V.

D'ALEMBERT'S PRINCIPLE.

History of the principle.

1. One of the most important principles for the rapid and convenient solution of the problems of mechanics is the *principle of D'Alembert*. The researches concerning the centre of oscillation on which almost all prominent contemporaries and successors of Huygens had employed themselves, led directly to a series of simple observations which D'ALEMBERT ultimately generalised and embodied in the principle which goes by his name. We will first cast a glance at these preliminary performances. They were almost without exception evoked by the desire to replace the deduction of Huygens, which did not appear sufficiently obvious, by one that was more *convincing*. Although this desire was founded, as we have already seen, on a miscomprehension due to historical circumstances, we have, of course, no occasion to regret the new points of view which were thus reached.

James Bernoulli's contributions to the theory of the centre of oscillation.

2. The first in importance of the founders of the theory of the centre of oscillation, after Huygens, is JAMES BERNOULLI, who sought as early as 1686 to explain the compound pendulum by the lever. He arrived, however, at results which not only were obscure but also were at variance with the conceptions of Huygens. The errors of Bernoulli were animadverted on by the Marquis de L'HOPITAL in the *Journal de Rotterdam*, in 1690. The consideration of velocities acquired in *infinitely small* intervals of time in place of velocities acquired in *finite* times—a consideration which the last-named mathematician suggested—led to the removal

of the main difficulties that beset this problem ; and in 1691, in the *Acta Eruditorum*, and, later, in 1703, in the *Proceedings of the Paris Academy* James Bernoulli corrected his error and presented his results in a final and complete form. We shall here reproduce the essential points of his final deduction.

James Bernoulli's deduction of the law of the compound pendulum from the principle of the lever.

A horizontal, massless bar AB (Fig. 166) is free to rotate about A; and at the distances r, r' from A the masses m, m' are attached. The accelerations with which these masses *as thus connected* will fall must be different from the accelerations which they would assume if their connections were severed and they fell freely. There will be one point and one only, at the distance x, as yet unknown, from A which will fall with the same acceleration as it would have if it were free, that is, with the acceleration g. This point is termed the centre of oscillation.

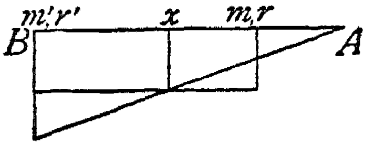

Fig. 166.

If m and m' were to be attracted to the earth, not proportionally to their masses, but m so as to fall when free with the acceleration $\varphi = gr/x$ and m' with the acceleration $\varphi' = gr'/x$, that is to say, if the *natural* accelerations of the masses were proportional to their distances from A, these masses would not interfere with one another when connected. In reality, however, m sustains, in consequence of the connection, an upward component acceleration $g - \varphi$, and m' receives in virtue of the same fact a downward component acceleration $\varphi' - g$; that is to say, the former suffers an upward force of $m\,(g - \varphi) = g(\overline{x - r}/x)\,m$ and the latter a downward force of $m'\,(\varphi' - g) = g\,(\overline{r' - x}/x)\,m'$.

Since, however, the masses exert what influence they have on each other solely through the medium of

The law of the distribution of the effects of the impressed forces, in James Bernoulli's example.

the lever by which they are joined, the upward force upon the one and the downward force upon the other must satisfy the law of the lever. If m in consequence of its being connected with the lever is held back by a force f from the motion which it would take, if free, it will also exert the same force f on the lever-arm r by reaction. It is this reaction pull alone that can be transferred to m' and be balanced there by a pressure $f' = (r/r')f$, and is therefore equivalent to the latter pressure. There subsists, therefore, agreeably to what has been above said, the relation $g\,(\overline{r'-x}/x)\,m' = r/r' \,.\, g\,(\overline{x-r}/x)\,m$ or, $(x-r)\,mr = (r'-x)\,m'r'$, from which we obtain $x = (mr^2 + m'r'^2)/(mr + m'r')$, exactly as Huygens found it. The generalisation of this reasoning, for any number of masses, which need not lie in a single straight line, is obvious.

The principle of John Bernoulli's solution of the problem of the centre of oscillation.

3. John Bernoulli (in 1712) attacked in a different manner the problem of the centre of oscillation. His performances are easiest consulted in his Collected Works (*Opera*, Lausanne and Geneva, 1762, Vols. II and IV). We shall examine in detail here the main ideas of this physicist. Bernoulli reaches his goal by conceiving the *masses* and *forces* separated.

The first step in John Bernoulli's deduction.

First, let us consider two simple pendulums of different lengths l, l' whose bobs are affected with gravitational accelerations proportional to the lengths of the pendulums, that is, let us put $l/l' = g/g'$. As the time of oscillation of a pendulum is $T = \pi\sqrt{l/g}$, it follows that the times of oscillation of these pendulums will be the same. Doubling the length of a pendulum, accordingly, while at the same time doubling the acceleration of gravity does not alter the period of oscillation.

Second, though we cannot directly alter the accel-

The second step in John Bernoulli's deduction.

eration of gravity at any one spot on the earth, we can do what amounts virtually to this. Thus, imagine a straight massless bar of length $2a$, free to rotate about its middle point; and attach to the one extremity of it the mass m and to the other the mass m'. Then the total mass is $m + m'$ at the distance a from the axis. But the force which acts on it is $(m - m')\, g$, and the acceleration, consequently, $(\overline{m - m'}/\overline{m + m'})\, g$. Hence, to find the length of the simple pendulum, having the ordinary acceleration of gravity g, which is isochronous with the present pendulum of the length a, we put, employing the preceding theorem,

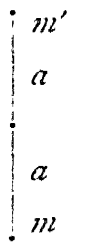

Fig. 167.

$$\frac{l}{a} = \frac{g}{\frac{m - m'}{m + m'}\, g}, \text{ or } l = a\,\frac{m + m}{m - m'},$$

The third step, or the determination of the centre of gyration.

Third, we imagine a simple pendulum of length 1 with the mass m at its extremity. The weight of m produces, by the principle of the lever, the same acceleration as half this force at a distance 2 from the point of suspension. Half the mass m placed at the distance 2, therefore, would suffer by the action of the force impressed at 1 the same acceleration, and a fourth of the mass m would suffer double the acceleration; so that a simple pendulum of the length 2 having the original force at distance 1 from the point of suspension and one-fourth the original mass at its extremity would be isochronous with the original one. Generalising this reasoning, it is evident that we may transfer any force f acting on a compound pendulum at any distance r, to the distance 1 by making its value rf, and any and every mass placed at the distance r to the distance 1 by making its value $r^2 m$, without changing the time of oscillation of the pendulum. If a force f act on a lever-arm a (Fig. 168) while at the distance r from the axis a mass m is attached, f will be equivalent to a force af/r impressed on m and will impart to it the linear acceleration af/mr and the angular acceleration af/mr^2. Hence, to find the angular acceleration of a compound pendulum, we divide the sum of the *statical moments* by the sum of the *moments of inertia*.

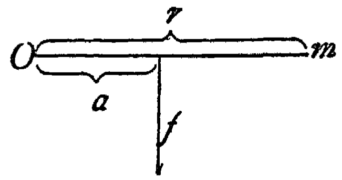

Fig. 168.

The researches of Brook Taylor.

BROOK TAYLOR, an Englishman,* also developed this idea, on substantially the same principles, but quite independently of John Bernoulli. His solution, however, was not published until some time later, in 1715, in his work, *Methodus Incrementorum.*

The above are the most important attempts to solve the problem of the centre of oscillation. We shall see that they contain the very same ideas that D'Alembert enunciated in a generalised form.

Motion of a system of points subject to constraints.

4. On a system of points M, M', M''. . . . connected with one another in any way,† the forces P, P', P''. . . . are impressed. (Fig. 169.) These forces would impart to the *free* points of the system certain determinate motions. To the *connected* points, however, *different* motions are usually imparted—motions which could be produced by the forces W, W', W''. . . . These last are the motions which we shall study.

Conceive the force P resolved into W and V, the force P' into W' and V', and the force P'' into W''

* Author of Taylor's theorem, and also of a remarkable work on perspective.—*Trans.*

† In precise technical language, they are subject to *constraints*, that is, forces regarded as infinite, which compel a certain relation between their motions.—*Trans.*

Statement of D'Alembert's principle.

and V'', and so on. Since, owing to the connections, only the components W, W', W''. . . . are effective, therefore, the forces V, V', V''. . . . must be *equilibrated* by the connections. We will call the forces P, P', P'' the *impressed* forces, the forces W, W', W''. . . ., which produce the actual motions, the *effective* forces, and the forces V, V', V'' . . . . the forces *gained and lost*, or the *equilibrated* forces. We perceive, thus, that if we resolve the impressed forces into the effective forces and the equilibrated forces, the latter form a system balanced by the connections. This is the principle of D'Alembert. We have allowed ourselves, in its exposition, only the unessential modification of putting forces for the momenta generated by the forces. In this form the principle was stated by D'ALEMBERT in his *Traité de dynamique*, published in 1743.

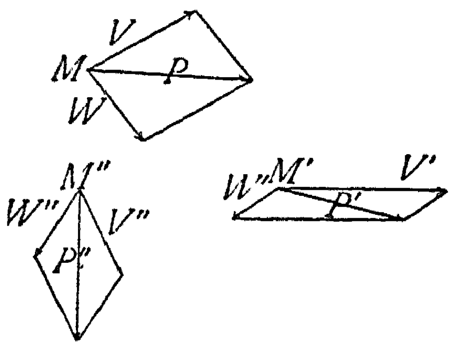

Fig. 169.

Various forms in which the principle may be expressed.

As the system V, V', V''. . . . is in *equilibrium*, the principle of *virtual displacements* is applicable thereto. This gives a second form of D'Alembert's principle. A third form is obtained as follows: The forces P, P'. . . . are the resultants of the components W, W'. . . . and V, V'. . . . If, therefore, we combine with the forces W, W'. . . . and V, V'. . . . the forces $-P$, $-P'$. . . ., equilibrium will obtain. The force-system $-P$, W, V is in equilibrium. But the system V is independently in equilibrium. Therefore, also the system $-P$, W is in equilibrium, or, what is the same thing, the system P, $-W$ is in equilibrium. Accordingly, if the effective forces with opposite signs be joined to the impressed forces, the two, owing to the connections, will balance. The principle of virtual displacements may also be applied to the system P, $-W$. This LAGRANGE did in his *Mécanique analytique*, 1788.

An equivalent principle employed by Hermann and Euler.

The fact that equilibrium subsists between the system P and the system $-W$, may be expressed in still another way. We may say that the system W is *equivalent* to the system P. In this form HERMANN (*Phoronomia*, 1716) and EULER (*Comment. Acad. Petrop.*, Old Series, Vol. VII, 1740) employed the principle. It is substantially not different from that of D'Alembert.

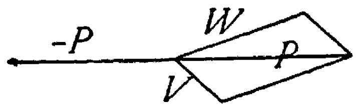

Fig. 170.

5. We will now illustrate D'Alembert's principle by one or two examples.

Illustration of D'Alembert's principle by the motion of a wheel and axle.

On a massless wheel and axle with the radii R, r the loads P and Q are hung, which are not in equilibrium. We resolve the force P into (1) W (the force which would produce the actual motion of the mass if this were free) and (2) V, that is, we put $P = W + V$ and also $Q = W' + V'$; it being evident that we may here disregard all motions that are not in the vertical. We have, accordingly, $V = P - W$ and $V' = Q - W'$, and, since the forces V, V' are in equilibrium, also $V . R = V' . r$. Substituting for V, V' in the last equation their values in the former, we get

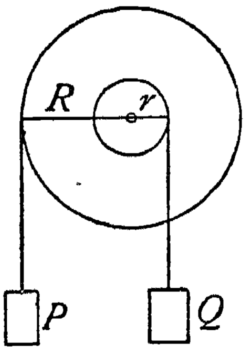

Fig. 171.

$$(P - W) R = (Q - W') r \quad \ldots \ldots \ldots \quad (1)$$

which may also be directly obtained by the employment of the second form of D'Alembert's principle. From the conditions of the problem we readily perceive