Chapter 9 Theoretical Constructions (II): Euler

Leonhard Euler

Although Euler devoted numerous works to the development of fluid mechanics, his most outstanding contribution to the theorisation of this discipline centres on three monographic papers that appeared in the eleventh volume of the *Mémoires de l'Académie de Sciences de Berlin*, 1755, published in 1757. Their titles were: 'Principes généraux de l'état d'équilibre des fluides', 'Principes généraux du mouvement des fluides' and 'Continuation des recherches sur la théorie de mouvement des fluides' ('General principles of the state of equilibrium of fluids', 'General principles of motion of fluids', 'Sequel to the researches on the motion of fluids'). The phased continuity of these titles, the fact that they follow a careful unity of expression and method, and the clarity of the argument all indicate that they were the result of settled reflection upon which Euler wished to establish the basis of the new theory of fluid mechanics. The ideas he expresses are not completely new with him, as he had already written forerunners to some of the works, particularly the 'Principia motus fluidorum' containing the nucleus of the theory, and which had been read in the Berlin Academy in 1752, although it was published in volume VI of the *Novi comentarii academiae scientarum petropolitanae*, in the year 1756/1757, but which appeared in 1761, i.e., after the three Memoirs.[1] The comparison of the contents of this work with the three previous ones enables us to understand the evolution of Euler's thought.[2]

The general principles upon which Euler bases himself are the Newtonian laws expressed in differential form, the complete acceptance of the concept of force, the use of pressure as force per surface unit, and the use of clearly defined systems of Cartesian coordinates. All are expressed with an absolute conceptual clarity, and with admirable accuracy in the formulation of the equations, so much so that, although some of the concepts that Euler deploys had already been underlined or used by previous treatise writers, the redefining, concision and accuracy to which he submits them greatly surpasses all his predecessors. Just as

[1] Cf. Truesdell, 'Rat. Fluid Mech.-12', p. LXII. The date he quotes is the 31 August of this year. The source is Eneström. Euler quotes this work in the monographs. Concerning this see the second one in §.17 and §.29.

[2] In translating Euler's works from Latin or French into English, we have taken as reference the translation made by Truesdell in the 'Rat. Fluid Mech.-12'.

in solid mechanics, many of his formulations come down to us almost without any alteration; and what is more, some of his discoveries have been attributed to other authors.[3]

In our attempts to follow the theoretical revolution of Euler, we begin with the ‘Principia motus fluidorum’, following with the other three works. As with d’Alembert, we shall limit ourselves only to the more relevant matters, as a detailed study would go significantly beyond the goals we have set ourselves.

Finally, a note of a general nature: d’Alembert, as we have seen in the previous chapter, arrived at the constitutive equations of motion as a consequence of the study of a particular problem, which was the search for a new theory for resistance. Euler, by contrast, attacked the problems concerning fluids in a general and very pure way, without reference to any specific application.

Principia motus fluidorum

It is in this work that the ideas of Euler on how to deal with the movement of fluids appear for the first time with clarity, although its scope is limited to non-compressible fluids. The work is divided into two parts: the first refers to the conditions of existence of motion and the second to the motion resulting when forces are applied. In both parts he begins with the assumption of two-dimensional movement, then proceeding to three-dimensional motion. There is no qualitative difference between one and the other, but only one of complexity of the calculations and formulas.

The first question Euler asks is how a fluid is to be understood, because the answer to this question depends on how we formulate the conditions of existence of motion, and how we distinguish possible and impossible motions:

> To this end we must find what characteristic is appropriate to possible motions, separating them from the impossible ones. When this is done, we shall have to determine which one of all possible motions in a certain case ought actually to occur. Plainly we must then turn to the forces which act upon the water, so that the motion appropriate to them may be determined from the principles of mechanics. [§.5][4]

---

[3] Specifically, the equations for perfect, non-compressible fluids continue to be used even today. On the other hand, the fluid mechanics equations are formulated nowadays with respect to fixed axis, called Eulerian, or in moving axes fixed to the actual particle, which we call Lagrangian, even when they are also due to Euler.

[4] Inasmuch as the contrary is not stated, the quotes between inverted commas refer to the ‘Principia motus fluidorum’.

The conditions of fluidity that he uses are contiguity and impenetrability, without any reference to whether the fluid is constituted by corpuscles or another type of particle. He supposes the fluid to be a continuous material, impenetrable and unable to be segregated, which is in accord with the hypothesis of the non-compressibility. Here there is a convergence with Clairaut and d'Alembert, in the sense of escaping from possible physical reality, which all understand as being corpuscular, in order to adopt a continuous mathematical form that persists up to the present day. In the light of the methods of calculation available at the time, the hypothesis of a continuous medium allows differential analysis, which was already well developed, to be used. It is worth mentioning how the three mathematicians distanced themselves from what they believed to be reality, namely the corpuscular nature of the fluid, to go into an imaginary construct, i.e., a continuous fluid.

Euler says:

> I assume the fluid to be such it is impossible for it to be forced into a lesser space, nor can its continuity be interrupted. I establish with certainly that no empty space remains in the middle of the fluid during movement, but that its continuity is conserved uninterruptedly. [§.6]

These conditions have to be established for the entire amount of the fluid and for any point of it whatsoever, and with this aim he calculates the mathematical conditions.

In order to study the continuity, Euler begins by looking at two-dimensional motion, that is to say motion in a plane. In this plane he takes a differential element of the fluid consisting in a rectangular triangle, and imposes the condition that the enclosed surface be constant during its temporal evolution. This is the equivalent of saying that the quantity of material contained in its interior must remain constant. Remember that d'Alembert had already imposed this condition with his requirement of the constancy of volume during motion.

Let the triangle of fluid be designated as *NML* (Fig. 9-1) at the instant t, and which in $t + dt$ had evolved up to $N'M'L'$, which would not necessarily be rectangular, but which has the same initial surface. If the velocity of point L is the vector $\boldsymbol{v}(x,y)$ and the components along the axes OX and OY are designated as $u(x,y)$ and $v(x,y)$, the following equations will be verified:

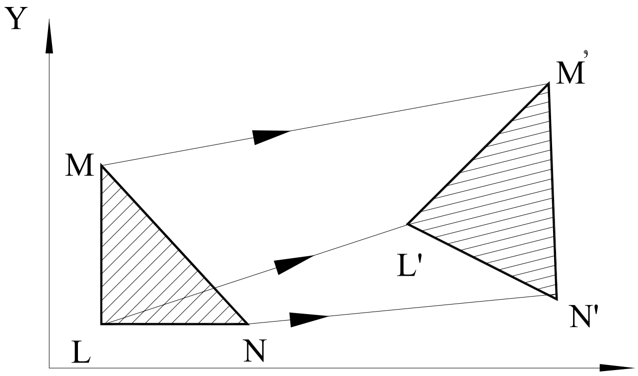

Fig. 9-1. Triangle evolution

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy \tag{9.1}$$

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy \tag{9.2}$$

He supposes them to be exact differentials, therefore the equality of the cross-derivatives must be established:

$$\frac{\partial^2 v}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y} \tag{9.3}$$

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 v}{\partial x \partial y} \tag{9.4}$$

With the help of the equations [9.1] and [9.2] it is possible to determine the velocities of the two vertices of the triangle, knowing the velocities of the other and its derivatives. Having chosen the vertex of the right angle L as base, he obtains[5]:

$$\vec{v}_L = u\vec{i} + v\vec{j} \tag{9.5}$$

[5] Hereinafter vectorial notation and the unitary vectors ***i*** and ***j*** will be used in order to simplify the presentation, although Euler wrote the components separately.

$$\vec{v}_N = \left(v + \frac{\partial v}{\partial x} dx \right) \vec{i} + \left(v + \frac{\partial v}{\partial y} dy \right) \vec{j} \qquad [9.6]$$

$$\vec{v}_M = \left(u + \frac{\partial u}{\partial x} dx \right) \vec{i} + \left(u + \frac{\partial u}{\partial y} dy \right) \vec{j} \qquad [9.7]$$

When a time dt has elapsed, each vertex will travel from its position $\boldsymbol{r}$ to $\boldsymbol{r} + \boldsymbol{v}dt$, both $\boldsymbol{r}$ and $\boldsymbol{v}$ being vectors. The result for each one of them will be:

$$\vec{r}_L = x\vec{i} + y\vec{j} \Rightarrow (x + udt)\vec{i} + (y + vdt)\vec{j} \qquad [9.8]$$

$$\vec{r}_N = (x + dx)\vec{i} + y\vec{j} \Rightarrow \left[x + dx + \left(u + \frac{\partial v}{\partial x} dx \right) dt \right] \vec{i} + \left[y + \left(v + \frac{\partial v}{\partial y} dy \right) dt \right] \vec{j} \qquad [9.9]$$

$$\vec{r}_M = x\vec{i} + (y + dy)\vec{j} \Rightarrow \left[x + \left(u + \frac{\partial u}{\partial x} dx \right) dt \right] \vec{i} + \left[y + dy + \left(v + \frac{\partial v}{\partial y} dy \right) dt \right] \vec{j} \qquad [9.10]$$

Where the initial coordinates and final coordinates of each vertex are expressed. The area of the initial triangle was ½$dxdy$, while that of the displaced one, after an involved calculation [§.19], turns out to be:

$$\frac{1}{2} dxdy \left[1 + \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) dt + \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial v}{\partial x}\frac{\partial u}{\partial y} \right) dt^2 \right] \qquad [9.11]$$

Which made equal with the first, leads to the following expression:

$$\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) + \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial v}{\partial x}\frac{\partial u}{\partial y} \right) dt \qquad [9.12]$$

Neglecting the terms of a higher order it simplifies to:

$$\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \qquad [9.13]$$

To which Euler adds that 'unless this condition holds, the motion of the fluid cannot take place' [§.20].

This last equation had already been found by d'Alembert, starting, just as Euler did, from kinematic conditions, although d'Alembert had also extended it to compressible fluids, something Euler would not do until his monographs of 1755.

After the development of the plane movement he goes on to the three-dimensional case [§.21]. The procedure is the same, now supposing that the initial element of fluid is a rectangular tetrahedron instead of a triangle. The mathematical calculation is considerably more involved and bothersome, as the factor adding a new dimension causes not only one more equation to appear, but the corresponding cross-equations. The final result is the following expression [§.35]:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}+\left(\frac{\partial(u,v)}{\partial(x,y)}+\frac{\partial(v,w)}{\partial(y,z)}+\frac{\partial(u,w)}{\partial(x,z)}\right)dt+\frac{\partial(u,v,w)}{\partial(x,y,z)}dt^2=0 \quad [9.14]$$

We note[6] that there now appear terms of the order dt^2, which is one order more than in the plane case. Neglecting these just as much as those of an inferior order, and justifying this on the grounds that they are differential quantities, he ends with the equation:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \quad [9.15]$$

With current calculus tools, if the field of velocities $\boldsymbol{v}(\boldsymbol{x},t)$ is assimilated to a vector field, which is certainly what it is, the last equation, nowadays called a

[6] The following algorithm is used in the formula that follows in order to simplify its writing:

$$\frac{\partial(u,v)}{\partial(x,y)}=\begin{vmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{vmatrix} \qquad \frac{\partial(u,v,w)}{\partial(x,y,z)}=\begin{vmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z}\\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z}\end{vmatrix}$$

'continuity' equation, could be written with the help of a divergence[7] operator such as:

$$div\ \vec{v} = \nabla \cdot \vec{v} = 0 \qquad [9.16]$$

Where *div* is the aforementioned operator which is the scalar product of the 'nabla' operator and the velocity,

Up to now, Euler has used kinematic arguments. The next step is to introduce dynamic conditions, that is to say forces, and, as a result, accelerations. In this respect, he makes the following reflection when beginning the second part:

> Once exposed these things that pertain only to possible motion, we now also investigate the nature of motion that can truly subsist in the fluid. [§.39]

This is the same as saying that the conditions [9.13] and [9.15] are necessary, but not sufficient. The process that follows begins by determining the accelerations, starting from the kinematics of motion, in order to continue to introduce the forces of pressure and gravity. At the end he generalises the results for other classes of mass forces.[8] He does this entire first in its two-dimensional aspect, and then afterwards in its three-dimensional aspect. In this approach he makes a clear separation between the kinematic aspects and the dynamic ones, which, according to Truesdell,[9] occurs for the first time. However, this separation is possible exclusively because of the incompressibility of the fluid, and in the case of compressible fluids the kinematics must be complemented by the mass properties as can be seen in the monographs of 1755.

---

[7] In a three-dimensional vector field, of which the two-dimensional is a particular case, divergence is defined as: $div\ \vec{f} = \dfrac{\partial f_x}{\partial x} + \dfrac{\partial f_y}{\partial y} + \dfrac{\partial f_z}{\partial z}$

The divergence is a scalar magnitude. A physical interpretation of it can be given as the tendency a particle would have of concentrating or diverging when moving throughout the field following its force lines. In the case that the field was of velocities, divergence would measure the tendency of the fluid to vary in volume, in such a way that the volume is invariable when the divergence is zero; which is precisely the case we are dealing with. It is customary to write $\nabla \cdot \vec{f}$ instead of $div\ \vec{f}$, where ∇ is the nabla operator which is defined as:

$$\nabla = \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}$$

This operator is applicable to scalar and vectors and we shall make use of it from henceforth.

[8] Mass forces are understood to be those whose magnitude is proportional to the mass of the particle upon which it acts.

[9] Cf. 'Rat. Fluid Mech. -12', p. LXXI.

In order to find the accelerations of any particle whose velocity components are *u* and *v*, he takes the variation of these when they move in a time *dt.* Deriving with respect to time and space he obtains the result:

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial t}dt \qquad [9.17]$$

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy + \frac{\partial v}{\partial t}dt \qquad [9.18]$$

As the particle in question moves precisely with the velocities *u* and *v*, the displacement will be $dx = udt$ and $dy = vdt$, which introduced into the previous equations lead to:

$$\frac{du}{dt} = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \frac{\partial u}{\partial t} \qquad [9.19]$$

$$\frac{dv}{dt} = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + \frac{\partial v}{\partial t} \qquad [9.20]$$

Which are the 'accelerating forces' along the axes OX and OY, 'by which the forces that are acting on particle of water must be equal' [§.41][10] and which he goes on to make equal to the forces acting upon it. Among the possible acting forces he enumerates three: gravity, friction and pressure. Of the first he says its effect,

---

[10] In the text of Euler it says precisely:

Vis acceleratirx secundum AL = 2(Lu + lv + L)
Vis acceleratirx secundum AB = 2(Mu + mv + M)

that correspond to the formulas above expressed with the exception of factor 2, whose reason to exist he radicates in the peculiar system of units which Euler employs, in which the value of the acceleration of gravity is ½. See his work 'Découverte d'un nouveau principe de méchanique' *Mém. Acad. Berlin VI (*1750), where he expounds for the first time the Newtonian equation of the second principle in a differential form to the time , in the following manner:

2MX = P; 2MY = Q; 2MZ = R;

The justification of this value is found in the 'Théorie plus complete des machines qui sont mises en mouvement par la reaction de l'eau' that appears in Vol. X of the *Mém. Acad. Berlin* (1754) He repeats the '2' again now. We shall ignore it. See also Truesdell, 'Rat. Fluid mech-12', p. XLIII.

> [I]f the plane of motion is horizontal, is to be taken as zero. But if instead the plane is inclined, and the axis *OY* follows that slope, due to gravity a constant accelerating force of magnitude α arises. [§.42][11]

He leaves friction to one side for the moment, and concentrates on pressure:

> Moreover, the pressure must be brought into the calculation. This pressure is the reciprocal action of the water particles upon each other. Each particle is pressed on every side by its neighbours, and as this pressure is not equal everywhere, to this extent motion is communicated to the particles. In all places the water will simply find itself in a certain state of compression similar to that of quiet water at a certain height finds itself. Therefore, this height (at which quiet water is found to be in a state similar to compression) can be conveniently employed to represent the pressure at an arbitrary point l of the fluid. Therefore let that height (or depth) expressing the state of compression at l be p; a certain function of the coordinates x and y, and if the pressure at l varies also with the time, then the time will also enter into the function p. [§.43]

This paragraph is important. On one hand Euler defines the concept of pressure as a force over a unit surface, although he still does not do this with total clarity. It is not that this is new, as it can be detected in d'Alembert, though he does not explain it so clearly, and other authors, such as Johann Bernoulli, liken the pressure to a total force upon a section of a fluid. As regards measuring the magnitude of pressure, Euler identifies it with the height of water column, which is not new either. We recall that Daniel Bernoulli introduced a water height manometer in his experiments, and that Pitot based his experiments on these apparatus. There are inklings of this idea even in Newton, but in Euler the idea of acquires the value of the measurement, and not of an equivalent force, a very important difference. Nevertheless, the concept will become even clearer in the succeeding monographs

In order to introduce the pressure equation he defines a differential element of fluid, which will now be a rectangle instead of triangle, designated as *NLMO* in Fig. 9-2, and which he imagines inside a fluid field whose pressure is the function of time and position. If the pressure on a vertex of this element is $p(x,y)$, then on the others it will be:

[11] This value depends on the choice of the system of axis.

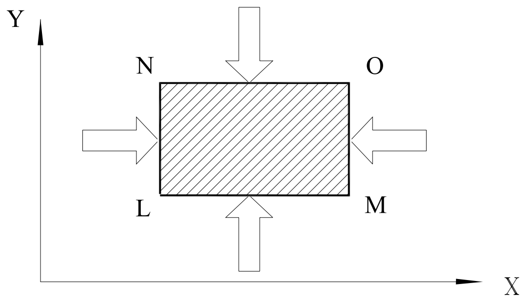

Fig. 9-2. Pressure forces

$$\mathbf{L}:\ p \qquad\qquad \mathbf{M}:\ p+\frac{\partial p}{\partial x}dx \qquad [9.21]$$

$$\mathbf{N}:\ p+\frac{\partial p}{\partial y}dy \qquad\qquad \mathbf{O}:\ p+\frac{\partial p}{\partial x}dx+\frac{\partial p}{\partial y}dy \qquad [9.22]$$

Therefore, the result of the forces produced by the pressures on the sides of the rectangle along both axis, will be:

$$\mathbf{OX}\text{ Axis:} \qquad -\frac{\partial p}{\partial x}dxdy \qquad [9.23]$$

$$\mathbf{OY}\text{ Axis:} \qquad -\frac{\partial p}{\partial y}dxdy \qquad [9.24]$$

These forces, plus the gravity along the axis *OX*, will be the 'accelerating forces', and by making them equal to the accelerations given in equations [9.19] and [9.20] the following two equations are obtained[12]:

$$g-\frac{1}{\rho}\frac{\partial p}{\partial x}=u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+\frac{\partial u}{\partial t} \qquad [9.25]$$

[12] In the formulas that follows the density, ρ, is introduced as the deviser of the pressure, the aim being to make the equations coherent.

$$-\frac{1}{\rho}\frac{\partial p}{\partial x} = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + \frac{\partial v}{\partial t} \qquad [9.26]$$

Moreover, the variation of pressure with time and space can be written as:

$$dp = \frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy + \frac{\partial p}{\partial t}dt \qquad [9.27]$$

Introducing the values of the two previous equations into this one, we arrive at the following expression for *dp:*

$$\frac{dp}{\rho} = gdx - \left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \frac{\partial u}{\partial t}\right)dx - \left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + \frac{\partial v}{\partial t}\right)dy + \frac{1}{\rho}\frac{\partial p}{\partial t}dt \qquad [9.28]$$

Which he says that it must be integrable. He states that 'the term *g* is *per se* integrable and nothing is defined for $\partial p/\partial t$, and by nature the differentials need to be exact' [§.46]. Therefore, it will be necessary to comply with the equality of the cross-derivatives between the other two terms:

$$\frac{\partial}{\partial y}\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \frac{\partial u}{\partial t}\right) = \frac{\partial}{\partial x}\left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + \frac{\partial v}{\partial t}\right) \qquad [9.29]$$

Which, after the corresponding manipulations become the following:

$$\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + \frac{\partial}{\partial t}\right)\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) = 0 \qquad [9.30]$$

That says [§.47] that it is completely satisfied by:

$$\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} \qquad [9.31]$$

This is an important question, as it is true that the fluids which fulfil this last condition will also verify the previous one, but the opposite is not true. This means that the condition [9.31] is sufficient, but not necessary. Euler limits the possible motions to a single category, which later would be called 'irrotational motions'. Later on, in successive works he rectifies having considered only this solution. D'Alembert had also found himself in a similar situation.[13] Truesdell insisted that Euler's mistake was due to d'Alembert's influence,[14] although one can easily interpret it as Euler having chosen the easiest and most obvious solution of the equation.

Before continuing, we must introduce a specification which Euler fails to mention. In equation [9.30], the sum $\partial u/\partial x + \partial v/\partial y$ is zero, as had already been found in equation [9.13] as a result of the continuity, which simplifies the formulation.

We take note that having started from the pressure as the only acting force, Euler arrived at some relationships in which this parameter disappeared in favour of the velocities. Now, within the irrotationality hypothesis the pressure returns, for which he introduces the results found in the expression containing the pressure, 'hence now we shall be able to ascertain the pressure p itself, which is absolutely necessary for the perfect determination of the motion of the fluid' [§.49]. With the condition [9.31], the pressure equation [9.28] becomes:

$$\frac{dp}{\rho} = gdx - udu - vdv - \frac{\partial u}{\partial t}dx - \frac{\partial v}{\partial t}dy \qquad [9.32]$$

The condition that $udx + vdy$ is an exact differential allows him to introduce the function S, which is the potential of the velocities.[15]

$$dS = udx + vdy \qquad [9.33]$$

And after some transformations he arrives at:

$$\frac{dp}{\rho} = gdx - udu - vdv - d\frac{\partial S}{\partial t} \qquad [9.34]$$

This is already an integrable equation whose result is:

---

[13] Cf. *Essai d'une nouvelle théorie de la résistance des fluides*, §. 48–49. Although what d'Alembert really did was to demonstrate if the expression [9.31] was substituted by another of the type: $\partial v/\partial x = \partial u/\partial y + \lambda$ this will only fulfill the conditions of potentiality if $\lambda = 0$.

[14] Cf. 'Rat. Fluid. Mech.-12', p. LXXIII.

[15] That is to say, it verifies $\vec{v} = \nabla S$.

$$\frac{p}{\rho} = gx - \frac{1}{2}(u^2 + v^2) - U + Cte \qquad [9.35]$$

As the total velocity at a point is $V = \sqrt{u^2 + v^2}$, what he obtains is Bernoulli's equation for a non-stationary motion. We shall come back to this potential function *S* once we have analysed the three-dimensional case.

If we are dealing with motion in three dimensions, the arguments will follow the same lines although, just as in the case of continuity, with a greater degree of complexity. There will be a third component of the velocity *w*, corresponding to the projection along the *OZ* axis, and on establishing the accelerations we shall have three equations that replace the two [9.19 and 9.20].

$$\frac{du}{dt} = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{\partial u}{\partial t} \qquad [9.36]$$

$$\frac{dv}{dt} = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + \frac{\partial w}{\partial t} \qquad [9.37]$$

$$\frac{dw}{dt} = u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} + \frac{\partial w}{\partial t} \qquad [9.38]$$

These equations may be written with vectorial notation as:

$$\vec{a} = \vec{v}\cdot\nabla\vec{v} + \frac{\partial\vec{v}}{\partial t} \qquad [9.39]$$

In which the nabla operator is used as the generator of the velocity gradient. It would be even simpler to use the concept of the substantial derivative, which would result in[16]:

[16] The gradient function is applied to a scalar field or to each component of a vectorial field. In the first of the cases if the field is represented by φ the gradient would be the vector:

$$\nabla\varphi = grad\varphi = \frac{\partial\varphi}{\partial x}\vec{i} + \frac{\partial\varphi}{\partial y}\vec{j} + \frac{\partial\varphi}{\partial z}\vec{k}$$

The direction of the gradient is the variation of the property φ when it moves through the field in such a way that $d\varphi = \nabla\varphi\cdot dr$ indicates the variation of φ when the position changes the distance *dr*. As regards the substantial derivative of the property φ this is defined as follows:

$$\vec{a} = \frac{D\vec{v}}{Dt} \qquad [9.40]$$

The condition that the expression dp/ρ is an exact differential, leads to three equalities among the cross-derivatives, which will be the equivalent of condition [9.29]. When he develops them he obtains the following three equations [§.59] equivalent to equation [9.30] of the two-dimensional case:

$$\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}+w\frac{\partial}{\partial z}+\frac{\partial}{\partial t}\right)\left(\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}\right)+\frac{\partial u}{\partial z}\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}\frac{\partial w}{\partial x}=0 \qquad [9.41]$$

$$\left(\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}+u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}+w\frac{\partial}{\partial z}+\frac{\partial}{\partial t}\right)\left(\frac{\partial v}{\partial z}-\frac{\partial w}{\partial y}\right)+\frac{\partial v}{\partial x}\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\frac{\partial u}{\partial y}=0 \qquad [9.42]$$

$$\left(\frac{\partial u}{\partial x}+\frac{\partial w}{\partial x}+u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}+w\frac{\partial}{\partial z}+\frac{\partial}{\partial t}\right)\left(\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\right)+\frac{\partial u}{\partial y}\frac{\partial w}{\partial z}-\frac{\partial w}{\partial y}\frac{\partial v}{\partial x}=0 \qquad [9.43]$$

A system which is sufficiently established with the following three values:

$$\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}=0;\quad \frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}=0;\quad \frac{\partial v}{\partial z}-\frac{\partial w}{\partial y}=0 \qquad [9.44]$$

Which correspond to an irrotational motion. The presentation of the equations obtained by Euler is simplified using modern vectorial notation. Firstly, the 'vorticity' is defined as the curl[17] of the velocity:

$$\frac{D\varphi}{Dt}=\frac{\partial\varphi}{\partial t}+\vec{v}\cdot\nabla\varphi$$

The significance is the variation of the property φ of a particle when this moves following a trajectory. That is to say with axis fixed to the particle.

[17] The *curl* of a vector field is another vector field defined as:

$$curl\ \vec{v}=\nabla\times\vec{v}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ u & v & w\end{vmatrix}$$

$$\vec{\omega} = curl\ \vec{v} = \nabla \times \vec{v} \qquad [9.45]$$

Which is a vector whose three components are precisely the first members of the antepenultimate equation [9.44]. With the help of vorticity, equations [9.41]–[9.43] can be written as the much simpler equation:

$$\frac{D\vec{\omega}}{Dt} = (\vec{\omega} \cdot \nabla)\vec{v} \qquad [9.46]$$

Examining the foregoing confirms that it satisfied the cases of $\omega = 0$, that is, an irrotational motion, but it does not do so necessarily, as these cases are only one class of the possible motions that satisfies equation [9.46], but obviously there are more possible motions. For the two-dimensional motion, the previous equation becomes:

$$\frac{D\vec{\omega}}{Dt} = 0 \qquad [9.47]$$

That corresponds to equation [9.30]. The disappearance of the term $(\vec{\omega} \cdot \nabla)\vec{v}$ is easy to explain as the curl vector is perpendicular to the plane of motion, and this plane contains the gradient vector, therefore the scalar product of both will be zero[18]

In order to capture the meaning of this vector field we suppose that small parallelepipeds move over the vector field. Now then, the curl will indicate the tendency to rotate upon themselves that these elements have. In the event that it as zero, they would shift without turning, which is designated as irrotational.

[18] These equations can be deducted starting from the equation of the momentum:

$$\frac{D\vec{v}}{Dt} = -\frac{1}{\rho}\nabla p + \vec{f} \qquad [1]$$

The first member can be written as:

$$\frac{D\vec{v}}{Dt} = \frac{\partial \vec{v}}{\partial t} + (\vec{v}\,\nabla)\vec{v} = \frac{\partial \vec{v}}{\partial t} + \nabla\left(\frac{v^2}{2}\right) - \vec{v} \times (\nabla \times \vec{v}) \qquad [2]$$

Introducing this in equation [1], together with the definition of vorticity, and applying the curl function to both members we end up with:

$$\nabla \times \frac{\partial \vec{v}}{\partial t} + \nabla \times \frac{\nabla v^2}{2} - \nabla \times (\vec{v} \times \vec{\omega}) = -\nabla \times \frac{\nabla p}{\rho} + \nabla \times \vec{f} \qquad [3]$$

As regards pressures, he repeats the process adding a new variable, thus arriving at the expression:

$$\frac{dp}{\rho} = gdx - udu - vdv - wdw - \frac{\partial u}{\partial t}dx - \frac{\partial v}{\partial t}dy - \frac{\partial w}{\partial t}dz \qquad [9.48]$$

This equation substitutes the two-dimensional one deduced previously [9.32].

An interesting detail, analysed by Euler, occurs where the field of velocities is integrable, ***S*** being its integral, i.e., there is a potential function.[19] Simplifying his transformations, the velocities will be:

$$u = \frac{\partial S}{\partial x}; \qquad v = \frac{\partial S}{\partial y}; \qquad w = \frac{\partial S}{\partial z} \qquad [9.49]$$

Therefore, it will also be established that:

$$\frac{d}{dt}(udx + vdy + wdz) = \frac{d}{dt}\left(\frac{\partial S}{\partial x}dx + \frac{\partial S}{\partial y}dy + \frac{\partial S}{\partial z}dz\right) = \frac{dS}{dt} \qquad [9.50]$$

Resulting in the expression for pressure:

$$\frac{p}{\rho} = C - gz - \frac{1}{2}V^2 - \frac{dS}{dt} \qquad [9.51]$$

In the assumption that the forces are derived from potential and that the density is constant, the two add-ins of the left are cancelled out, as is the second on the right. Therefore, recalling the definition of vorticity, we arrive at:

$$\frac{\partial \vec{\omega}}{\partial t} = \nabla \times (\vec{v} \times \vec{\omega}) \qquad [4]$$

Going on to the substantial derivative we end with:

$$\frac{D\vec{\omega}}{Dt} = (\vec{\omega} \cdot \nabla)\vec{v} - \vec{\omega}\nabla \cdot \vec{v} \qquad [5]$$

Which like the density is constant $\nabla\vec{v} = 0$, giving the [9.46] results. We note that Truesdell [Rat. Fluid Mech.-12, p. LXXII] points out this transformation. However, he presents the previous equation [5] as being equivalent to equations [9.41] and [9.42] of Euler, which would only be true if the density were not constant.

[19] Euler takes two addends, one of which he calls *U*, which is variable with the time. We shall skip this step.

which is the generalisation of the equation of Bernoulli for non-stationary motions.

On the other hand, he recalls the existence condition of the motion, expressed in equation [9.15], in which he introduces the velocities derived from the potential *S*, as presented in equation [9.49]. The resulting equation is:

$$\frac{\partial^2 S}{\partial x^2} + \frac{\partial^2 S}{\partial y^2} + \frac{\partial^2 S}{\partial z^2} = 0 \qquad [9.52]$$

This is an equation which is called the potential equation or 'Laplacian' and which is usually written as $\nabla^2 S = 0$ or $\Delta S = 0$.

As a complement to this last formulation, Euler tries to find some kind of solution [§.68-ff] supposing that the function *S* takes the form:

$$S = (Ax + By + Cz)^n \qquad [9.53]$$

He applies the previous condition [9.52] to this one and he arrives at the following relation between the parameters:

$$n(n-1)(A^2 + B^2 + C^2)(Ax + By + Cz)^{n-2} = 0 \qquad [9.54]$$

He devotes a lot of attention to these functions, in particular to the solution corresponding to $n = 1$, where he finds that it is the equivalent to a shift in space at constant velocity, as can be easily deduced by applying equation [9.49] to the function $S = Ax + By + Cz$; and where the fluid behaves like a rigid solid. Following this thread Euler ponders whether 'it is legitimate to suspect in other cases that the motion of the fluid can also be assimilated to the motion of a solid body, whether rotational or with any other anomaly' [§.75]. With this aim, he launches himself into the search of relations having the velocities that makes possible the motion of the fluid as a solid rotation, A situation complementary to that of the shift. After a series of calculations, he finds that for this type of motion to be possible, the matrix $\left|\partial v_i / \partial x_j\right|$ must be anti-symmetrical, that is:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = \frac{\partial z}{\partial w} = 0 \qquad [9.55]$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}; \qquad \frac{\partial u}{\partial z} = -\frac{\partial w}{\partial x}; \qquad \frac{\partial v}{\partial z} = -\frac{\partial w}{\partial y}; \qquad [9.56]$$

Now the last three equalities contradict the condition found for the existence of fluid motion, which was [9.44], which indicates that this motion will not be compatible with these conditions unless the velocities are constant. In response, he ends by saying: ‘thus, it is obvious that it is only in this case [$v_i = cte$] where the motion of a fluid can be assimilated to that of a solid body’ [§.77] The interest of this statement lies in the fact that the solid rotation is an example of motions that are not covered by the conditions of existence.[20]

The next step he takes is to extend the forces to ones other than weight and pressure, these being the only ones he has handled up to now. In order to do this, he extends the theory to assumptions where other external forces exist. Instead of using a new approach to the equations, what Euler does do is to introduce an acceleration potential *T*, so that:

$$T = \frac{1}{2}(u^2 + v^2 + w^2) + \frac{\partial S}{\partial t} \qquad [9.57]$$

If the new external forces are of the type $Qdx + qdy + \varphi dz$, the expression for pressure will result in:

$$\frac{p}{\rho} = C + \int (Qdx + qdy + \quad dz) - T \qquad [9.58]$$

In concluding our treatment of the ‘Principia motus fluidorum’, the last point of significance is the specification he makes for fluids moving in ducts, declaring that ‘everything which has hitherto been said concerning the motion of a fluid through tubes is easily derived from these principles’ [§.87]. The final equation he arrives at is:

$$\frac{p}{\rho} = C + \int (Qdx + qdy + \quad dz) - \frac{S_0^2}{S^2}V_0^2 - 2\frac{dV_0}{dt}\int \frac{S_0}{S}\, ds \qquad [9.59]$$

[20] Truesdell (‘Rat Fluid Mech-12’, note 2, p. LXXIV) quotes Professor Kuert’s remark that makes plain his puzzlement that neither Euler nor d’Alembert found counter examples to their theories in these motions. That a solid rotation is not irrotational is very easy to see. A irrotational motion requires that a particle does not turn in its movement, which in turn requires that the law of velocities be inverse to the distance for the center of rotation, that is to say of the $v = k/r$ type. Now, in a solid rotation the velocity is proportional to the distance, that is, $v = kr$.

Where S_0 and $S(s)$ are the cross sections of the pipe and V_0 the velocity in the cross section S_0 taken as a reference. This equation is an extension of the one given by d'Alembert, and of course turns out to be the equation of Bernoulli, an equation that begins to occupy second place with respect to the general hypotheses of hydrodynamics.

Finally, it is useful to compare Euler's method with that of d'Alembert. They clearly have differing approaches to the problem. The latter begins by obtaining an equation that links the pressures[21] to the velocities for a pipe with a very narrow current tube; next, using kinematic and dynamic considerations, he obtains the field of velocities defined by differential equations in which only the velocities intervene, and whose solution depends on the shape of the body. Once these are solved, the pressures at each point in the fluid can be deduced. By contrast, Euler first introduces the pressures as forces, and with these, together with the equations of general dynamics, he establishes some relations among the velocities alone, just like d'Alembert. He goes back to introduce the pressures at the end of the Bernoulli's equations, which will give the pressures at specific points once the velocities are obtained. The elements brought into play are the same, although in a different order, and with a different methodology, in which Euler deals with the dynamic concepts with greater clarity. Apart from this, Euler tackles the three-dimensional problem, while d'Alembert limits himself to this last case, be it in the plane or axisymmetric case.

General principles of the state of equilibrium of the fluids

This first of the three monographs of the series is dedicated to hydrostatics. Euler begins by a declaration of his aims:

> Here I propose to develop the principles upon which all hydrostatics, or the science of the equilibrium in fluids, is founded. … I include in my investigations not only fluids that have the same density in all their parts … but also those fluids composed of particles of different density. … Moreover, I shall not limit my investigations to the single force of gravity, but will extend them to any forces. [§.1][22]

As a consequence of the general nature of the research, the earlier explanations 'are only a very particular case of those which I am going to establish here' [§.2] he notes however

[21] See Chapter 8, 'Body in flowing currents' of this book.
[22] The quotes between brackets follow the monograph 'Principes généraux de l'état d'equilibre des fluides'.