptical motion about a center of force located at the focus of the ellipse. Kepler had demonstrated that a planet *P* moves in an elliptical orbit about the sun *S* located at a focal point of the ellipse, and in Theorem 3 Newton demonstrated that the nature of the force required to maintain that motion is inversely proportional to the square of the distance *SP* , which is the mathematical statement of the law of universal gravitation. Problem 3 clearly has important physical significance. Preliminary to the solution of that very important problem, however, Newton presented the solutions to two other direct problems: find the force that generates (1) circular motion with a center of force on the circumference of the circle and (2) elliptical motion with a center of force at the center of the ellipse (see fig. 5.1). Problems 1 and 2, however, have no clear physical significance, that is, they have no direct application to physical phenomena such as Problem 3 does to the motion of planets.[1] They appear as relatively simple preliminary mathematical exercises in the application of Theorem 3, and they serve only to prepare the reader for the more complex solution that follows them in Problem 3. In an attempt to make clear the general method employed by Newton in applying Theorem 3 to these specific examples, I repeat the suggestion I

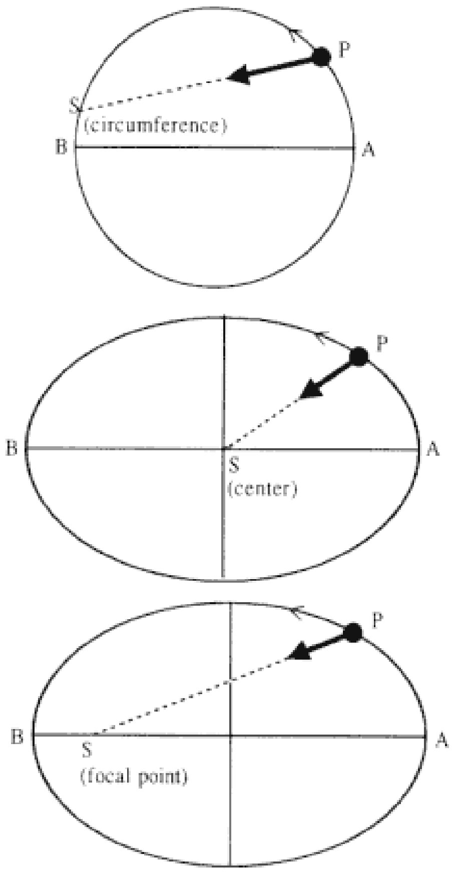

Figure 5.1
In the three direct problems analyzed in *On Motion* ,
a body *P* moves on a given path *APB* about a
given center of force *S* : (1) circular
path/circumference, (2) elliptical path/center, and
(3) elliptical path/focus.

made in chapter 2 that Newton's exemplar solutions for direct problems be distilled into the following general pattern of analysis:

Step 1. The Diagram . A drawing is provided that identifies the specific orbit corresponding to the general orbit *QPA* in Theorem 3. The immediate position *P* of the body is located, and the line of force *SP* is constructed that connects the body *P* with the force center *S* . Then the future position *Q* of the body is located, and the two lines *QR* and *QT* are constructed. Thus, all three elements of the linear dynamics ratio $QR / QT^2 \times SP^2$ are identified in the diagram.

Step 2. The Analysis . Given the full diagram, Newton begins the search for the geometric relationships that will reduce the discriminate ratio QR / QT^2 to a form in which it is expressed as a function of the distance *SP* alone. It is in this search that Newton displays his command of geometry, conic sections, and mathematical insight; and it is here that the reader must be careful not to lose sight of the general structure of the dynamics in the flurry of mathematical details.

Step 3. The Limit . The general theorem holds only in the limit as the future point *Q* approaches the immediate point *P* . Thus, Newton need not search for exact geometric relationships, but only for those that will reduce to the desired functional form in that limit as the point *Q* approaches the point *P* . Such relationships will eventually be sought out by others employing the methods of the calculus, but here Newton employed his unique geometric/limiting technique that serves in its stead.

I divide the following detailed discussion of each of the three problems into the three steps just outlined. No such explicit formal division exists in Newton's tract, but I offer it here as a guide for the first-time reader.

Problem 1—
A Circular Orbit with the Center of Force on the Circumference of the Circle

As the first example of the application of the paradigm of Theorem 3 to the solution of direct problems, Newton determined the nature of the force required to maintain a circular orbit with the center of force on the circumference of the circle. This problem had no obvious physical application, but it is the simplest of the three examples he presented and thus serves the reader in understanding the more difficult examples to follow.

> Problem 1. *A body orbits on the circumference of a circle; there is required the law of centripetal force being directed to some point on the circumference* .

Step 1— The Diagram

Figure 5.2 is based on the diagram that accompanies this problem in Newton's manuscript. The dynamic elements of Theorem 3 are evident in this figure: the projected tangential displacement *PR* , the actual circular arc *PQ* , and the deviation *QR* . The center of force *S* is located on the circumference of the circular arc *SQPA* . The chord *QL* and the normal *QT* are also elements in the analysis of the force.

> [1-A] *Let* SQPA *be the circumference of a circle* , S *the center of centripetal force* , P *a body carried on the circumference, and* Q *a nearby position into which it will be moved* .

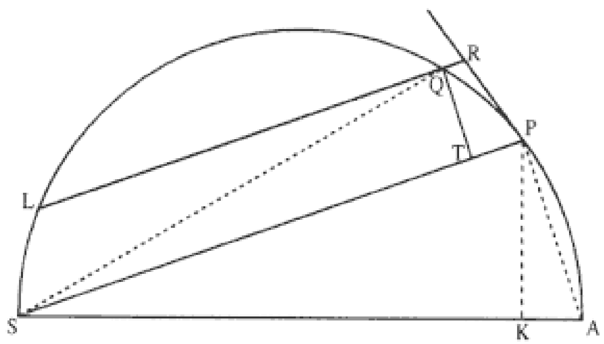

Figure 5.2
Based on Newton's diagram for Problem 1. A body *P* moves in a circular orbit *APQLS* about a center of force *S* located on the circumference of the circle.

The general curve *QPA* from Theorem 3 appears here as a portion of the specific circular path *SQPA* of Problem 1.

> [1-B] *To the diameter* SA *and to* SP *drop the perpendiculars* PK *and* QT,

The line *QT* , which is constructed perpendicular to the line *SP* , is one of the two elements in the discriminate ratio QR / QT^2 from Theorem 3, which is needed to express the force law in terms of the radius *SP* .

> [1-C] *and through* Q *draw* LR *parallel to* SP, *reaching the circle at* L *and the tangent* PR *at* R.

The second element of the determinate ratio *QR* and the chord of the circle *QL* are defined.

Step 2— The Analysis

In what follows, Newton expressed the discriminate ratio QR / QT^2 in terms of the radius *SP* and the given constant diameter of the circle *SA* .

1. He first demonstrated from similar triangles that $SA^2 / SP^2 = RP^2 / QT^2$.
2. Then he called upon a proposition from Euclid to show that $RP^2 = QR \times LR$.
3. Finally, he argued that the line *LR* can be replaced by the line *SP* in the limit as point *Q* approaches point *P* .

Thus, SA^2 / SP^2 = RP^2 / QT^2 = (QR × LR) / QT^2 = (QR × SP) / QT^2, which can be solved for the discriminate ratio (i.e., QR / QT^2 = SA^2 / SP^3). When that is done, the linear dynamics ratio QR / (QT^2 × SP^2), and hence the force F, can be expressed in terms of SP.

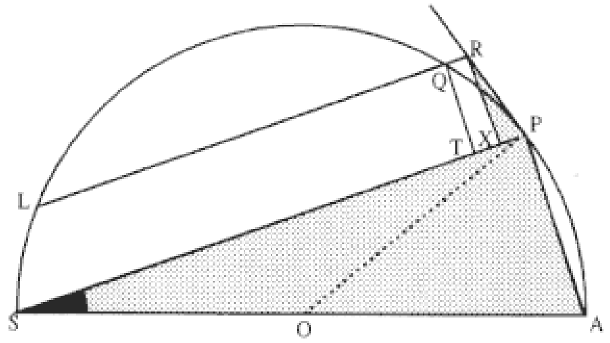

Figure 5.3A
A revised diagram for Problem 1. The perpendicular *RX* and the radius *OP* are added.

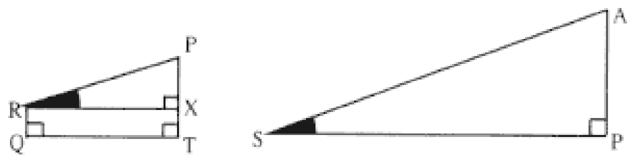

Figure 5.3B
The triangle *RPX* is similar to the triangle *SAP*.

1. From Theorem 3, $F\mu QR$ / (QT^2 × SP^2) = (QR / QT^2) (1 / SP^2).

2. Substituting for QR / QT^2 from above, $F\mu$ (SA^2 / SP^3) (1 / SP^2) = SA^2 / SP^5.

3. Because *SA* is the given constant diameter of the circle, the force is inversely proportional to the fifth power of the radius, that is, $F\mu$ 1/SP^5.

Thus, the proportional dependence of the force *F* on *SP* is known and so the solution to Problem 1 is given (i.e., $F\mu$ 1/SP^5). What follows is Newton's detailed analysis for this problem.

[2-D] *There will be* RP^2 (*that is*, QR × LR) *to* QT^2 *as* SA^2 *to* SP^2.

First, one must demonstrate that RP^2 / QT^2 = SA^2 / SP^2. See figure 5.3A for a revised diagram[2] that defines another perpendicular to *SP* at a point *X*. From the similar triangles *RXP* and *SPA* (see above) one has *RP* / *RX* = *SA* /*SP*, and from the parallelogram *RPXQ* one has *RX* = *QT*. Thus, *RP* /*QT* = *SA* / *SP*. Newton expresses the ratio RP^2 / QT^2 = SA^2 / SP^2 as squares because QT^2 is needed in the discriminate ratio, QR / QT^2.

Similarity: (1) Angle *OPR* = 90° because *OP* is a radius of the circle and *PR* is the tangent to the circle. Angle *PXR* = 90° by construction. Thus,

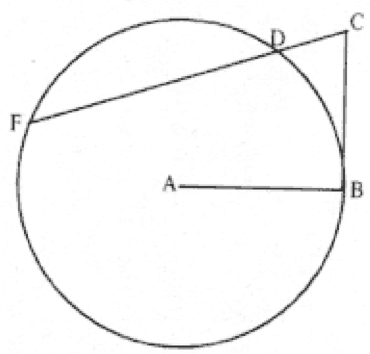

Figure 5.4A
From Proposition 36 of Book 3 of
Euclid's *Elements* : $CF / CB = /CB / CD$.

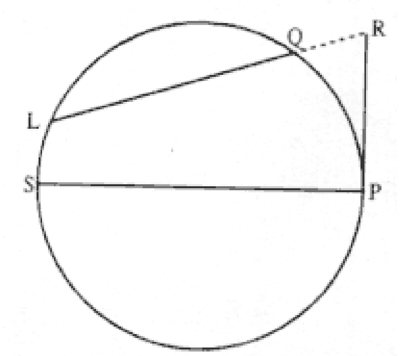

Figure 5.4B
Thus, $RL / RP = RP / QR$ or
$RP^2 = QR \times RL$ as required in Problem 1.

angle *SPR* is the complement of both angles *OPS* and *PRX* and thus angles *OPS* and *PRX* are equal.

(2) Angle *PSO* = angle *OPS* because *OS* and *OP* are both radii of the circle. Thus, angle *PSO* = angle *PRX* .

(3) Angle *APS* = 90° because *PS* and *PA* are chords of the circle terminating on the diameter *SA* . Thus, triangles *RXP* and *SPA* have all angles equal and are thus similar. Note in figure 5.3B that line *RX* equals line *QT* and thus the ratio $RP /RX = RP /QT = SA /SP$ as required above.

Second, one must demonstrate that $RP^2 = QR \times LR$, a relationship that introduces the deviation *QR* into the analysis. Newton obtained this result from Proposition 36 of Book 3 of Euclid's *Elements* to relate the following lines (see fig. 5.4A): *CB* (the tangent), *CD* (the deviation), and *CF* (the chord plus the deviation). Euclid's basic relationship is given as $CF /CB = CB /CD$, or $BC^2 = CD \times CF$. Figure 5.4B extends Euclid's relationship to the diagram for Problem 1, where $CB = RP$, $CD = QR$, and $CF = LR$. Thus, $QR / RP = RP / LR$, or $RP^2 = QR \times LR$, as required above.

[2-E] *Therefore* $(QR \times LR) \times (SP^2 / SA^2) = QT^2$.

The first result from [2-D] gives $RP^2 / QT^2 = SA^2 / SP^2$ or $QT^2 = (SP^2 / SA^2) RP^2$. The second result from [2-D] gives $RP^2 = (QR \times LR)$. Substituting the second value for RP^2 into the first expression gives the desired result (i.e., $QT^2 = (QR \times LR) (SP^2 / SA^2)$). Thus, Newton has obtained an expression relating the two elements QT^2 and *QR* of the discriminate ratio QR /QT^2 , which is required in the linear dynamics ratio $QR /(QT^2 \times SP^2)$ to measure the force.

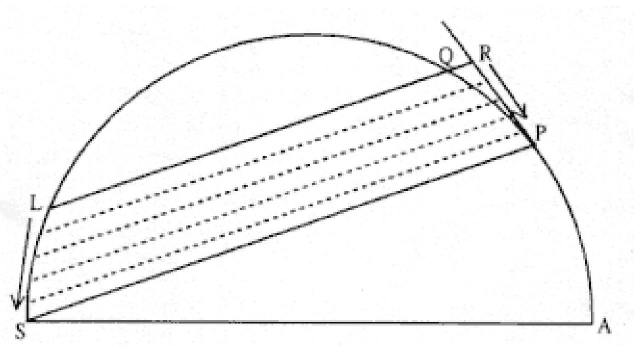

Figure 5.5
As the point *Q* approaches the point *P* , the line segment *LR* approaches the line segment *SP* .

Step 3— The Limit

[3-F] *Multiply these equals by* SP^2 / QR, *and, with the points* P *and* Q *coalescing, let* SP *be written in place of* LR. *Thus* , $SP^5 / SA^2 = QT^2 \times SP^2 / QR$.

From [2-E], $QT^2 = (QR \times LR)(SP^2 / SA^2)$. The linear dynamics ratio can be obtained if both sides of the expression above are multiplied by SP^2 / QR (i.e., $QT^2 \times SP^2 / QR = (QR \times LR)(SP^2 / SA^2)(SP^2 / QR)$) which, upon canceling *QR* and combining powers of *SP* , reduces to $QT^2 \times SP^2 / QR = LR \times SP^4 / SA^2$. In the limit, "with the points *P* and *Q* coalescing," *Q* ®*P* , then *LR* ®*SP* (see fig. 5.5). When *SP* is substituted for *LR* , the expression above reduces to $QT^2 \times SP^2 / QR = LR \times SP^4 / SA^2$® SP^5 / SA^2 , or $SP^5 / SA^2 = QT^2 \times SP^2 / QR$ as required above.

Conclusion

[3-G] *Therefore the centripetal force is reciprocally as* SP^5 / SA^2 , *that is* (*because* SA^2 *is given*), *as the fifth power of the distance* SP. *Which was to be proven* .

Or, *Fμ* $1/SP^5$. The centripetal force *F* is given by the linear dynamics ratio $QR / (SP^2 \times QT^2)$, which from [3-F] is equal to (SA^2 / SP^5). Since *SA* is the given constant diameter of the circle, the force *F* is inversely proportional to the fifth power of the distance *SP* and the solution to Problem 1 is given (i.e., *F μ* $1/SP^5$).

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In this case and in other similar cases, it must be understood that after the body reaches the center S, *it will no longer return to its orbit, but it will depart along the tangent.*

Recall that the force center was located on the circumference of the circle and therefore the body will eventually pass through the force center. In such a singular situation both the force and speed would increase without limit and, as Newton states, the body would shoot off along the tangent. The statement is not carried through to the 1687 edition of the *Principia* .

In a spiral which cuts all the radii at a given angle, the centripetal force being directed to the beginning of the spiral is reciprocally in the tripled ratio of the distance, but at that beginning no straight line in a fixed position touches the spiral.

A full solution to the problem of the equiangular spiral appears as Proposition 9 in the 1687 edition of the *Principia* with the force center at the pole (or "beginning") of the spiral. At the pole, no tangent can be defined, but elsewhere the force is proportional to the inverse cube of the distance *SP* .

Newton has thus demonstrated that the centripetal force required to maintain a circular orbit with a center of force located on the circumference of that circle is inversely proportional to the fifth power of the distance *SP* . As has been previously noted, such an orbit with such a center of force does not serve any particular physical situation. The example is intended purely as a demonstration of how the general paradigm in Theorem 3 can be applied to a very simple problem. The analysis requires only one set of similar triangles and a single reference to a proposition from Euclid. (In contrast, the analysis for the Kepler problem in Problem 3 has some sixteen sub-steps.) The particular example of Problem 1 is carried forward to the 1687 edition of the *Principia* in much the same form as given in the tract *On Motion* .

Problem 2—
An Elliptical Orbit with the Center of Force at the Center of the Ellipse

As the second example of the application of the paradigm of Theorem 3 to direct problems, Newton determined the nature of the centripetal force required to maintain an elliptical orbit with the force center located at the *center* of the ellipse. Figure 5.6 is based on the diagram that Newton provided for this problem. As in the previous problem, the dynamic elements of Theorem 3 can be identified: the projected tangential path *PR* , the elliptical arc *PQ* , and the deviation *QR* , here constructed correctly as being parallel to the line of force *PC* . I break Newton's statement of the problem and its solution down into three steps as outlined earlier and provide a line-by-line commentary. I have not given a discussion of Problem 2 as detailed as that in the preceding problem or as detailed as that I shall give in the distinguished Kepler problem (Problem 3). Readers may

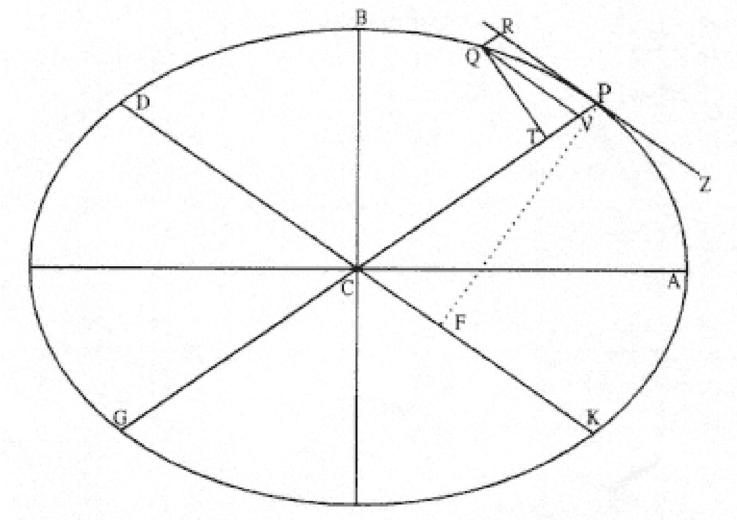

Figure 5.6
Based on Newton's diagram for Problem 2. A body *P* moves on an elliptical orbit *APQB* about a center of force *C* located at the center of the ellipse.

want simply to follow the flow of the solution to Problem 2 and reserve their energy for the solution to Problem 3.

> Problem 2. *A body orbits on a classical ellipse; there is required the law of centripetal force being directed to the center of the ellipse* .

Step 1— The Diagram

> [Problem 2] *Let* CA *and* CB *be the semi-axes of the ellipse;* GP *and* DK *conjugate diameters;* PF *and* QT *perpendiculars to these diameters;* QV *ordinate to the diameter* GP; *and* QVPR *a parallelogram* .

> [1-A] *Let* CA *and* CB *be the semi-axes of the ellipse;*

The semi-major axis *CA* and the semi-minor axis *CB* are constructed perpendicular to each other (see fig. 5.6). Note also that the conjugate diameters *GP* and *DK* (defined next) are not, in general, mutually perpendicular.

> [1-B] GP *and* DK *conjugate diameters;*

The transverse diameter *GP* is constructed from the point *P* through the center *C* of the ellipse to the point *G* opposite point *P* (see fig. 5.6). The conjugate diameter *DK* is constructed parallel to the tangent *PR* , and it passes through the center *C* . The diameters *GP* and *DK* are said to be a

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conjugal (united) pair. Many properties of the ellipse can be expressed in terms of these conjugate diameters.

> [1-C] PF *and* QT *perpendiculars to these diameters;*

PF is normal to *DK* and will be used in the calculation of the circumscribed area to the ellipse given in Lemma 1. The line *QT* is constructed perpendicular to the diameter *GP* and will be used in the discriminate ratio QR / QT^2 (see fig. 5.7).

> [1-D] QV *ordinate to the diameter* GP;

QV is constructed parallel to the tangent *PR* and hence is parallel to the conjugate diameter *DK* and furthermore is said to be ordinate to the transverse diameter *GP* (see fig. 5.7).

[1-E] *and* QVPR *a parallelogram* .

The deviation *QR* must be constructed parallel to the line of force *PC* , and is thus parallel to the conjugate diameter *PV* . The line segment *QV* is parallel to the tangent *RP* from [1-D]. Thus, *QVPR* is a parallelogram (see fig. 5.7).

Step 2— The Analysis

In the analysis to follow, Newton expresses the discriminate ratio QR / QT^2 in terms of the given constant circumscribed area of the ellipse ($BC \times AC$) and the radius *PC* . Note that the particular radius *PC* from Problem 2 is equal to the general radius *SP* from Theorem 3. When the discriminate ratio was determined in terms of *PC* , then the linear dynamics ratio was also expressed in terms of *PC* . Thus, the proportional dependence of the force *F* on *PC* was known, and so the direct problem was solved ($F \mu QR / (QT^2 \times PC^2)$).

[2-A] *After these have been constructed, there will be* [*from the* Conics] $PV \times VG$ *to* QV^2 *as* PC^2 *to* CD^2 *and* $QV^2 / QT^2 = PC^2 / PF^2$, *and on combining these proportions* $PV \times VG / QT^2 = PC^2 / (CD^2 \times PF^2)/PC^2$.

Find QT^2 : An expression for the line *QT* is obtained from the similarity of triangles *QTV* and *PFC* (i.e., $QT / QV = PF / PC$ or $QT^2 = QV^2 (PF^2 / PC^2)$). Then, a relationship from Apollonius (i.e., $PV \times VG / QV^2 = PC^2 / CD^2$ or $QV^2 = (PV \times VG) (CD^2 / PC^2)$) is used to eliminate QV^2 from the expression for QT^2 (i.e., $QT^2 = (PV \times VG) (PC^2 / CD^2) (PC^2 / PF^2)$).

[2-B] *Write* QR *in place of* PV *and* $BC \times CA$ *in place of* $CD \times PF$,

Find *QR* : The relationship $PV = QR$ (from the parallelogram *QRPV*) is used to introduce the deviation *QR* into the expression for QT^2 and to obtain

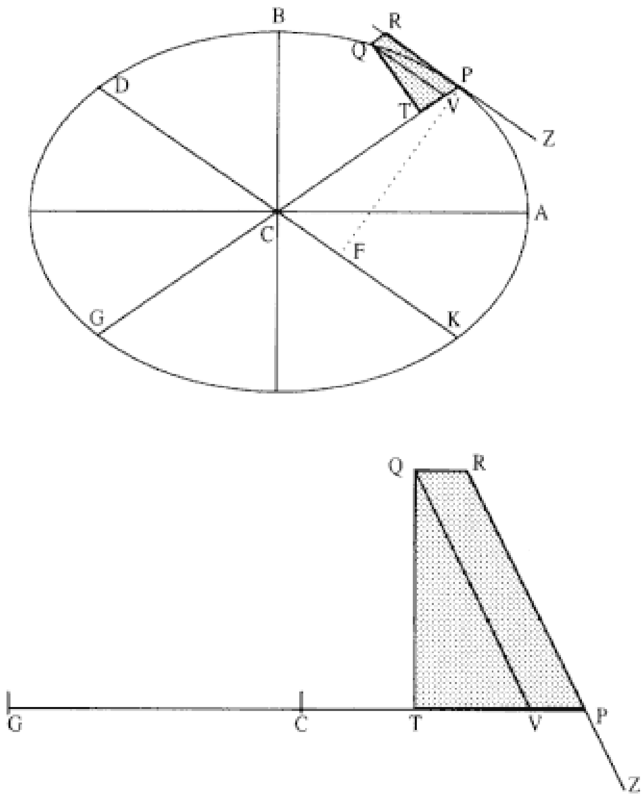

Figure 5.7
The bottom figure is abstracted from the full figure above. The line *QT* is constructed perpendicular to the conjugate diameter *GP* , and the line *QV* is constructed parallel to the tangent *ZPR* .

the following expression: $QR / QT^2 = PC^4 / (VG)(CD \times PF)^2$, where $(CD \times PF)$ is a constant equal to area $(CA \times CB)$.

Step 3—
The Limit

> [3-A] *and in addition* (*with the points* P *and* Q *coalescing*) 2PC *in place of* VG, *and, when the ends and middles are multiplied into each other, there will result* $QT^2 \times PC^2 / QR = 2BC^2 \times CA^2 / PC$.

Find $QR / (QT^2 \times PC^2)$: Finally, when the point *Q* approached the point *P* , then the line *VG* approached the value 2*PC* , and the expression for the discriminate ratio becomes $QR / QT^2 = PC^3 / 2(CA \times CB)^2$.

> [3-B] *The centripetal force is therefore reciprocally as* $2BC^2 \times CA^2 / PC$, *that is* (*because* $2BC^2 \times CA^2$ *is given*), *as* 1/PC, *that is, directly as the distance* PC. *Which was to be found* .

Dividing both sides by PC^2 gives the linear dynamics ratio $QR / (QT^2 \times PC^2) = PC / 2(CA \times CB)^2$, which, because the area $(CA \times CB)$ is a constant, gives the ratio as directly proportional to *PC* .

Conclusion

From Theorem 3, $F \mu \, QR / (QT^2 \times PC^2) \, \mu PC$, or the centripetal force *F* required to maintain elliptical motion about a center of force located at the center *C* of the ellipse is directly proportional to the distance *PC* , which is the solution for Problem 2. Newton has thus demonstrated that the centripetal force required to maintain an elliptical orbit about a center of force located at the center of the ellipse is directly proportional to the first power of the distance *PC* . As noted earlier, such an orbit with such a center of force did not represent any particular physical situation. Newton intended this example purely as a demonstration of how the general paradigm in Theorem 3 was to be applied to yet another direct problem. The analysis for Problem 2 is more complicated than that for Problem 1, but it is relatively simpler than that for Problem 3. The particular example of Problem 2 is carried forward to the 1687 edition of the *Principia* in much the same form as given above for the tract *On Motion* .

Problem 3—
An Elliptical Orbit with the Center of Force at a Focus of the Ellipse

The challenge set forth in Problem 3 is a distinguished one: find the solution to the "Problem of the Planets." Kepler had demonstrated in 1609 that Mars moved in an elliptical orbit with the sun at a focus. The question that remained to be answered, however, was the mathematical nature of the force required to maintain that motion. In *On Motion* , the solution follows the two preliminary examples without any fanfare. In the *Principia* , however, Newton called attention to "the dignity of the problem and its use in what follows" by separating Problem 3 from the preceding direct problems and giving it a place of honor at the beginning of a new section. It is the keystone of both works in terms of the dignity of the problem and in the degree of mathematical difficulty.

As in all the previous problems, Newton's diagram identifies the dynamic elements to be employed in the demonstration (see fig. 5.8 in the following statement of the problem): the projected tangential path *PR* , the elliptical arc *PQ* , and the deviation *QR* from the point *R* on the tangential path to the point *Q* on the elliptical arc. Note in this diagram that

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the deviation *QR* is constructed parallel to the line of force *SP* , which is directed toward the focus *S* , and that the line segment *QT* is constructed perpendicular to *SP* . The challenge for this problem is, as it was for the two previous problems, to express the discriminate ratio of the force QR / QT^2 in terms of the radius *SP* and / or constants of the orbit. The final expression of the solution appears deceptively simple: the discriminate ratio QR / QT^2 is found to be proportional to the reciprocal of the constant *latus rectum L* of the ellipse (where $L = 2BC / AC$), and therefore the force is proportional to the inverse square of the distance, $1/SP^2$.

$$F \propto QR / (QT^2 \times SP^2) \propto 1 / (L \times SP^2) \propto 1 / SP^2$$

where *L* is a constant of the ellipse. Thus, once it is demonstrated that the discriminate ratio *QR* / QT^2 is a constant for the elliptical/focal motion, then the problem is solved. That initial demonstration, however, requires a number of steps, and the reader must remember that the goal is to demonstrate that the discriminate ratio *QR* / QT^2 is a constant. Once that goal has been achieved, it is a simple matter to determine from Theorem 3 that the force is inversely proportional to the inverse square of the distance *SP* .

I have divided Step 2, the analysis, into some sixteen sub-steps between line [2-B] and line [2-Q]. The journey is not for the faint of heart; but then if it had not been challenging there would have been no reward for Newton. In honor of the dignity of the problem, I give the statement of the entire proposition and follow it with a line-by-line analysis. I suggest reading the full statement to get an overview of the problem without attempting to justify each point and then following the details of the proof in the line-by-line analysis.

Problem 3. *A body orbits on an ellipse; there is required the law of centripetal force directed to a focus of the ellipse* .

Let S *be a focus of the ellipse above. Draw* SP *cutting the diameter of the ellipse* DK *at* E. *It is clear that* EP *is equal to the semi-major axis* AC, *seeing that, when from the other focus* H *of the ellipse the line* HI *is drawn parallel to* CE, *because* CS *and* CH *are equal* , ES *and* EI *are equal, and hence* EP *becomes half the sum of* PS *and* PI, *that is, of* PS *and* PH *which are conjointly equal to the total axis* 2AC. [See fig. 5.8.]

Let drop the perpendicular QT *to* SP, *and, after calling the principal latus rectum* (*or* $2BC^2$ / AC) *of the ellipse* L, *there will be*

L × QR *to* L × PV *as* QR *to* PV, *that is, as* PE (*or* AC) *to* PC;
and L × PV *to* GV × VP *as* L *to* GV;
and GV × VP *to* QV^2 *as* CP^2 *to* CD^2 ;
and QV^2 *to* QX^2 *as, say* , M *to* N;
and QX^2 *is to* QT^2 *as* EP^2 *to* PF^2 , *that is, as* CA^2 *to* PF^2 , *or as* CD^2 *to* CB^2 .

And when all these ratios are combined ,

L × QR / QT^2 *will be equal to* (AC / PC) × (L / GV) × (CP^2 / CD^2) × (CD^2 / CB^2),

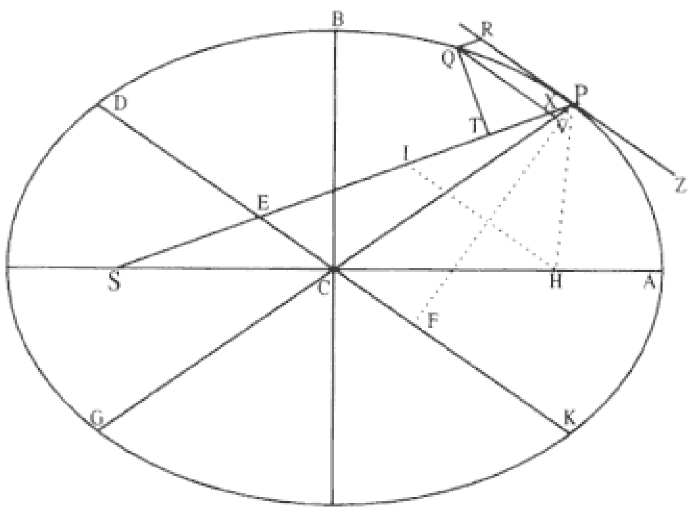

Figure 5.8
Based on Newton's diagram for Problem 3. A body *P* moves in an elliptical orbit *APQB* about a center of force *S* located at a focus of the ellipse.

that is, as AC × L (*or* $2BC^2$) / (PC × GV) × (CP^2 / CB^2) × (M / N) *or as* (2PC / GV) × (M / N).

But with the points P *and* Q *coalescing, the ratios* 2PC / GV *and* M / N *become equal* [*and approach unity*], *and therefore the combined ratio of these* L × QR / QT^2 [*equals unity (i.e.,* L = QT^2 / QR)]. *Multiply each part by* SP^2 / QR *and there will result* L × SP^2 = SP^2 × QT^2 / QR.

Therefore the centripetal force is reciprocally as $L \times SP^2$ *, that is ,* [*reciprocally*] *in the doubled ratio of the distance. Which was to be proven .*

The concluding statement of Problem 3 sets forth the mathematical nature of the gravitational force; that is, it is reciprocally as the square of the distance *SP* between the sun and the planet. It is this result that provided the answer to the problem of the planets. I now give a detailed analysis of each portion of Newton's demonstration.

Step 1— The Diagram

[1-A] *Let* S *be a focus of the ellipse above. Draw* SP *cutting the diameter of the ellipse* DK *at* E.

The body is located at a general point *P* , and the center of force is located at a specific point *S* , the focus of the ellipse (see fig. 5.8). The diameter *DK* is one of the conjugate diameters to the point *P* , and it is drawn parallel to the tangent *PR* and through point *C* , the center of the ellipse. The

intersection of the diameter *DK* with the line of force *SP* defines the point *E* . Note that the diameter *PG* is not perpendicular to the diameter *DK* but that the line segment *PF* is constructed perpendicular to the diameter *DK* .[3]

Step 2— The Analysis ([2-B] to [2-Q])

In Problem 3, as in the solutions to the two previous direct problems, Newton expressed the discriminate ratio QR / QT^2 in terms of the radius *SP* and / or constants of the figure. When the discriminate ratio was determined in terms of *SP* , then the linear dynamics ratio $QR / QT^2 \times SP^2$ was also expressed in terms of *SP* . So, the proportional dependence of the force *F* , given by the linear dynamics ratio, was also known, and the direct Kepler problem was solved.

As an overview for the reader, I summarize Newton's determination of the discriminate ratio QR / QT^2 here in an effort to provide a general guide to the multiple details in the analysis.

Find *QR* : A relationship for the deviation *QR* is obtained from the similarity of triangles *PXV* and *PEC* , where $QR = PX$ by construction (i.e., $PE / PC = PX / PV = QR / PV$ or $QR = PV (PE / PC)$). Newton also demonstrated that the line *PE* (defined in fig. 5.8) is equal to the semi-major axis *AC* (a useful relationship for an ellipse that was not found in the standard works on conics). Thus, *QR* can be written as $PV (AC / PC)$. From the same proposition of Apollonius's used in the solution of Problem 2, Newton obtained an expression for *PV* (i.e., $PV \times VG / QV^2 = PC^2 / DC^2$ or $PV = (QV^2 / GV) (PC^2 / DC^2)$), and he used it to eliminate *PV* from the expression for *QR* (i.e., $QR = (QV^2 / GV) (PC^2 / DC^2) (AC / PC)$).

Find QT^2 : From the similarity of triangles *EPF* and *XQT* , he obtained an expression for the second element of the discriminate ratio QT^2 (i.e., $QT / QX = PF / PE$ or $QT^2 = QX^2 (PF^2 / AC^2)$), where as above $PE = AC$). From Euclid's relationship for circumscribed areas (also used in the solution of Problem 2) one has $PF / AC = BC / DC$, and thus $QT^2 = QX^2 (BC^2 / DC^2)$.

Find QR / QT^2 : The discriminate ratio QR / QT^2 can then be written from $QR = (QV^2 / GV) (PC^2 / DC^2) (AC / PC)$ and $QT^2 = QX^2 (BC^2 / DC^2)$ or $QR / QT^2 = (QV^2 / QX^2) (PC / GV) (AC / BC^2)$. Given the definition of the constant *latus rectum* $L = 2BC^2 / AC$, the discriminate ratio becomes $(QV^2 / QX^2) (PC / GV) (2 / L)$.

In the limit as the point *Q* approaches the point *P* , the line *QV* approaches the line *QX* , and the line *GV* approaches a value of $2PC$. Thus, the discriminate ratio QR / QT^2 approaches $1/L$, and the linear dynamics ratio *QR* /

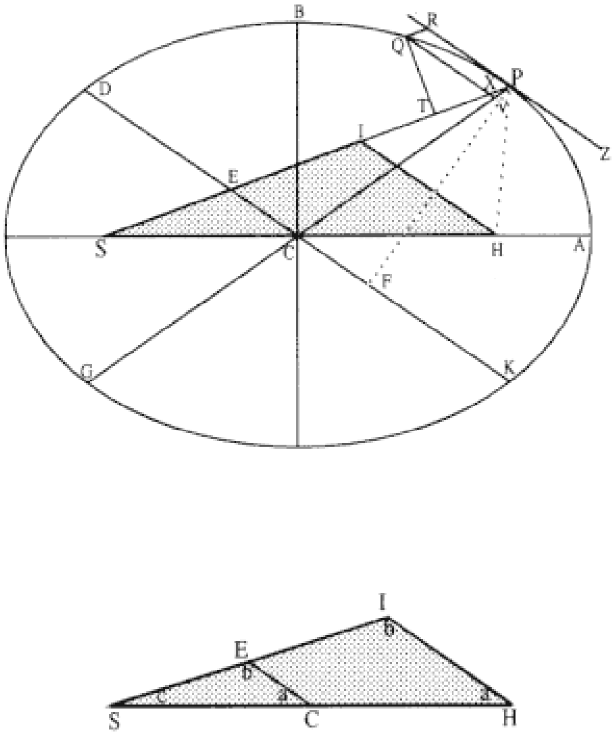

Figure 5.9
The triangle *SIH* below is abstracted from the shaded area of the ellipse above. Line *EC* is parallel to line *IH* and thus triangles *SEC* and *SIH* are similar.

$QT^2 \times SP^2$ approaches $1 / (L \times SP^2)$. The force, therefore, is inversely proportional to the square of the distance *SP* , as was to be demonstrated. What follows is a detailed analysis of Newton's demonstration of the solution.

[2-B] *It is clear that* EP *is equal to the semi-major axis* AC,

At least it will be "clear" that *EP* = *AC* after Newton's demonstration in the next three steps ([2-C] to [2-E]). This particular relationship for an ellipse, *EP* = *AC* , is not found in the standard works on conic sections and appears to be original with Newton.**[4]**

[2-C] *seeing that, when from the other focus* H *of the ellipse the line* HI *is drawn parallel to* CE, *because* CS *and* CH *are equal* , ES *and* EI *are equal* .

Or, *ES* = *EI* . The lines *CS* and *CH* are equal because they both locate a focus (*S* or *H*) relative to the center *C* . The equality of *ES* and *EI* follows from the similarity of triangles *SEC* and *SIH* and the equality of line segments *CS* and *CH* (see fig. 5.9).

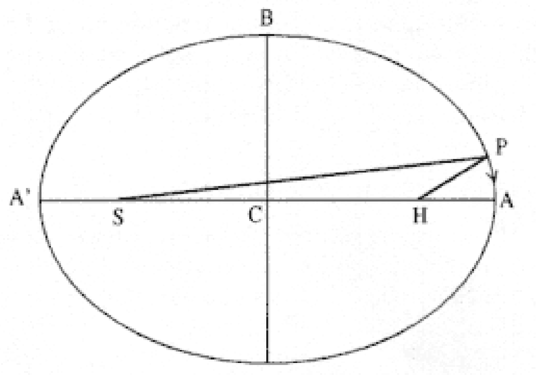

Figure 5.10
The sum (*SP* + *PH*) is a constant for any given ellipse,

and when the point P goes to A that sum is equal to
$2AC$.

Similarity: HI was constructed parallel to CE , and these two parallel lines are cut by the two transversals SCH and SEI , as demonstrated in figure 5.9 by the small diagram abstracted from the full diagram. Hence, all the angles *a, b* , and *c* in triangles *SEC* and *SIH* are equal and thus the triangles are similar.

Because the triangles are similar, C bisects SH , then E bisects SI , and thus $ES = EI$, as was to be demonstrated.

[2-D] *and hence* EP *becomes half the sum of* PS *and* PI,

Or, $EP = 1/2(PS + PI)$. From figure 5.9, $PS = EP + ES = EP + EI$ (because $ES = EI$ from [2-C]). Moreover, $PS + PI = (EP + EI) + PI = EP + (EI + PI) = 2EP$. Thus, $EP = 1/2(PS + PI)$.

[2-E] *that is, of* PS *and* PH *which are conjointly equal to the total axis* 2AC.

Or, $2EP = (PS + PI) = (PS + PH) = 2AC$. From the properties of conics (Apollonius, Proposition 48, Book 3)[5] the angles made by the tangent and focal lines (i.e., angles RPS and APH) are equal, and since the line IH is constructed parallel to the tangent RPZ , the line $PI = PH$. An ellipse is a curve such that the sum of the distances from two fixed points (the foci) to a general point is given (i.e., the sum $(PS + PH)$ is a constant). When the general point P is on the major axis A (see fig. 5.10) then the sum $(PS + PH)$ equals the sum $(AS + AH)$ equals $(2AC)$ because $AH = A'S$. Thus, $(PS + PH) = 2AC$. Therefore, from [2-D] $2EP = (PS + PI) = (PS + PH) = 2AC$, or "clearly" $EP = AC$, as was to be demonstrated in [2-B].

[2-F] *Let drop the perpendicular* QT *to* SP, *and, after calling the principal* latus rectum (*or* $2BC^2 / AC$) *of the ellipse* L,

---

In [2-F], Newton calls upon the Apollonian definition of the *latus rectum L* , which states that the *latus rectum L* is in the same ratio to the minor diameter $2BC$ as the minor diameter $2BC$ is to the major diameter $2AC$ (i.e., $L : 2BC :: 2BC : 2AC$ or $L / 2BC = 2BC / 2AC$) or $L = 2BC^2 / AC$ as given above.

[2-G] *there will be* L × QR *to* L × PV *as* QR *to* PV,

The deviation QR is introduced in the format $(L \times QR) / (L \times PV) = (QR) / (PV)$, which will be extended next. Given $(QR) / (PV)$, then simply multiply by (L / L), which can be written as $(L / L)(QR / PV) = (L \times QR) / (L \times PV) = (QR) / (PV)$.

[2-H] *that is, as* PE (*or* AC) *to* PC;

Or, $L(QR) / L(PV) = AC / PC$. The expression for the deviation QR is extended. Triangles PEC and PXV are similar because XV and EC are parallel lines (see fig. 5.11). Thus, from the ratio of sides of the similar triangles, $PX / PV = PE / PC$. Moreover, $PX = QR$ because $QRPX$ is a parallelogram, and $PE = AC$, as demonstrated in [2-E]. Therefore, the ratio of sides can be written as $QR / PV = AC / PC$. Thus, from this result and [2-H], one obtains $L(QR) / L(PV) = AC / PC$, as required.

[2-I] *and* L × PV *to* GV × VP *as* L *to* GV

The element PV is introduced in a similar format to that employed to introduce QR in [2-G]. Given L / GV , simply multiply by PV / PV , which can be written as $(PV / PV)(L / GV) = (L \times PV) / (GV \times VP) = L / GV$.

[2-J] *and* GV × VP *to* QV^2 *as* CP^2 *to* CD^2 ;

Or, $GV \times VP / QV^2 = PC^2 / DC^2$. This property of an ellipse will be used to eliminate PV from the expression for QR . The particular reference is to Proposition 15 of Book 1 of Apollonius's *Conics* , in which it is demonstrated that any chord parallel to a diameter of an ellipse is bisected by the conjugate diameter and, moreover, its square is equal to the product of the portions of the conjugate diameter. In figure 5.12, if the chord is given by QVQ' and the conjugate diameter by PVG , then $(PV \times VG) = (QV \times VQ')$, where $QV = VQ'$. The same relationship holds for the chord DK, that is ($DC \times$

CK) = ($BC \times CG$), where $DC = CK$ and $BC = CG$. Thus, the ratio $PV \times VG / QV^2$ about point V is equal to the ratio PC^2 / CD^2 about point C , as required above.

[2-K] *and* QV^2 *to* QX^2 *as, say* , M *to* N;

The ratio *M* / *N* is simply a definition of the ratio of *QV* / *QX* , a step that Newton eliminates in the 1687 edition of the *Principia* .

[2-L] *and* QX^2 *is to* QT^2 *as* EP^2 *to* PF^2 ,

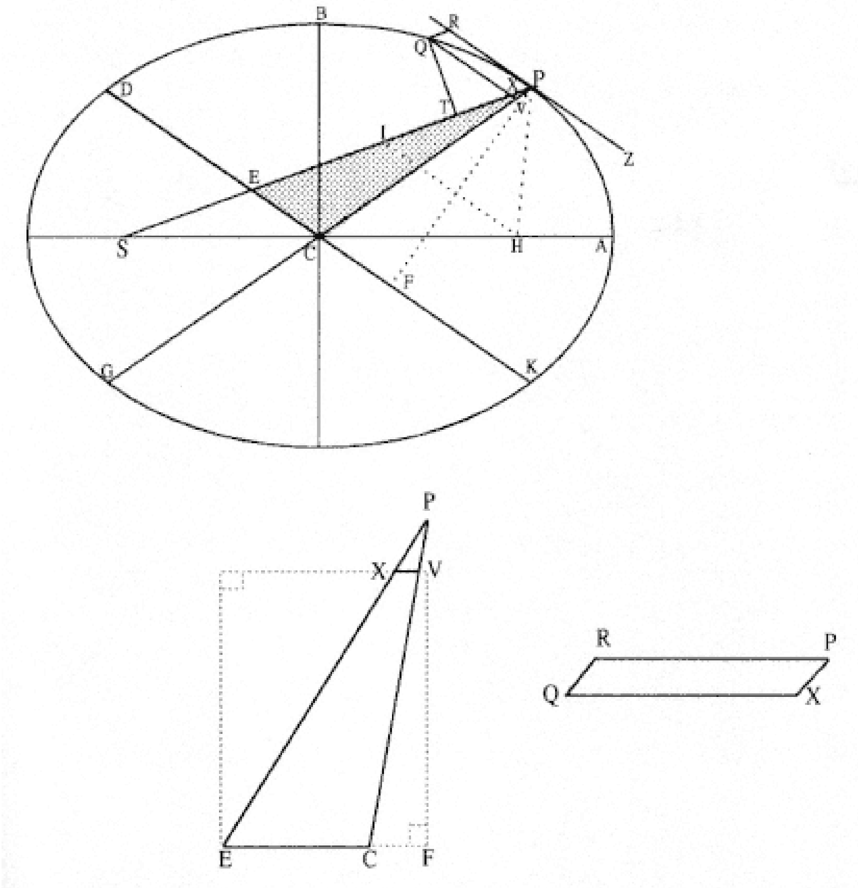

Figure 5.11
The shaded area *CEP* from the ellipse above is abstracted as the similar triangles *PXV* and *BEC* below. The parallelogram *QRPX* from the ellipse is shown enlarged below.

The second element of the discriminate ratio QT^2 is now introduced. Because triangle *EPF* is similar to triangle *XQT* (see fig. 5.13), the ratio of similar sides gives the ratio $QX / QT = EP / PF$.

Similarity: Lines *QX* and *EF* are parallel lines cut by the transversal *EP* , as demonstrated in the small drawing abstracted from the full drawing in figure 5.13. Thus, angles *PEF* and *QXT* are equal, as are the right angles *EFP* and *XQT* . Thus, all the angles of triangles *EPF* and *XQT* are equal and the triangles are similar.

Thus, $QX / QT = EP / PF$ and as given above, $QX^2 / QT^2 = EP^2 / PF^2$.

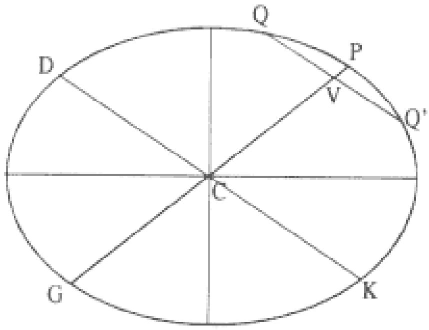

Figure 5.12
The diameter *PG* bisects the chords *QQ* ' and *DK* . From Proposition 15 of Book 1 of Apollonius's *Conics* , the ratio of $PV \times VG / QV^2$ is equal to the ratio PC^2 / DC^2 .

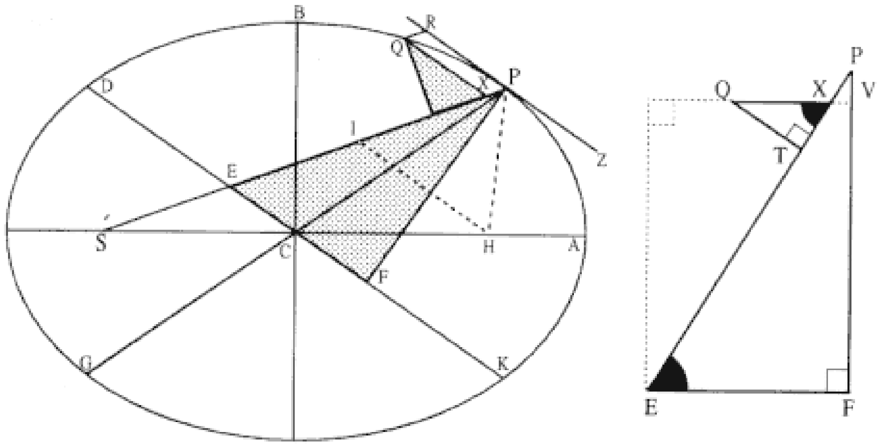

Figure 5.13
The shaded area in the ellipse is abstracted as the triangles *EPF* and *XQT* . Since the lines *QX* and *EF* are parallel, the triangles are similar.

[2-M] *that is, as* CA^2 *to* PF^2 , *or as* CD^2 *to* CB^2 . [*Finally, that is* , QX^2 *to* QT^2 *as* CD^2 *to* CB^2 .]

From [2-B] or [2-E], *EP* = *CA* , thus *EP* / *PF* = *CA* / *PF* . Also, from Lemma 1, area 4(*CA*) (*CB*) = area 4(*CD*) (*PF*) (see fig. 5.14), or *CA* / *PF* = *CD* / *CB* . Thus, from [2-L] *QX* / *QT* = *EP* / *PF* = *CA* / *PF* = *CD* /*CB* , which can be written in the square as in line [2-L] (i.e., $QX^2 / QT^2 = CD^2 / CB^2$).

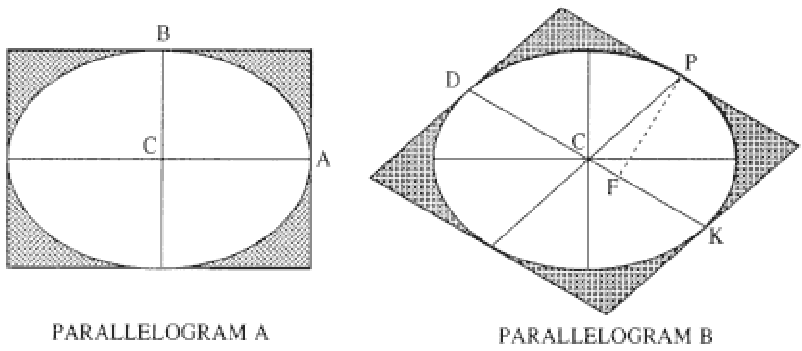

Figure 5.14
The area of parallelogram A (= 4*BC* × *CA*) is equal to the area of parallelogram B (= 4*CD* × *PF*).

[2-N] *And when all these ratios are combined* , $L \times QR / QT^2$

Multiply together the five preceding relationships (i.e., lines [2-H] to [2-L]) and consider the products of the left- and right-hand sides separately. The product of the left-hand side of [H] [I] [J] [K] [L] is equal to the product ($L \times QR$) ($L \times PV$) ($GV \times PV$) (QV^2) (QX^2) divided by the product ($L \times PV$) ($GV \times PV$) (QV^2) (QX^2) (QT^2). By simple cancellation of equals, this ratio of products reduces to ($L \times QR$) / QT^2 .

[2-O] *will be equal to* (AC / PC) × (L / GV) × (CP^2 / CD^2) × (M / N) × (CD^2 / CB^2),

The product of the right-hand side of [H] [I] [J] [K] [L] equals the product (AC) (L) (PC^2) (M) (CD^2) divided by the product (PC) (GV) (CD^2) (N) (CB^2), which is equal to (AC /PC) × (L /GV) × (CP^2 / CD^2) × (M /N) × (CD^2 / CB^2).

[2-P] *that is, as* AC × L (*or* $2BC^2$) / (PC × GV) × (CP^2 / CB^2) × (M / N)

That is, by simple cancellation of CD^2 in the ratio given in [2-O], one has (AC) (L) (PC^2) (M) divided by (PC) (GV) (N) (CB^2). Also, from the definition of the *latus rectum* L (see [2-F]), $AC \times L = 2BC^2$. Substituting $2BC^2$ for $AC \times L$, the ratio can be further reduced to ($2BC^2$) (CP^2) (M) divided by (PC) (GV) (N) (CB^2). This ratio can be rearranged into the ratio given above (i.e., [($2BC^2$) / ($PC \times GV$)] × [(CP^2 / CB^2) × (M /N)]).

[2-Q] *or as* (2PC / GV) × (M / N) *or* [(L × QR) *to* QT^2]

Upon canceling BC^2 and PC , [2-P] reduces to (2PC / GV) (M /N), as in [2-Q]. Thus, the left-hand side [2-N] equals the right-hand side [2-Q], or ($L \times QR$) / QT^2 = (2PC / GV) (M / N).

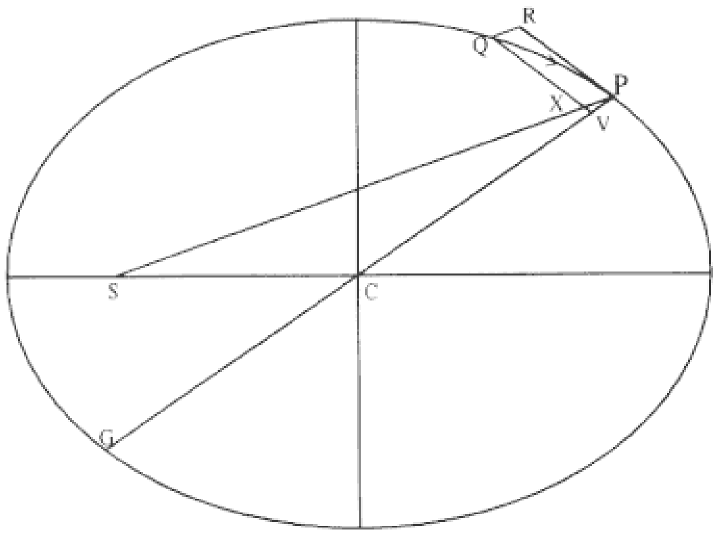

Figure 5.15
As the point Q approaches the point P , the line QV approaches the line QX and the line GV approaches the line GP , which is equal to 2PC .

Step 3— The Limit

[3-R] *But with the points* P *and* Q *coalescing, the ratios* 2PC / GV *and* M / N *become equal [and approach unity]* ,

In the limit as the point Q approaches the point P , then the line GV approaches the line PG = 2PC (see fig. 5.15), and thus the ratio 2PC / GV approaches unity. Also in the limit, the line QV approaches the line QX and from [2-K], (QV / QX)^2 = M / N , thus the ratio M / N , also approaches unity.

[3-S] *and therefore the combined ratio of these* L × QR / QT^2 [*equals unity (i.e* ., L = QT^2 / QR)]

At long last, $QR / QT^2 = 1/L$! Thus, from [2-Q], one has $(L \times QR) / QT^2 = (2PC / GV)(M / N)$ ® 1, or $L = QT^2 / QR$. Recall that Newton concluded Theorem 3 by stating that the force was to be computed from the ratio $QR / QT^2 \times SP^2$. Everything in this proof has been directed toward showing that the discriminate ratio QR / QT^2 is a constant equal to the reciprocal of the constant *latus rectum* L . The long journey is over.

[3-T] *Multiply each part by* SP^2 / QR *and there will result* $L \times SP^2 = SP^2 \times QT^2 / QR$.

To obtain the reciprocal of the linear dynamics ratio ($QR / SP^2 \times QT^2$), simply multiply the expression for the discriminate ratio in [3-S] by SP^2 . Thus, $SP^2 (L) = SP^2 (QT^2 / QR) = 1$ / linear dynamics ratio.

Conclusion

[U] *Therefore the centripetal force is reciprocally as* $L \times SP^2$ *, that is,* [*reciprocally*] *in the doubled ratio of the distance. Which was to be proved* .

From Theorem 3, the force is proportional to the linear dynamics ratio $QR / SP^2 \times QT^2$, or reciprocally as $L (SP^2)$, where the *latus rectum* L is a constant of the ellipse. Thus, the force is proportional to the inverse square of the distance SP .

Scholium

[V] *Therefore the major planets orbit in ellipses having a focus at the center of the sun, and with their radii having been constructed to the sun describe areas proportional to the times, exactly as Kepler supposed* .

By 1609, Kepler had obtained both the area law and the general ellipticity of solar planetary orbits by analyzing Tycho's observations of the motion of the planet Mars. In 1664, Newton demonstrated that both relationships can represent motion in the absence of any resistance. Note, however, that Newton demonstrated only that if the planet moves in an ellipse, and if the force center is at a focus, then the force is as the inverse square of the distance (a solution to the direct problem). He has not here demonstrated that if the force is as the inverse square of the distant, then it necessarily follows that the orbit is an ellipse with the force center at a focus of the ellipse (a solution to the inverse problem). In the solution to Problem 4 (to follow) he demonstrates that if the body is initially moving in an arbitrary ellipse under the action of an inverse square force, and if the speed and angle of inclination are changed to take on all possible values, then the resulting motion will be an ellipse, a hyperbola, or a parabola. Newton did not formally address the question of the uniqueness of the solution of the direct problem in the 1678 edition of the *Principia* ; however, he did speak to that question in the 1713 edition (to be discussed in chapter 10).

[W] *Moreover, the* latera recta *of these ellipses are* QT^2 / QR, *where the distance between the points* P *and* Q *is the least possible and, as it were, infinitely small* .

For infinitely small displacements the discriminate ratio QT^2 / QR is equal to the constant *latus rectum L* . Newton considers this property of elliptical motion (derived in [3-S]) to be so important that it is elevated to the position of a scholium for easy future reference. Given the detailed analysis required to provide the demonstration, it is a result worthy of the elevation.

Conclusion

Thus, Newton demonstrated that the centripetal force required to maintain a body P in elliptical motion about a focal point was inversely proportional to the square of the distance SP . From this solution he developed the concept of a force of universal gravitation that controlled the motion of the planets as they sweep around the sun. All the mathematical astronomers to follow used and extended the solution and, from its beginning here in Newton's reply to Halley's request of 1684, it has developed into the sophisticated gravitational astronomy of the eighteenth, nineteenth, and twentieth centuries. Of all the contributions that Newton made to mathematics, to optics, to astronomy, and to areas other than science, no other received as much public recognition as this solution of the Kepler problem. It elevated both Newton and mathematical astronomy to new heights. One wonders, however, when (if ever) Newton would have made his results public had not Halley happened to visit Cambridge after his discussion with Robert Hooke and Christopher Wren concerning such a proof. As one historian has put it (no pun intended, I am sure), "by 1684, the general question of gravitation was in the air."[6] It is reasonable to assume, therefore, that Newton would eventually have become aware of the general interest in his demonstration. Nevertheless, Halley asked his question at the right time and of the right person, and Newton's answer caught the attention of the academic world.

Six— The Paradigm Extended: *On Motion* , Theorem 4 and Problem 4

Contemporary textbooks in astronomy or physics attribute three laws governing planetary motion to the work of Kepler: the first law states that a planet moves in an elliptical path about the sun located at a focus of the ellipse; the second law states that a line joining the sun and a planet sweeps out equal areas in equal times; and the third law states that the period of a planet about the sun is proportional to the three-halves power of the transverse axis of the elliptical orbit. In Theorem 1 of *On Motion* , Newton demonstrates that the second law is valid for any central force; in Problem 3 he demonstrates that, given the first law, it follows that the force will be inversely proportional to the square of the distance; and in Theorem 4, he demonstrates the third law.

Theorem 4— The Three-Halves Power Law

> Theorem 4. *Supposing that the centripetal force is reciprocally proportional to the square of the distance from the center, the squares of the periodic times in ellipses are as the cubes of their transverse axes* .

Figure 6.1 is based on the diagram that accompanies the theorem in the tract *On Motion* . In the following, I consider each line of Newton's demonstration of the theorem in detail.

> [A] *Let* AB *be the transverse axis of an ellipse* , PD *the other axis* , L *the* latus rectum, S *one of the foci* ;

Thus, *AB* is the major axis and *PD* is the minor axis. Newton employed the Apollonian definition of the *latus rectum L* , which states that the *latus rectum L* is in the same ratio to the minor diameter *PD* as the minor diameter *PD* is to the major diameter *AB* (i.e., *L* : *PD* :: *PD* : *AB* or *L* / *PD* = *PD* / *AB*

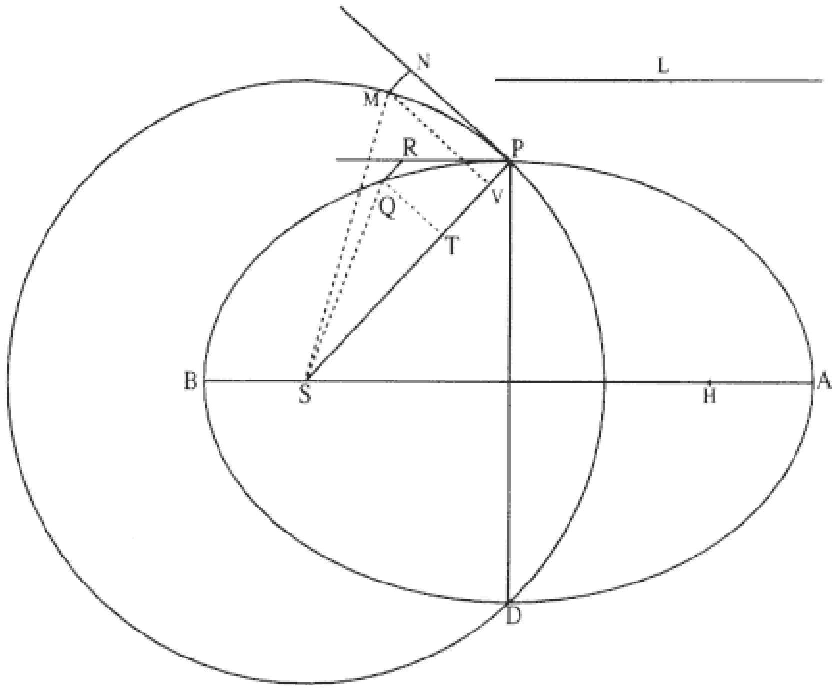

Figure 6.1
Based on Newton's diagram for Theorem 4.

or $L = PD^2 / AB$). From the definition of an ellipse, the distance ($SP + PH$) is a constant equal to the principal axis AB . In this construction $SP = PH$, therefore $AB = 2SP$ and thus $L = PD^2 / 2SP$, a relationship that will be employed in the following.

[B] *let the circle* PMD *be described with* S *as the center and* SP *as the radius* .

Thus, the circle *PMD* has a radius *SP* and its diameter, 2*SP* , is equal to the major axis of the ellipse, *AB* .

[C] *And at the same time let two orbiting bodies describe an elliptical arc* PQ *and the circular arc* PM, *with the centripetal force directed to the focus* S.

It will be demonstrated next that the period of the body orbiting on the circle is equal to the period of the body orbiting on the ellipse when the diameter of the circle is equal to the transverse axis of the ellipse.

[D] *Let* PR *and* PN *be tangent to the ellipse and circle at the point* P. *Parallel to* PS *draw* QR *and* MN *meeting those tangents at* R *and* N.

The deviation *QR* measures the elliptical departure from the tangential inertial motion along *PR* (see fig. 6.2), and the deviation *MN* measures the

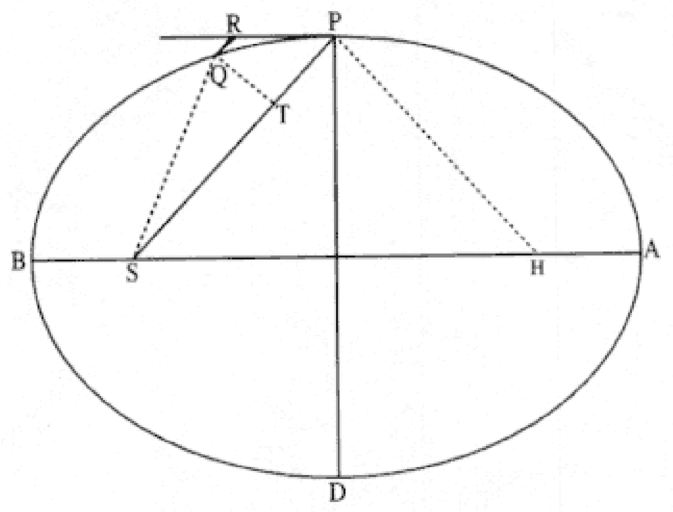

Figure 6.2
The ellipse abstracted from the general diagram for Problem 4.

circular departure from the tangential inertial motion along *PN* (see fig. 6.3). Both are employed in the linear dynamics ratio as a measure of the force.

[E] *But let the figures* PQR *and* PMN *be indefinitely small, so that (by the scholium to Problem 3) there results* $L \times QR = QT^2$ *and* $2SP \times MN = MV^2$.

In the scholium to Problem 3, Newton demonstrates that the ratio QT^2 / QR approaches the *latus rectum L* of the ellipse as the point *Q* approaches the point *P* , or ultimately $L \times QR = QT^2$. For a circle, the major axis, the minor axis, and the *latus rectum* are all equal to the diameter, 2*SP* . Thus, for the circle, $L = MV^2 / MN = 2SP$ or $2SP \times MN = MV^2$.

[F] *On account of their common distance* SP *from the center* S *and therefore equal centripetal forces* , MN *and* QR *are equal* .

The centripetal force depends only upon the distance *SP* , which is the same for the common point *P* on both orbits (see fig. 6.1). Thus, for a given time, the displacements *MN* and *QR* are equal because the forces are equal.

[G] *Consequently* QT^2 *is to* MV^2 *as* L *to* 2SP, *and so* QT *to* MV *as the mean proportional between* L *and* 2SP;

Taking $QT^2 = L \times QR$ and $MV^2 = 2SP \times MN$ from [E], $QT^2 / MV^2 = L \times QR / 2SP \times MN = L / 2SP$, since $MN = QR$ from [F]. Thus, $QT^2 / MV^2 = L / 2SP$, or what is equivalent, *QT* / *MV* is the mean proportional between *L* and 2*SP* .

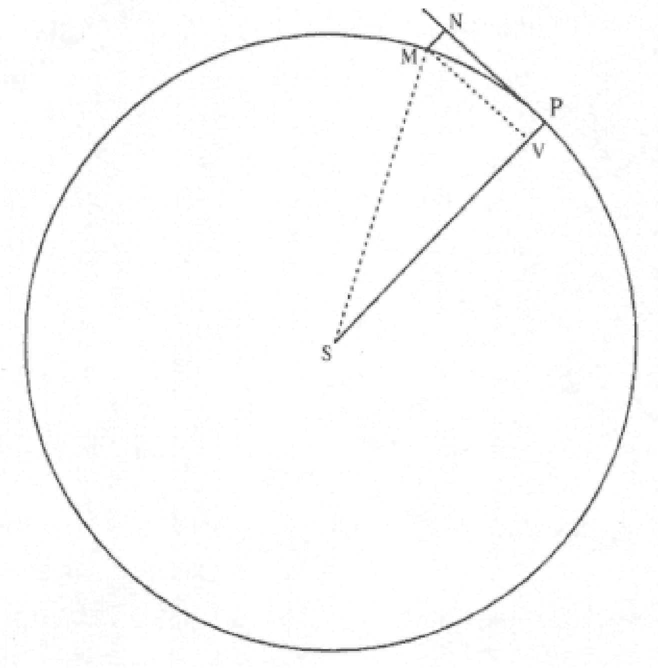

Figure 6.3
The circle abstracted from the general diagram for Problem 4.

[H] [*that is* , PD *to* 2SP]

From [G], QT^2 / MV^2 = L /(2SP). From [A], L = (PD ^{2} / AB) and AB = (2SP), therefore L = PD^2 / (2SP). Thus, L / (2SP) = PD^2 / (2SP)^2 . Therefore, QT / MV = PD / (2SP).

[I] *for this reason the area* SPQ *is to the area* SPM *as the total area of the ellipse to the total area of the circle* .

The area of the ellipse equals p (AB / 2) (PD / 2) = (1/4)pAB × PD , and the area of the circle equals pSP ^{2} . Hence, the ratio of areas = AB × PD /4SP^2 = PD /2SP because AB = 2SP from line [B]. The ratio of the triangular areas SPQ = (1/2)SP × QT and SPM = (1/2) SP × MV reduces to QT / MV , which from line [H], is as PD / 2SP . Thus areas (SPQ / SPM) = PD / 2SP = areas (ellipse / circle).

[J] *But the parts of the areas generated at individual moments are as the areas* SPQ *and* SPM, *and hence as the total areas* ,

The incremental areas of the circle DA_C and the ellipse DA_E are in the same ratio as the total areas given in [I] (i.e., DA_C /DA $_E$ = A_C / A_E).

[K] *and consequently when multiplied by the number of moments they will likewise turn out equal to the total areas* .

The total areas equal the sums of the N incremental areas or A = SDA = NDA . Thus, DA_C / DA_E = (N_C DA_C) / ($N_E D$ A_E) = A_C / A $_E$.

[L] *Revolutions on ellipses, therefore, are accomplished at the same time as those on circles whose diameters are equal to the transverse axes of the ellipses* .

From line [K], the number of moments N must be equal (i.e., N_C = N_E). Moreover, the number of moments to sweep out the total area equals the period T divided by the size of the equal time increment D t (i.e., N = T /D t). Thus, N_C / N_E = (T_C /Dt) / (T_E /Dt) = T_C /T $_E$ or T_C = T_E .

[M] *The squares of the periodic times in circles (by Corollary 5 of Theorem 2) are as the cubes of their diameters. And hence also in ellipses. Which was to be proven* .

In the 1687 edition, Newton proves this result (Kepler's third law) directly from the limiting relationship given in the scholium to Problem 3 (i.e., QR / QT^2® L) without using the concentric circle employed here in *On Motion* (1684).

When Newton published the 1687 edition of the *Principia* , which builds directly upon *On Motion* , he adds a third book devoted to the analysis of the actual observations of celestial phenomena. Thus, he brings to life rather abstract mathematical demonstrations of the nature of gravitational force by comparing them to the actual data compiled by astronomers. In the following, Newton adds a scholium to the theorem just discussed, Theorem 4, that suggests one way in which the measurements of the periods of the planets can be used to determine the dimensions of their orbits.

Scholium

Hence in the celestial system from the periodic times of the planets we come to know the proportions of the transverse axes of their orbits. We shall assume one axis, from which others will be given.

The assumed axis usually will be the diameter of the earth's orbit. Then, using Theorem 4, the ratio of the other axes will be known from their observed periods.

[Scholium] *When the axes have been given, however, the orbits will be determined in this way. Let* S *be the position of the sun, or one focus of the ellipse* ; A, B, C, D *positions of the planets found by observation; and* Q *the transverse axis of the ellipse. With center* A *and radius* Q – AS *let the circle* FG *be described, and the other focus of the ellipse will be in its circumference* . [See fig. 6.4.]

The points *A, B, C* , and *D* all lie on the planetary ellipse. It is a property of an ellipse (see [2-E] in Problem 3) that the sum of the distances from any point on the ellipse to the foci is a constant equal to the length of the

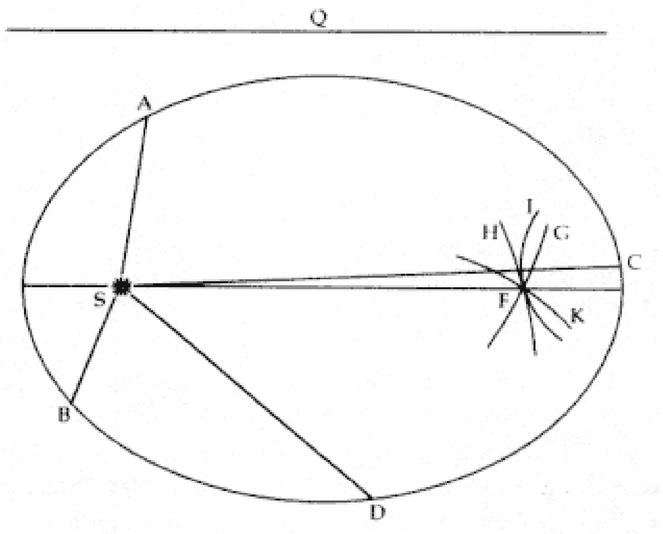

Figure 6.4
The first diagram for the scholium to Theorem 4.

transverse axis, *Q* . Thus *AS* + *AF* = *Q* or *AF* = *Q* – *AS* . Thus, the circle *FG* of radius *Q* – *AS* about the point *A* will intersect the transverse axis at the focus *F* .

[Scholium] *Likewise, with centers* B, C, D, *etc., and intervals* Q – BS, Q – CS, Q – DS, *etc., let any number of other circles be described, and that other focus will be in all their circumferences and hence at the common intersection of all of them. If all of the intersections do not coincide, a mean point for the focus must be taken. The advantage of this procedure is that as many observations as possible may be made to elicit a single conclusion, and they may be expeditiously compared with one another* .

The transverse axis *Q* and the positions *A, B, C, D* , etc., are only known approximately. Thus, the various circles will not intersect at a single point. If all the points are known with equal accuracy, however, then the mean point for *F* could be defined as the simple arithmetic mean.**[1]**

[Scholium] *Halley has shown, however, how to find the individual positions* A, B, C, D, *etc., of a planet from pairs of observations, once the great orbit of the earth is known.***[2]***If that great orbit is not yet considered to be exactly enough determined, then by knowing it approximately, the orbit of any other planet, like Mars, will be determined more closely. Then from the orbit of the earth, the orbit of the planet will be determined more accurately by far than before. And so in turn, until the intersections of the circles in the focus of each orbit concur exactly enough* .

By this method we may determine the orbits of the earth, Mars, Jupiter, and Saturn; but for the orbits of Venus and Mercury, do the following .

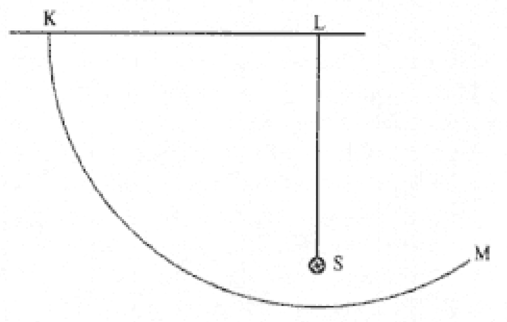

Figure 6.5
The second diagram for the scholium to Theorem 4.

The method Newton describes employs pairs of solar oppositions to fix the position and relative sizes of the planetary radii. Since the two inner planets, Venus and Mercury, can never be in opposition to the sun, then another method, such as the following, must be used for them.

[Scholium] *From observations made in the greatest digression of the planets from the sun, let tangents of the orbits be obtained. To such a tangent* KL *let the perpendicular* SL *be dropped from the sun, and with center* L *and radius half the axis of the ellipse let the circle* KM *be described; the center of the ellipse will be in its circumference* . [See fig. 6.5.]

Whiteside suggests that Newton may here assume that his reader would know an equivalent proposition from Apollonius to demonstrate that the center of the ellipse would lie on such a circle.[3]

[Scholium] *and thus when several circles of this kind are described, it will be found at the intersection of all of them. Then, when the dimensions of the orbits are known, the lengths of these planets will be determined more exactly from their passage through the disc of the sun .*

This technique only gives proportions relative to an assumed size of the earth's orbit. The reference here is to the determination of the absolute size of the distance from the sun to the earth by observations of the transits of Mercury and Venus across the solar disc. Halley had traveled to the island of St. Helena (latitude 16°S) in October of 1677 to observe a transit of Mercury, and he may have called Newton's attention to the possible use of such observations during his famous visit to Cambridge in August of 1684.[4]

Problem 4—
Given an Inverse Square Force, Find the Resulting Conic Section

In Problem 3, the elliptical orbit and focal force center were given, and the force function was found to be inversely proportional to the square of

the distance. In Problem 4, the force function is given as inversely proportional to the square of the distance and the orbit is to be determined. Newton considers a body projected at a given point, with a given speed, acted upon by an inverse square force whose absolute value (gravitational constant) is given. From the solution to Problem 3, it is known that one possible orbit under such conditions is that of an ellipse with the center of force at a focus of the ellipse. Newton begins with the assumption that the body is moving in such an ellipse. Moreover, he assumes the existence of an auxiliary circular orbit that is centered on the focus of the ellipse.[5] He then proceeds to determine all the relevant parameters of the initial ellipse from the given elements of the problem. Having done so, he then allows the initial speed of the body to increase and demonstrates that the initial elliptical path becomes a parabolic path and then a hyperbolic path. Newton makes no explicit claim that this solution provides a demonstration that the conic sections exhaust all possible types of motion under the action of an inverse square force (i.e., that it is a solution of the inverse problem). But others saw the possibility of such a solution in his demonstration of Problem 4, which also appears as Proposition 17 in the *Principia* . As Bertoloni Meli points out, support came from both the English mathematician John Keill and the continental mathematician Leonhard Euler.

In 1716 Keill claimed that Proposition 17 [Problem 4] contained a demonstration of the inverse theorem. In his reply through his student Johann Kruse, however, Johann Bernoulli objected that Proposition 17 [Problem 4] assumes the result rather than proving it. Surprisingly Johann Bernoulli's most talented pupil and possibly the most gifted mathematician of the Enlightenment sided with Keill: Leonhard Euler, in his 1736 *Mechanica* , claimed that the inverse problem of central forces could be solved on the basis of Proposition 17 [Problem 4].**[6]**

The composite nature of the solution, however, is such that it has been the subject of considerable controversy, and the debate still continues.**[7]**

I give the statement of the problem and then follow it with a line-by-line analysis of Newton's demonstration of the solution. Figure 6.6 is based on Newton's diagram for Problem 4.

Problem 4. *Supposing that the centripetal force be made reciprocally proportional to the square of the distance from its center, and that the absolute quantity of that force is known; there is required an ellipse which a body will describe, when released from a given position with a given speed along a given straight line* .

[A] *Let the centripetal force directed to point* S *be that which makes the body* P *orbit in a circle* pq *described with center* S *and any radius* Sp.

The body *p* moves in a reference circle *pq* of radius *Sp* about the center of force *S* . For a given "absolute quantity of force" (i.e., a given force con-

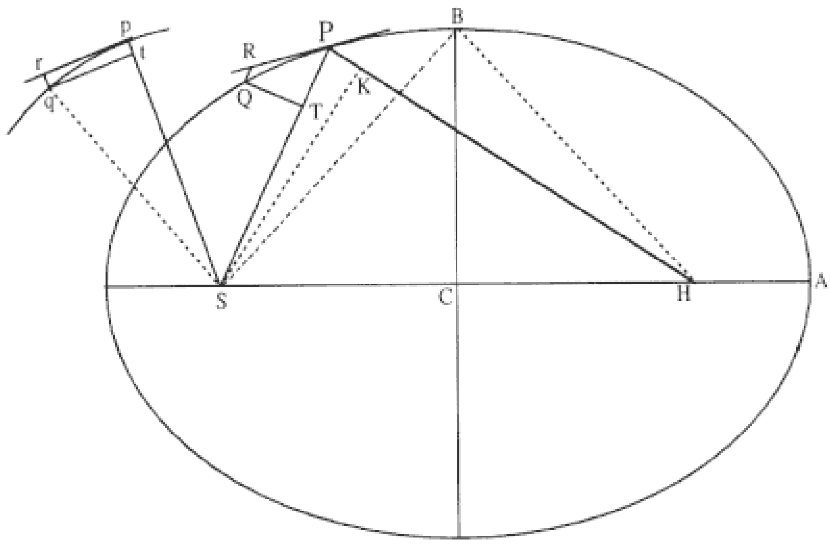

Figure 6.6
Based on Newton's diagram for Problem 4.

stant) there exists a particular constant tangential speed for which the body *p* will move uniformly in a circle, as has been demonstrated in Theorem 2. In what follows, Newton assumes that the proper set of initial conditions has been met and he will relate various aspects of this constant uniform circular motion to the more general conic motion of the orbit *PQ* (see fig. 6.6).

[B] *Let the body* P *be released from the position* P *along line* PR, *and soon after let it deflect under the compulsion of the centripetal force into the ellipse* PQ. *The straight line* PR, *therefore, will touch this at* P.

This description of the elements of the diagram constitute Newton's standard method of analysis of direct problems. The line *PR* represents the tangential linear displacement the body would have made had no force acted on it. The possibility of elliptical motion under an inverse square centripetal force has been demonstrated in Problem 3, and here Newton assumed that the proper set of initial conditions had been met for the body *P* to move on an ellipse. Moreover, from Theorem 1, the motion will be such that the radius will sweep out equal areas in equal times because the force is directed to a fixed center. Newton was aware that other initial conditions will produce other motions than elliptical ones, and he addressed those alternate motions in what follows.

[C] *In the same way let the straight line* pr *touch the circle at* p,

The tangent is defined for the reference circle.

[D] *and let* PR *be to* pr *as the initial speed of the body* P *sent out to the uniform speed of the body* p.

The implicit assumption is made that the tangential displacements *PR* and *pr* take place in equal times. If v_p is the projection speed at *P* and v_p is the uniform circular speed at *p* , then the displacements *PR* and *pr* are equal to the product of the speed and the time, or are proportional to those speeds in the same time *t* (i.e., $PR / pr = v_p t / v_p t = v_p / v_p$).

[E] *Let* RQ *and* rq *be drawn parallel to* SP *and* Sp, *the latter meeting the circle at* q, *the former the ellipse at* Q,

The deviations *QR* and *qr* are the first elements required for the respective linear dynamics ratios.

[F] *and let the perpendiculars* QT *and* qt *be drawn from* Q *and* q *to* SP *and* Sp.

The perpendiculars *QT* and *qt* are the second elements required for the respective linear dynamics ratios.

[G] RQ *is to* rq *as the centripetal force at* P *is to the centripetal force at* p:

For a given time, the force F_p is proportional to the deviation *QR* and the force F_p is proportional to the deviation *qr* , as has been discussed in detail in the previous problem solutions.

[H] *that is, as* Sp^2 *to* SP^2 , *and hence that ratio is given* .

From the statement of the problem, $F_p / F_p = (C / SP^2) / (C / Sp^2) = Sp^2 / SP^2$, where *C* is a constant set by the given absolute value of the force. Since the ratio of the displacements *RQ* /*rq* is given in [D] by the initial projection speed, and since that ratio equals the ratio of the forces [from line [G]], then ratio Sp^2 / SP^2 is also given, and it equals *QR* / *qr* .

[I] *The ratio* QT *to* qt *is also given* .

The ratio of the areas, $(QT \times SP) / (qt \times Sp)$, is equal to the ratio of the times (Theorem 1: Kepler's law of equal areas in equal times). Since the times are equal, then $(QT \times SP) = (qt \times Sp)$, or $QT / qt = Sp / SP$, and the ratio QT / qt is thus given (because the ratio Sp / SP is given in line [H]).

[J] *From this latter ratio doubled let the given ratio* QR *to* qr *be taken away, and there will remain the given ratio of* QT^2 / QR *to* qt^2 / qr,

Divide the square of the given ratio of $(QT / qt)^2$ by the given ratio of (QR / qr), and there results the given ratio of $(QT^2 / QR) / (qt^2 / qr)$.

[K] *that is* (*by the scholium to Problem 3*) *the ratio of the* latus rectum *of the ellipse to the diameter of the circle;*

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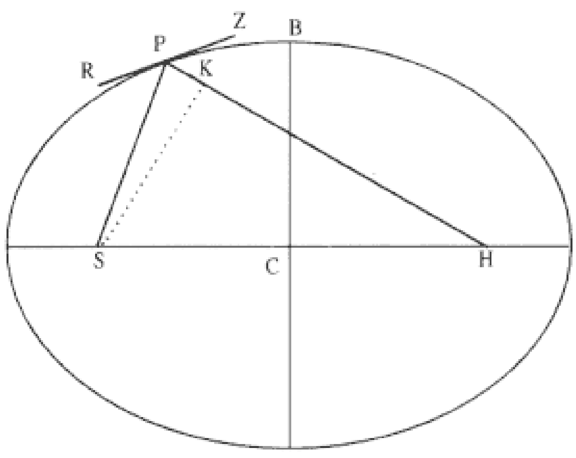

Figure 6.7
Angle *RPS* equals angle *ZPH* , and thus angle *RPH* is the supplement of angle *RPS* .

That is, $(QT^2 / QR) / (qt^2 / qr) = L / Sp$. The scholium to Problem 3 states that the limiting value for the ratio QT^2 / QR for an ellipse as *Q®P* is the *latus rectum, L* . Thus, since the ratio in line [J] is known, the ratio of L_E for the ellipse to L_C for the circle is known. The *latus rectum* L_C for the circle, however, is its diameter *Sp* . (From the definition, L = (minor diameter)^2 / (major diameter); for a circle the two diameters are equal; thus, L_C = diameter.)

[L] *and therefore the* latus rectum *of the ellipse is given. Let that be* L.

Thus, since the diameter of the circle *Sp* is given, the *latus rectum L* of the ellipse is known (i.e., from [K] $L / Sp = (QT^2 / QR) / (qt^2 / qr)$, where the final ratio is given from line [J]).

[M] *In addition, the focus of the ellipse* S *is given* .

The focal point *S* is given in the opening statement in line [A].

[N] *Let angle* RPH *be the complement of angle* RPS *to two right angles* [*i.e., angle* RPH *is the supplement of angle* RPS],

From figure 6.7, angle *RPH* + angle *HPZ* = 180°. From the properties of conics (Apollonius, Proposition 48, Book 3) the angles *RPS* and *ZPH* that are made by the tangent and focal lines are equal.**[8]** Thus, the angle *RPH* = 180° – *ZPH* = 180° – *RPS* , or angle *RPH* is the supplement of angle *RPS* .

[O] *and there will be given in position the line* PH *in which the other focus* H *is located* .

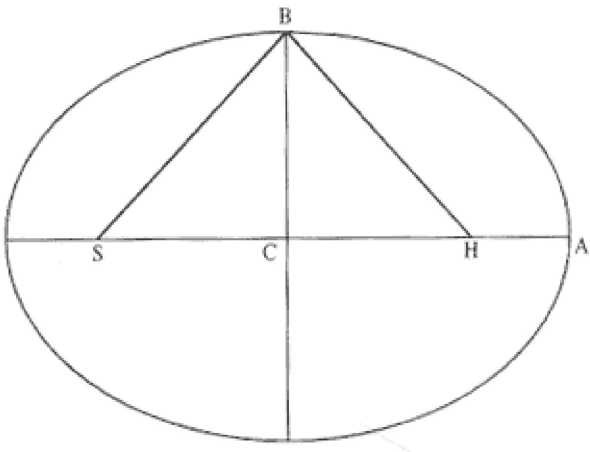

Figure 6.8
The focal distances *SC* and *CH* are equal, and thus *SH* can be given as 2*CH* .

The angle *RPS* is known from the projection velocity. From line [P] the angle *SPH* is known; it is 180° – 2*RPS* . Thus, the direction of the line on which the second focus *H* is located is known relative to the line *SP* , or as Newton puts it, "There will be given in position the line *PH* ." The distance *PH* , however, is not yet known. Lines [P] to [V] are required to determine the length of *PH* and thus to locate the second focus *H* .

[P] *After the perpendicular* SK *is let down to* PH *and the semi-minor axis* BC *is erected, there is* $SP^2 - 2KP \times PH + PH^2 = SH^2$

This result is found in Euclid, Book 2, Proposition 13.**[9]** (Otherwise, from the law of cosines, $SH^2 = SP^2 - 2SP \times PH \times \cos(\text{angle } SPH) + PH^2$. From figure 6.7, cos (angle *SPH*) = *KP* / *SP* . Thus, $SP^2 - 2KP \times PH + PH^2 = SH^2$.)

[Q] $SH^2 = 4BH^2 - 4BC^2$

From figure 6.8, $SH = SC + CH = 2CH$ or $SH^2 = 4CH^2$. From the right triangle BCH , one has $CH^2 = BH^2 - BC^2$. Thus, $4CH^2 = 4(BH^2 - BC^2)$ or $SH^2 = 4BH^2 - 4BC^2$.

[R] $4BH^2 - 4BC^2 = (SP + PH)^2 - L \times (SP + PH)$

From the definition of an ellipse, $(SP + PH) = 2AC = 2BH$, thus $4BH^2 = (SP + PH)^2$. From the definition of the *latus rectum* , $L = 2BC^2 / AC$, thus $4BC^2 = L \times 2AC = L \times (SP + PH)^2$. Thus, $4BH^2 - 4BC^2 = (SP + PH)^2 - L \times (SP + PH)$ as given above.

[S] $(SP + PH)^2 - [L \times (SP + PH)] = (SP^2 + 2SP \times PH + PH^2) - [L \times (SP + PH)]$.

Simply expand the square, $(SP + PH)^2 = (SP^2 + 2SP \times PH + PH^2)$.

[T] *To each side add* $2KP \times PH + L \times (SP + PH) - SP^2 - PH^2$ *and there will result* $L \times (SP + PH) = 2SP \times PH + 2KP \times PH$,

Starting at line [P], $SP^2 - 2KP \times PH + PH^2$ has been shown to be equal at line [S] to $(SP^2 + 2SP \times PH + PH^2) - L \times (SP + PH)$, which upon cancellation and rearrangement becomes $L \times (SP + PH) = 2SP \times PH + 2KP \times PH$.

[U] *or* $(SP + PH)$ *to* PH *as* $(2SP + 2KP)$ *to* L.

Dividing both sides by $L \times PH$, one has $(SP + PH) / PH = (2SP + 2KP) / L$

[V] *From which the other focus* H *is given* .

The direction of the line *PH* was determined in line [P]. The length of *PH* was determined by the ratio in line [U], because the elements, *SP, L* , and *KP* are given. ($KP = SP \cos(SPH)$, where the angle *SPH* is given in line [P].) Thus, the direction and distance of *PH* relative to *S* is determined, and therefore the second focus *H* is known.

[W] *Given the foci, however, along with the transverse axis* SP + PH, *the ellipse is given. As was to be proven* .

Knowledge of the position of both foci and the distance $(SP + PH)$ permits construction of the ellipse. A simple demonstration is accomplished by putting a tack at each of the foci and joining them with a loose string of length $(SP + PH)$. When a pencil is moved so that the string remains taut, the ellipse is drawn.

[X] *This argument holds when the figure is an ellipse. But it can happen that a body moves in a parabola or a hyperbola* .

The initial conditions set in line [B] restricted the motion to an ellipse, but in [X] Newton demonstrates that the motion under an inverse square force could also be a parabola or a hyperbola (i.e., any conic section). In doing so, he permits the projection speed v_p of the body *P* to take on greater values than the specific value that produced elliptical motion at a given point under an inverse square centripetal force with a given force constant. He demonstrates that the resulting motion will be one of the conic sections.

[Y] *But if the speed of a body is so great that the* latus rectum L *is equal to* 2SP + 2KP, [*then*] *the figure will be a parabola having its focus at the point* S *and all its diameters parallel to the line* PH.

As the second focus *H* recedes to infinity, the figure becomes a parabola. From line [*N*], as *PH* increases without limit, $(SP + PH) / PH$® 1 or the *latus rectum* *L* approaches $(2SP + 2PK)$.

[Z] *But if the body is released at an even greater speed,* [*then*] *it will be moved in a hyperbola having one focus at the point* S, *the other at the point* H *taken on the opposite side of the point* P, *and its transverse axis equal to the difference of the lines* PS *and* PH.

The second focus *H* , having receded to infinity, now approaches the other focus *S* from the opposite side of the general point *P* .

Newton did not address the full question of the uniqueness of this solution to Problem 4 (i.e., do the three conic sections exhaust all the possible solutions for motion under an inverse square force?). Perhaps he thought the affirmative answer self-evident, since for every initial speed, angle of projection, and value of force constant, a unique conic is defined: an ellipse, a parabola, or a hyperbola.[10] Moreover, the subject of the uniqueness of the solution was not discussed in the revised solution to Problem 4 that appeared as Proposition 17 in the 1687 edition of the *Principia* . In Propositions 11, 12, and 13, however, he presented the solutions for the direct problems of each of the conic orbits: an ellipse, a hyperbola, and a parabola. In the 1687 edition, he included the following corollary to Proposition 13, which directly assumed uniqueness.

Corollary. *From the last three propositions it follows that if any body* P *should depart from Position* P *along any straight line* Pr, *with any velocity, and is at the same time acted upon by a centripetal force that is reciprocally proportional to the square of the distance from the center, this body will be moved in one of the sections of conics having a focus at the center of forces; and conversely* .

In the 1713 edition, however, he added a statement to the corollary in Proposition 13 that was intended to address and to justify the assumption of uniqueness. (See chapter 10 for a discussion of that revised corollary.) That demonstration is important because, if established, the solution to the direct problem in combination with this solution serves as a solution to the inverse problem. Following Newton, it has been demonstrated to the satisfaction of the most demanding of mathematicians that the conic sections are the unique set of solutions for motion under an inverse square force. In *On Motion* and in the 1687 edition of the *Principia* , however, Newton is satisfied with a much more intuitive demonstration of uniqueness.

In the scholium to Theorem 4, Newton discussed the theorem's application to the computation of the relative size of planetary orbits from the observation of planetary periods. In the scholium to Problem 4, he applies his techniques developed in Problem 4 to the determination of orbits of comets. Again, he applies the ideal mathematical world of the tract to the actual observational world of the heavens. The topic of comets caused Newton much concern. In a letter to Halley, written more than a year after sending the following scholium on comets, Newton states that "in Autumn

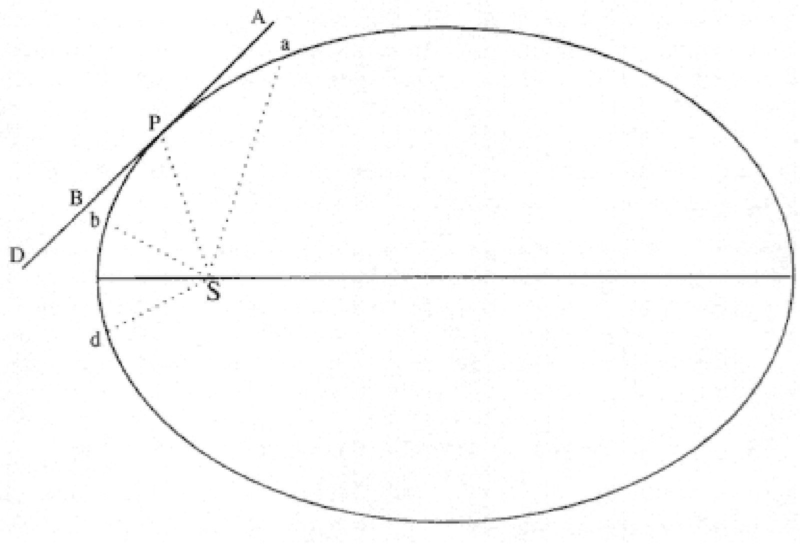

Figure 6.9
The diagram for the scholium to Problem 4.

last I spent two months in calculations to no purpose for want of a good method." Not until Newton replaces the linear approximation discussed next with a parabolic approximation for the elliptical path does the "good method" appear as Proposition 41 of Book Three of the 1687 *Principia* .

Scholium

Now indeed with the help of this problem when it has been solved, it is possible to define the orbits of comets, and from that the times of their revolutions, and from a comparison of the magnitude of their orbits, eccentricities, [*perihelia*], *inclinations to the ecliptic plane and their nodes, to know whether the same comet returns to us frequently.*

The most celebrated example of a returning comet is the comet of 1682, which Halley correctly identified as the return of the comet of 1531 and 1607 (a period of 75 or 76 years) and which he correctly predicted to return in 1758 (after his death).

[Scholium] *To be sure, from the four observations of the comet's position, according to the hypothesis that a comet moves in a straight line, its rectilinear path must be determined. Let that* [*path*] *be* APBD, *and let* A, P, B, D *be positions of the comet on that path at the times of observations, and let* S *be the position of the sun. Imagine that with the speed at which it regularly traverses the straight line* AD, *the comet is released from one of its positions* P, *and, soon forced by centripetal force, it is deflected from the straight path and goes off into the ellipse* Pbda. [See fig. 6.9.] [11]

By the time of this analysis of 1684, Newton had come to accept the universality of the gravitational force (i.e., that the force derived in Problem 3 for planetary motion also applied to all celestial bodies, such as comets). Previously there had been a question in Newton's mind concerning the curved nature of cometary paths. The rectilinear hypothesis (that is, comets move in straight lines), however, is simply an approximation: one that will be replaced by a more sophisticated method in the *Principia* .[12]

[Scholium] *This ellipse must be determined as in the problem above. In it let a* , P, b, d *be positions of the comet at the times of observations. Let the longitudes and latitudes of these positions from the earth be known. As much as the observed longitudes and latitudes are greater or less than these, let new longitudes and latitudes be taken greater or less than the observed ones. From these new* [*measurements*] *let the rectilinear path of the comet again be found, and from that the elliptical path as before. And the four new positions on the elliptical path, having been augmented or diminished by previous errors now will agree sufficiently precisely with the observations* .

In the determination of the elliptical orbit in Problem 4 above, one must assume a value for the magnitude of the gravitational force (see the statement of the problem). Moreover, it is not clear how one is to know that the new position will produce convergence with the actual path of the comet. Clearly, Newton is struggling with the procedure.

[Scholium] *But if perchance palpable errors should still remain, it is possible for the entire task to be repeated. And, so that the computations may not be annoying to the astronomers, it will suffice to determine all these things by a geometrical procedure* .

There then follows another paragraph in which Newton discusses the difficulty in assigning areas proportional to the times. Whiteside observes that the method outlined in this scholium is a "makeshift construction . . . more optimistic of a chance success than solidly reasoned."[13] In fact, the entire subject of the motion of comets was a challenge. Newton claimed that "this discussion about comets is the most difficult in the whole book."[14] This challenge continued to receive his attention and would eventually be met in the *Principia* .

In Book Three of the *Principia* Newton will consider many more such observational problems. The following is an example taken from the 1687 *Principia* in which Newton discusses the application of the theorems from Book One to the phenomena discussed in Book Three. In *On Motion* , and in its extension into Book One of the *Principia* , Newton has discussed the ideal case of a single planet orbiting about a single stationary center of force in the sun. In the following proposition from Book Three he discusses the influence of adjacent planets upon their mutual motion (that is, a three-body versus a two-body problem). Now Newton is less a mathematician and more of a practical astronomer. He must call into action estimates of the relative masses of the sun and planets as well as their relative

distances. He calls on demonstrations in Propositions 66 and 67 in Book One concerning motion about the center of mass to relate the ideal mathematical problem with a fixed center of force to the actual problem of the sun and planet moving about their center of mass.

Book Three
Proposition 13
Theorem 13

Book Three. Proposition 13. *Planets move in ellipses having a focus in the center of the sun, and with radii having been constructed to that center they describe areas proportional to the times* .

We have talked above about these motions from the phenomena. Now that we have understood the principles of motions, from these [*principles*] *we infer celestial motions a priori. Since the weights of the Planets toward the sun are reciprocally as the squares of the distances from the center of the sun; if the sun were at rest and the other Planets were not acting on each other,* [*then*] *their orbits would be elliptical having the sun in a common focus and areas would be described proportional to the times* (*by Propositions 1 and 11 and Corollary 1 Proposition 13 Book One* [Theorem 1 and Problem 3 in *On Motion*]). *But the actions of the Planets reciprocally on each other are very small* (*so that they might be able to be neglected*), *and they disturb the motions of the Planets in ellipses around the sun in motion less* (*by Proposition 66, Book One*) *than if those motions were being performed around the sun at rest* .

Indeed, the action of Jupiter upon Saturn must not be altogether neglected. For the gravity toward Jupiter is to the gravity toward the sun (*the distances being equal*) *as 1 to 1,100* ;

Modern values of the masses of the sun and Jupiter give a ratio of 1 to 1,047, which compares to Newton's value of 1 to 1,100 (a value that appears as 1,067 in the revised *Principia*), values which Newton obtained using Kepler's third law (Theorem 4) and measurements of the periods of their satellites (Venus for the sun, its moons for Jupiter).**[15]**

[Proposition 13] *and therefore in the conjunction of Jupiter and Saturn, since the distance of Saturn from Jupiter is to the distance of Saturn from the sun, almost as 4 to 9;*

Modern value of the semi-major axis of Jupiter is 5.203 AU, for Saturn it is 9.539 AU, and thus the distance between Jupiter and Saturn at conjunction is the difference, or 4.336 AU. Thus, the ratio of distances is 4.336 to 9.539 or, as Newton states, 4 to 9.

[Proposition 13] *the gravity of Saturn toward Jupiter will be to the gravity of Saturn toward the sun as 81 to 16 × 1,100; or 1 to about 217* .

The gravitational force is directly proportional to the mass and inversely proportional to the square of the distance. Thus, the force on Saturn due to Jupiter is as $1/4^2$ and that due to the sun as $1{,}100/9^2$, or a ratio of (1/16) / (1,100/81) = 81/(16 × 1,100) = 1/217.

[Proposition 13] *Yet the whole error in the motion of Saturn around the sun, arising from so great a gravity toward Jupiter, can be avoided by constructing a focus of the orbit of Saturn in a common center of gravity of Jupiter and the sun (by Proposition 67, Book One) and hence when the error is greatest it hardly exceeds two minutes. But in the conjunction of Jupiter and Saturn, the accelerative gravities of the sun toward Saturn, of Jupiter toward Saturn, and of Jupiter toward the sun, are almost as 16, 81, and 16 × 81 × 2,360/25 or 122,342* ,

Using modern notation, the forces between two masses can be expressed as

$$GM_1M_2 / R_{12}^2,$$

where G is the universal gravitational constant and R_{12} is the distance between their centers. The accelerative gravities given are those forces divided by the mass of the force center (that is, the "accelerative gravity" of the sun toward Saturn is $GM_{Saturn}/9^2$, of Jupiter toward Saturn is $GM_{Saturn}/4^2$, and the sun toward Jupiter is $GM_{Jupiter}/5^2$). Thus, the ratios are as $GM_{Saturn}/9^2 : GM_{Saturn}/4^2 : GM_{Jupiter}/5^2$. Newton simplifies the results by dividing by $(GM_{Saturn}) / (9^2 \times 4^2)$, which gives the result $4^2 : 9^2 : (M_{Jupiter}/M_{Saturn})/5^2$, where the ratio of $M_{Jupiter}/M_{Saturn}$ is given by 2,360 in the first edition and by 3,021 in the revised editions.

[Proposition 13] *and thus the differences of gravities of gravities of the sun toward Saturn, and of Jupiter toward Saturn, is to the gravity of Jupiter toward the sun as 65 to 122,342 or as 1 to 1,867. But the greatest power of Saturn to disturb the motion of Jupiter is proportional to this difference: and hence the perturbation of the orbit of Jupiter is far less than that of Saturn's. The perturbations of the remaining orbits are even far less* .

Thus, does Newton the mathematician and Newton the practical astronomer relate the demonstrations of the theorems and solutions of abstract problems to the observations of the motion of celestial bodies. The applications and examples are increased in the editions of the *Principia* that follow the tract *On Motion* . The demonstration of a universal gravitational force that depends upon the inverse square of the distance is to be found in the heavens as well as in the pages of the *Principia* .

Conclusion

Following his receipt in London of Newton's tract *On Motion* in November of 1664, Halley returned to Cambridge to consult with Newton once again. He found Newton occupied with the task of rewriting the work in more detail. That project would eventually require two years to complete. Under Halley's personal encouragement, careful editorial eye, and financial support, it would ultimately result in publication as the first edition of the *Principia* in 1687. Upon his return to London, Halley reported to the Royal Society at its meeting on 10 December 1684, about his interchange with Newton and of his intent to have the tract entered in the *Register* of the

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Society, which it was. It was not until two years later, however, at the meeting of the Royal Society on 21 April 1686, that Halley announced that the work was almost ready for the press. At the following meeting, on 28 April 1686, he presented the manuscript of Book One to the society. He delivered Book One to the printer that same month, but it was not until March of the following year that Halley delivered Books Two and Three to the printer.[16] On 5 July 1687, Halley wrote to Newton about the details of the final publication figures and to announce, "I have at length brought your Book to an end."[17]

Six— The Paradigm Extended:On Motion , Theorem 4 and Problem 4
