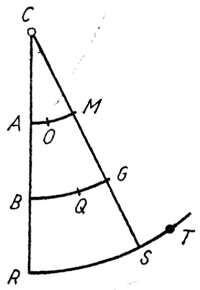
Module 2

Principles of mechanics in the XVIII century



UNIVERSITY OF COPENHAGEN

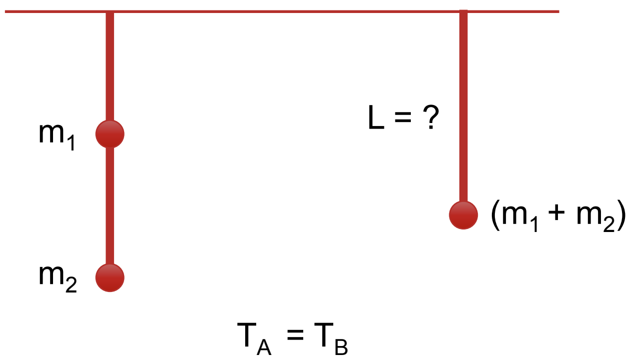
Mechanics in the XVIII century

From Newton's *Principia* (1687) to Lagrange's *Mécanique analytique* (1788)

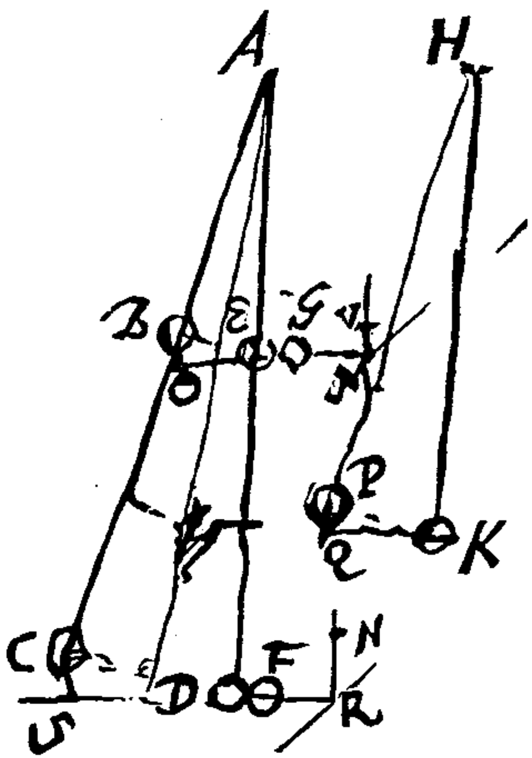
- Newtonian paradigm (central forces, point particles)
- Many challenges: e.g. interacting bodies subjected to constrains, continuous mechanics;
- Fruitful interplay between physics and mathematics;
- The "analytical revolution": Leibniz's calculus applied to mechanics (Varignon, Bernoullis, Euler's Mechanics (1736));
- New principles: e.g. laws of impact, principle of *vis viva*, D'Alembert's, Least Action - scalar (energy) formulations;
- Controversies: Notion of force, conservation of mv or mv^2

Center of oscillation problem

- A simple pendulum with all its mass concentrated at that point will have the same period of oscillation as the compound pendulum.



Center of oscillation problem (Huygens)



1. Isochrony: $v_B/v_C/v_P = b/c/x$

2. Heights fallen and lengths: $PQ/CS = HK/AD$

3. Heights fallen, velocities exchanged, heights acquired:
$RN/PQ = v^2_C/v^2_P$ and $MV/PQ = v^2_B/v^2_P$

4. Huygens assert: the composite center of gravity of the spheres G and F, after they have taken motion from E and D and have converted it upward as far as they can, that this center rises to the same height as that of the center of gravity of the spheres B and C.

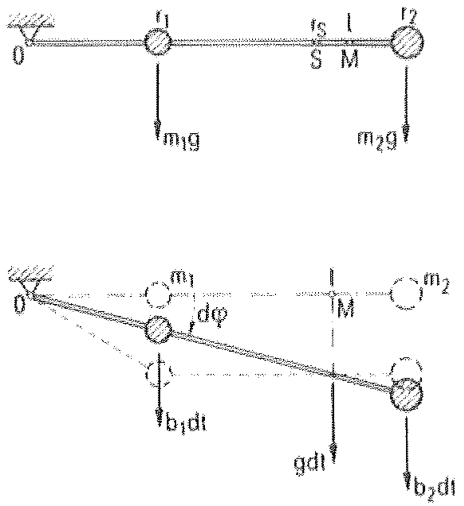
$$RN.F + MV.G = CS.C + BO.B$$

Combining 1., 2., 3. and 4.

$$x = (C.c^2 + B.b^2)/(C.c + B.b)$$

Details in Mahoney (1980)

Center of oscillation problem (James) Bernoulli



In an infinitesimal *dt*:

- m_1 experiences an "upward force"
- m_2 experiences a "downward force"
- There is a point M that falls "freely"

Geometry:

- $a_1 = (r_1/\ell)g \qquad a_2 = (r_2/\ell)g \qquad (1)$

Principle: "Lost" and "Gained" forces in equilibrium

- $m_1 .(g - a_1). r_1 + m_2 .(g - a_2). r_2 = 0 \qquad (2)$

Substituting (1) in (2)

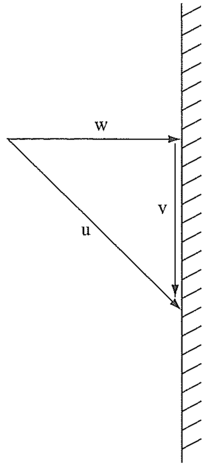
$$\ell = (m_1 r_1^2 + m_2 r_2^2)/(m_1 r_1 + m_2 r_2)$$

For n masses

$$\ell = \Sigma(m_i r_i^2)/\Sigma(m_i r_i)$$

D’Alembert’s principle (3 motions)

Collision of a “hard” particle



u velocity before

w velocity “lost”

If the body were animated with the lost velocity alone, equilibrium would subsist.
Thus,

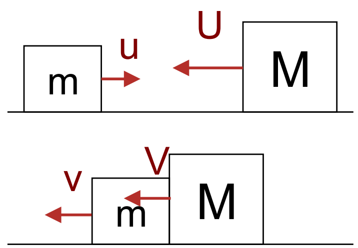
w ⊥ wall

v velocity after

Inelastic collision

before

after



u = v + (u – v)

U = V + (U – V)

u impressed velocity

v actual velocity

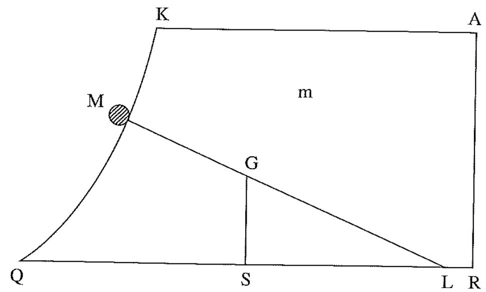
(u – v) “lost” velocity

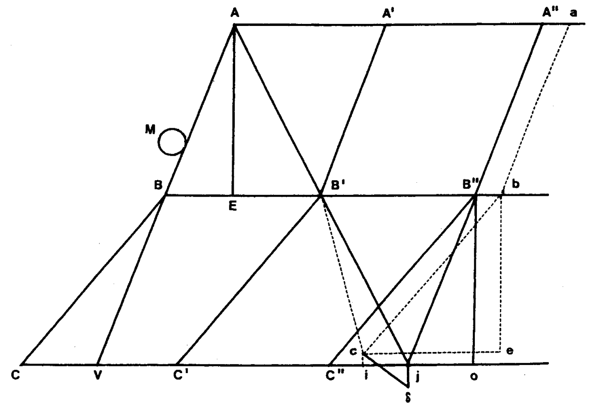
The application of the “lost” velocities to m and M must produce equilibrium.
Thus,

m(u – v) + M(U – V) = 0

v (or V) = (mu + MU)/(m + M)

Problem X





$B'B''$: composed of $B'b$ and $-(B''b)$
$B'\delta$: composed of $B'c$ and $c\delta$.

Impressed = Actual + Lost/Gained

Details in Fraser (1985)

D’Alembert’s principle

Let A, B, C, etc.... be the bodies that constitute the system and suppose that the motions *a, b, c*, etc. ... are **impressed** on them; let there be forces, arising from their mutual action, which **change** these into the [actual] motions *a’*, *b’*, *c’*, etc… It is clear that the motion *a* impressed on the body A can be compounded of the motion *a’* which it acquires and another motion α. In the same way the motions *b*, *c*, etc.... can be regarded as compounded of the motions *b’* and β, *c’* and χ , etc ... From this it follows that the motions of the bodies A, B, C, etc … would be the same, among themselves, if instead of their having been given the impulses *a*, *b*, *c*, etc.... they had been simultaneously given the twin impulsions *a’* and α, *b’* and β, *c*’ and χ , etc... Now, by supposition, the bodies A, B, C, etc… have assumed, by their own action, the motions *a*’, *b*’, *c*’, etc… Therefore the motions α, β, χ , etc ... must be such that they do not disturb the motions *a*’, *b’*, *c’,* etc… in any way. That is to say, **that if the bodies had only received the motions α, β, χ, etc... these motions would have been cancelled out among themselves, and the system would have remained at rest**.

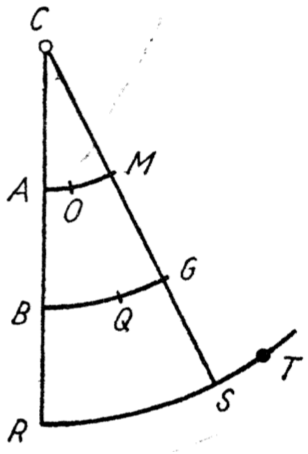
Traité de Dynamique (1743)

D'Alembert's principle

From this results the following principle for finding the motion of several bodies which act upon each other. Decompose each of the motions *a*, *b*, *c*, etc... which are impressed on the bodies into two others, *a'* and α, *b'* and β, *c'* and χ, etc… which are such that if the motions *a', b', c',* etc... had been impressed on the bodies, they would have been retained unchanged; and if the motions α, β, χ, etc. ... alone had been impressed on the bodies, **the system would have remained at rest**. It is clear that *a', b', c'*, etc... will be the motions that the bodies will take because of their mutual action. This is what it was necessary to find (QED).

Traité de Dynamique (1743)

Center of oscillation problem (D'Alembert's principle)



"Problem. — To find the velocity of a rod CR fixed at C, and loaded with as many weights as may be desired, under the supposition that these bodies, if the rod had not prevented them, would have described infinitely short lines AO, BQ, RT, perpendicular to the rod, in equal times.

Impressed velocities (*a*, *b*, *c*): *AO*, *BQ*, *RT*

Actual velocities (*a'*, *b'*, *c'*): *AM*, *BG*, *RS*

Impressed = Actual + Lost/Gained

RT = *RS* + *ST*
BQ = *BG* - *GQ*
AO = *AM* - *MO*

By our principle, the lever *CAR* **would have remained in equilibrium** if the bodies *R*, *B*, *A* had received the motions *ST*, - *GQ*, - *MO* alone.

Therefore

$$A.MO.AC + B.QG.BC = R.ST.CR$$

$AO = a$, $BQ = b$, $RT = c$, $CA = r$, $CB = r'$, $CR = \rho$

$$R\,(c\text{-}z)\,\varrho = Ar\left(\frac{zr}{\varrho} - a\right) + Br'\left(\frac{zr'}{\varrho} - b\right)$$

Isolating z:

$$z = \frac{Aar\varrho + Bbr'\varrho + Rc\varrho^2}{Ar^2 + Br'^2 + R\varrho^2}$$

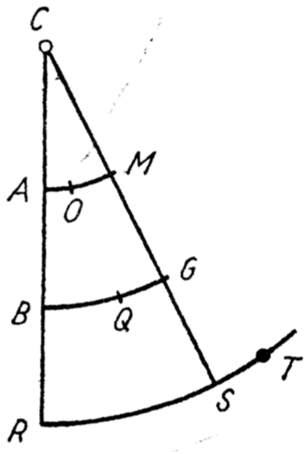
If F, f and φ are the motive (applied) forces at A, B and R:

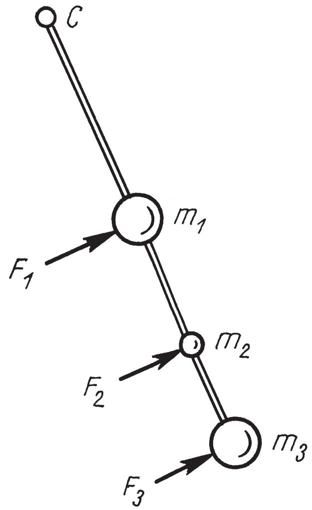
$$\frac{Fr + fr' + \varphi\varrho}{Ar^2 + Br^2 + R\varrho^2} \times \varrho$$

Element of arc = ds and velocity of R = u

$$\frac{Fr + fr' + \varphi\varrho}{Ar^2 + Br^2 + R\varrho^2}\,\varrho ds = udu$$

Center of oscillation problem (modern solution)





A.MO.AC + B.QG.BC = R.ST.CR

$$z = \frac{Aar\varrho + Bbr'\varrho + Rc\varrho^2}{Ar^2 + Br'^2 + R\varrho^2}$$

$$\frac{Fr + fr' + \varphi\varrho}{Ar^2 + Br^2 + R\varrho^2} \times \varrho$$

$$\frac{Fr + fr' + \varphi\varrho}{Ar^2 + Br^2 + R\varrho^2}\,\varrho ds = udu$$

Torque = moment of inertia x angular acceleration

$$F_1 r_1 + F_2 r_2 + F_3 r_3 = \left(m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2\right)\frac{\mathrm{d}\omega}{\mathrm{d}t}$$

$$\frac{\sum F_i r_i}{\sum m_i r_i^2} = \frac{\mathrm{d}\omega}{\mathrm{d}t}$$

Multiply by $r_3 ds$

$$\frac{\mathrm{d}\omega}{\mathrm{d}t} r_3 \mathrm{d}s = \frac{\mathrm{d}s}{\mathrm{d}t}\mathrm{d}\left(r_3\omega\right) = u_3\,\mathrm{d}u_3$$

$$\frac{\sum F_i r_i}{\sum m_i r_i^2} r_3\,\mathrm{d}s = u_3\,\mathrm{d}u_3$$

D’Alembert’s principle (Appraisal by Lagrange)

The *Traité de Dynamique* of d' Alembert, which was published in 1743, put an end to this type of challenge by giving a general and direct method to solve or at least to put into the form of equations all the problems of dynamics that can be imagined. This method reduces all the laws of the motions of bodies to those of their equilibrium and thus reduces dynamics to statics. We have already observed that the principle used by James Bernoulli in his research on the center of oscillation had the advantage of making this research dependent on the equilibrium conditions for the lever. But it was reserved for d'Alembert to conceive this principle in a general fashion and to give it all the simplicity and fecundity which it merits (Lagrange, 1788).

D’Alembert’s principle (Critic by Lagrange)

Thus by combining this [D’Alembert’s] principle with the ordinary principles for the equilibrium of the lever or of the composition of forces, the equations for each problem can always be found. But the difficulty of determining the forces which must be equilibrated as well as the laws of equilibrium between these forces often makes the application of this principle awkward and difficult, and the solutions obtained are almost always more complicated than if they were deduced from less simple and direct principles as is evident from the second part of the *Traité de Dynamique*.

D'Alembert's principle (Formulation by Lagrange)

2. The formula is composed of two different parts which must be considered separately. The first part contains the terms

$$\mathrm{S}\left(\frac{\mathrm{d}^2 x}{\mathrm{d}t^2}\delta x + \frac{\mathrm{d}^2 y}{\mathrm{d}t^2}\delta y + \frac{\mathrm{d}^2 z}{\mathrm{d}t^2}\delta z\right)m$$

which results solely from the inertia forces of the bodies. The second part is composed of the terms

$$\mathrm{S}(P\,\delta p + Q\,\delta q + R\,\delta r + \cdots)m$$

and is due to the accelerating forces P, Q, R, etc. which are assumed to act effectively on each body in the directions of the lines p, q, r, etc. and which have a tendency to shorten these lines. The sum of these two quantities, when equated to zero, constitutes the general formula of dynamics (SECTION II, Article 5).

D'Alembert's principle (Modern formulation)

$$\sum_i (\mathbf{F}_i - m_i \mathbf{a}_i) \cdot \delta \mathbf{r}_i = 0,$$

i	is an integer used to indicate (via subscript) a variable corresponding to a particular particle in the system,
$\mathbf{F}_i$	is the total applied force (excluding constraint forces) on the i-th particle,
m_i	is the mass of the i-th particle,
$\mathbf{a}_i$	is the acceleration of the i-th particle,
$m_i \mathbf{a}_i$	together as product represents the time derivative of the momentum of the i-th particle, and
$\delta \mathbf{r}_i$	is the virtual displacement of the i-th particle, consistent with the constraints.

Lagrange Equations

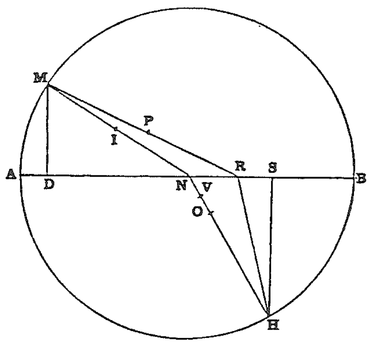
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0 \qquad L = T - V$$

Questions for discussion

- D'Alembert uses the notion of forces in his original text, when did the notion of force enter the physical sciences and what meaning does it have at the time of D'Alembert?
- When D'Alembert mentions the accelerating force, what precisely does he mean by this?
- How was Hume’s idea of causation received in the scientific society at the time? Did the physicists believe their grounding of physics was gone due to this discovery? Did this effect D'Alembert and Maupertius as they talk about bodies colliding and changing course due to the collision?

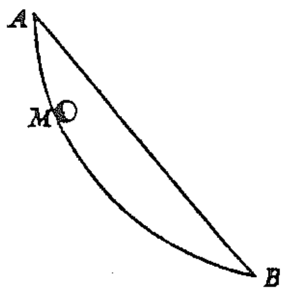
Principle of least action

- Fermat – “The synthesis of refractions” (1662)



“Nature operates by means and ways that are easiest and fastest”

- Brachystochrone problem (John Bernoulli’s challenge 1696)



“Find the curve joining two points in a vertical plane along which a frictionless beam will descend in the least possible time”

Principle of least action (Maupertuis)

Derivation of the laws of motion and equilibrium from a metaphysical principle (1746)

I. Assessment of the Proofs of God's Existence that are Based on the Marvels of Nature/II. Need to Identify Proofs of God's Existence in the General Laws of Nature

- Whether we stay locked up in our own thoughts, or venture out to survey the marvels of the universe, we find so many arguments for the existence of an all-powerful and all-wise Being, that we don't need to increase their number; rather, we should distill them down to a few solid proofs.

- Many things in the universe suggest that it is governed by a blind power. On all sides, we see consequences of effects leading to some destination; but this does not prove intelligent design. We must rather seek signs of God's wisdom in the *goals* of His designs;

- Let us see whether we can find a better use for mathematics. Mathematical arguments for God's existence would have the obvious certainty characteristic of geometrical truths. Those who doubt metaphysical reasoning would believe a mathematical argument more readily, whereas those exposed to the usual arguments would find mathematical arguments more elevating and precise.

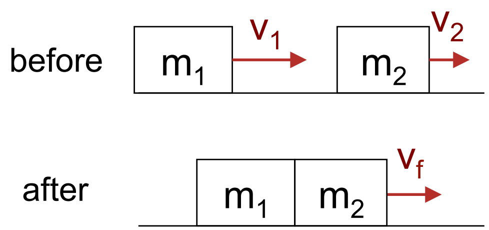
Principle of least action (Maupertuis)

General Principle

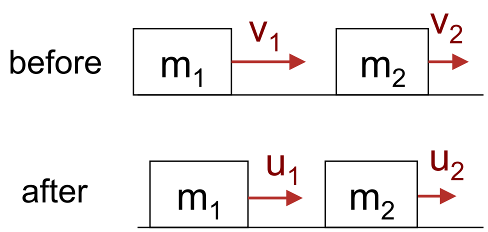
When a change occurs in Nature, the quantity of action necessary for that change is as small as possible.

The quantity of action is the product of the mass of the bodies times their speed and the distance they travel. $A = m.v.s$

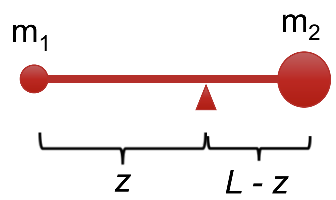
Inelastic collision



Elastic collision



Equilibrium



Changes in action: $m_1(v_f - v_1)^2$ and $m_2(v_f - v_2)^2$

Total change should be *minimized* ($f'(v_f) = 0$)

$$m_1 v_1^2 - 2m_1 v_1 v_f + m_1 v_f^2 + m_2 v_f^2 - 2m_2 v_2 v_f + m_2 v_2^2$$

$$-2m_1 v_1 dv_f + 2m_1 v_f dv_f + 2m_2 v_f dv_f - 2m_2 v_2 dv_f = 0$$

$$v_f = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2}$$

Changes in action: $m_1(u_1 - v_1)^2$ and $m_2(u_2 - v_2)^2$

Total change should be *minimized* ($f' = 0$)

$$m_1 v_1^2 - 2m_1 v_1 u_1 + m_1 u_1^2 + m_2 u_2^2 - 2m_2 v_2 u_2 + m_2 v_2^2$$

$$-2m_1 v_1 du_1 + 2m_1 u_1 du_1 + 2m_2 u_2 du_2 - 2m_2 v_2 du_2 = 0$$

For elastic collisions, relative speeds remain

$u_2 - u_1 = v_1 - v_2$ or $u_2 = u_1 + v_1 - v_2$ thus $du_2 = du_1$

Actions: $m_1 z^2$ and $m_2(L - z)^2$

Total action should be *minimized* ($f' = 0$)

$$m_1 z^2 + m_2 L^2 - 2m_2 Lz + m_2 z^2$$

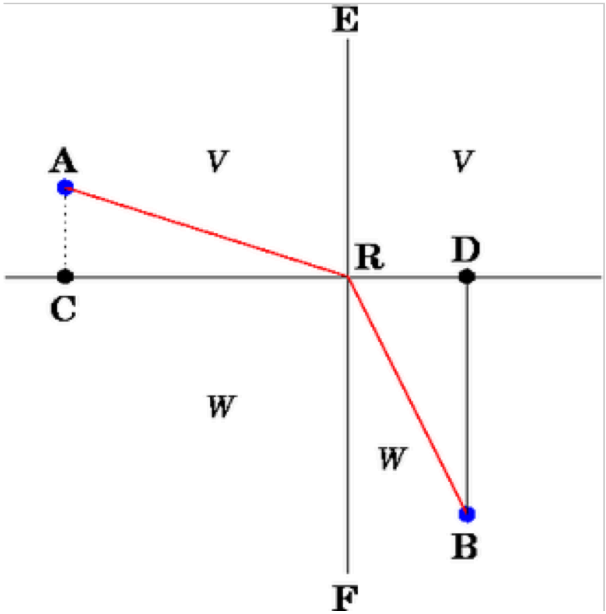
$$2m_1 Ldz - 2m_2 Ldz + 2m_2 zdz = 0$$

$$z = \frac{m_2 L}{m_1 + m_2}$$

Principle of least action (Maupertuis)

Accord between different laws of Nature that seemed incompatible (1744)

Now I have to define what I mean by "action". When a material body is transported from one point to another, it involves an action that depends on the speed of the body and on the distance it travels. However, the action is neither the speed nor the distance taken separately; rather, it is proportional to the sum of the distances travelled multiplied each by the speed at which they were travelled. Hence, the action increases linearly with the speed of the body and with the distance travelled.



To find the point **R** at which the light is bent, I seek a point that minimizes the action, i.e., *V.AR* + *W. RB* should be minimized

$$V\cdot\sqrt{AC^2+CR^2}+W\cdot\sqrt{BD^2+CD^2-2CD\times CR+CR^2}$$

Since *AC*, *BD* and *CD* are constants, minimization yields

$$\frac{V\cdot CR\,dCR}{\sqrt{AC^2+CR^2}}-\frac{W\cdot(CD-CR)\,dCR}{\sqrt{BD^2+DR^2}}=0$$

$$\frac{V\cdot CR}{AR}=\frac{W\cdot DR}{BR}\cdot\frac{CR}{AR}:\frac{DR}{BR}::W:V$$

Find the error

I know the distaste that many mathematicians have for *final causes* applied to physics, a distaste that I share up to some point. I admit, it is risky to introduce such elements; their use is dangerous, as shown by the errors made by Fermat and Leibniz in following them. Nevertheless, it is perhaps not the principle that is dangerous, but rather the hastiness in taking as a basic principle that which is merely a *consequence* of a basic principle.

Mach's sharp criticism to Maupertuis

It will thus be seen that Maupertuis really had no principle, properly speaking, but only a vague formula, which was forced to do duty as the expression of different familiar phenomena not really brought under one conception. I have found it necessary to enter into some detail in this matter, since Maupertuis's performance, though it has been unfavorably criticized by all mathematicians, is, nevertheless, still invested with a sort of historical halo. It would seem almost as if something of the pious faith of the church had crept into mechanics. However, the mere endeavor to gain a more extensive view, although beyond the powers of the author, was not altogether without results. Euler, at least, if not also Gauss, was stimulated by the attempt of Maupertuis.

Principle of least action (Euler, 1744 – App. 2)

A very different approach compared with Maupertuis…

1. Since all natural phenomena obey a certain maximum or minimum law; there is no doubt that some property must be maximized or minimized in the trajectories of particles acted upon by external forces. However, it does not seem easy to determine which property is minimized from metaphysical principles known *a priori*. Yet if the trajectories can be determined by a direct method, the property being minimized or maximized by these trajectories can be determined, provided that sufficient care is taken. After considering the effects of external forces and the movements they generate, it seems most consistent with experience to assert that the integrated momentum (i.e., the sum of all momenta contained in the particle's movement) is the minimized quantity. This assertion is not sufficiently proven at present; however, if I can show it to be connected with some truth known a priori, it will carry such weight as to utterly vanquish every conceivable doubt. If indeed it's truth can be verified, this assertion will make it easier to investigate the deepest laws of Nature and their final causes, and also easier to identify a firmer rationale for this assertion.

Principle of least action (Euler, 1744 – App. 2)

2. Let the mass of a moving particle be M, and let its speed be v while being moved over an infinitesimal distance ds. The particle will have a momentum Mv that, when multiplied by the distance ds, gives $Mvds$, the momentum of the particle integrated over the distance ds. Now **I assert** that the true trajectory of the moving particle is the trajectory to be described (from among all possible trajectories connecting the same endpoints) that minimizes $\int Mvds$ or (since M is constant) $\int vds$. Since the speed v resulting from the external forces can be calculated *a posteriori* from the trajectory itself, a method of maxima and minima should suffice to determine the trajectory *a priori*. The minimized integral can be expressed in terms of the momentum (as above), but also in terms of the kinetic energy. For, given an infinitesimal time dt during which the element ds is traversed, we have $ds = vdt$. Hence, $\int Mvds = \int Mv^2dt$ is minimized, i.e., the true trajectory of a moving particle minimizes the integral over time of its instantaneous kinetic energies. Thus, this minimum principle should appeal both to those who favor momentum for mechanics calculations and to those who favor kinetic energy.

Principle of least action (Euler, 1744 – App. 2)

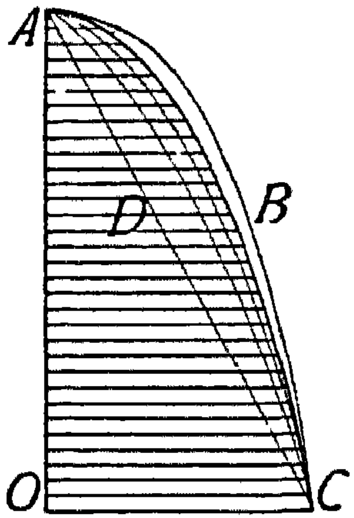
3. For our first example, consider a moving particle free of external forces, which has a constant speed, denoted b. By our principle, such a particle describes a trajectory that minimizes $\int b ds$ or $\int ds = s$. Hence, the true path of a free particle has the minimum length of all paths connecting the same endpoints; this path is a straight line, just as the first principles of Mechanics postulate. I do not present this example as evidence for the general principle, since the integral of any function of the constant speed would, upon minimization, produce a straight line. I begin with this simple case merely to illustrate the reasoning.

Principle of least action (Euler, 1744 – App. 2)

4. Let us proceed to the case of uniform gravity or, more generally, to the case in which a moving particle is acted upon by a downwards force of constant acceleration *g.*

Goal: Minimize $\int v ds$

Mach's qualitative reasoning:
Why is the parabola ABC better than the straight line ADC?



Analytically – Calculus of variations

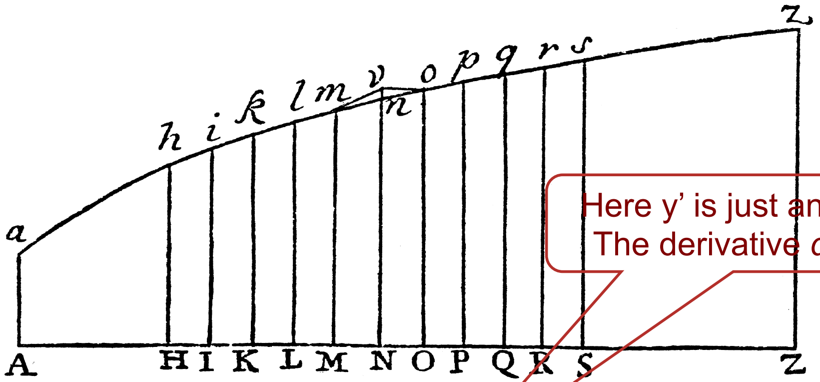
Calculus of variations (Euler, 1744)

General problem: Conditions for the integral to be a maximum/minimum (stationary)

$$\int_a^b Z\left(x, y, y', \ldots, y^{(n)}\right) dx$$

Here y' means the derivative of y with respect to x

Euler's original (be careful with notation)



Here y' is just an ordinate. The derivative *dy/dx* is *p*

Let *AM* = *x*, *AN* = *x'*, *AO* = *x''* and *Mm* = *y*, *Nn* = *y'*, *Oo* = *y''*

Let *p* = *dy/dx* $\quad p = \dfrac{y'-y}{dx}, \qquad p' = \dfrac{y''-y'}{dx}$

Problem: Minimize (stationary) the integral

$$\int_a^b Z(x,y,p)\,dx \qquad \delta\int_a^b Z(x,y,p)\,dx = 0$$

Euler treats the integral as a sum

$$\ldots +, Zdx + Zdx + Z'dx + Z''dx + \ldots = 0$$

Let *y'* increase by the infinitesimal "particle" *nv*

$$dy' = nv \qquad dp = \frac{(y'+nv)-y}{dx} - \frac{y'-y}{dx} = \frac{nv}{dx} \qquad dp' = -\frac{nv}{dx}$$

The change in *y'* will change only *Z* and *Z'*

$$dZ = M\,dx + N\,dy + P\,dp \qquad dZ' = M'\,dx + N'\,dy' + P'\,dp'$$

$$dZ = P\cdot\frac{nv}{dx} \qquad dZ' = N'\cdot nv - P'\cdot\frac{nv}{dx}$$

If the integral is stationary, the total change is 0

$$dZ + dZ' = 0 \qquad P.\frac{nv}{dx} + N'nv - P'.\frac{nv}{dx} = 0$$

Making *N'* = *N* and *P* – *P'* = *dP* in the limit

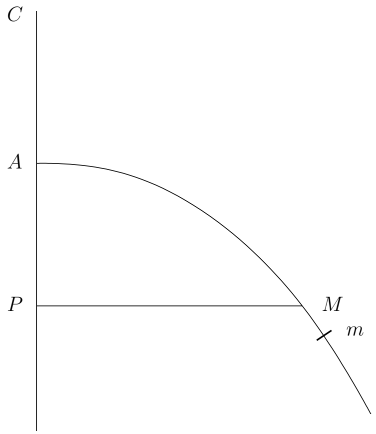
$$N - \frac{dP}{dx} = 0$$

$$\frac{\partial Z}{\partial y} - \frac{d}{dx}\frac{\partial Z}{\partial y'} = 0 \qquad \text{Euler equation}$$

Principle of least action (Euler, 1744 – App. 2)

Item 4. (Euler 1744): Downwards force with constant acceleration *g*

Analytically – Mach (p. 370)   General reasoning: Minimize $\int v\,ds$



$$\int \varphi(x, y) \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \,.\, dx$$

General case: velocity is some function of position and *ds* is expressed in terms of *dx* and *dy*

$v^2 = v_x^2 + v_y^2$
v_y^2 is constant
$v_x^2 = 2gx$ (free fall)

$AP = x \quad PM = y$

A bit confusing because we are used to the opposite convention

Putting the origin of the coördinates at A, reckoning the abscissas x vertically downwards as positive, and calling the ordinates perpendicular thereto y, we obtain for the expression to be minimised

$$\int_0^x \sqrt{2g(a+x)} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \,.\, dx,$$

where g denotes the acceleration of gravity and a the distance of descent corresponding to the initial velocity. As the condition of minimum the calculus of variations gives

Meaning Euler equation (slide 26)

Typo: this 2 should be outside the parenthesis

$$\frac{\sqrt{2g(a+x)}\,\frac{dy}{dx}}{\sqrt{1 + \left(\frac{dy^2}{dx}\right)}} = C \text{ or}$$

$$\frac{dy}{dx} = \frac{C}{\sqrt{2g(a+x) - C^2}} \text{ or}$$

$$y = \int \frac{C\,dx}{\sqrt{2g(a+x) - C^2}},$$

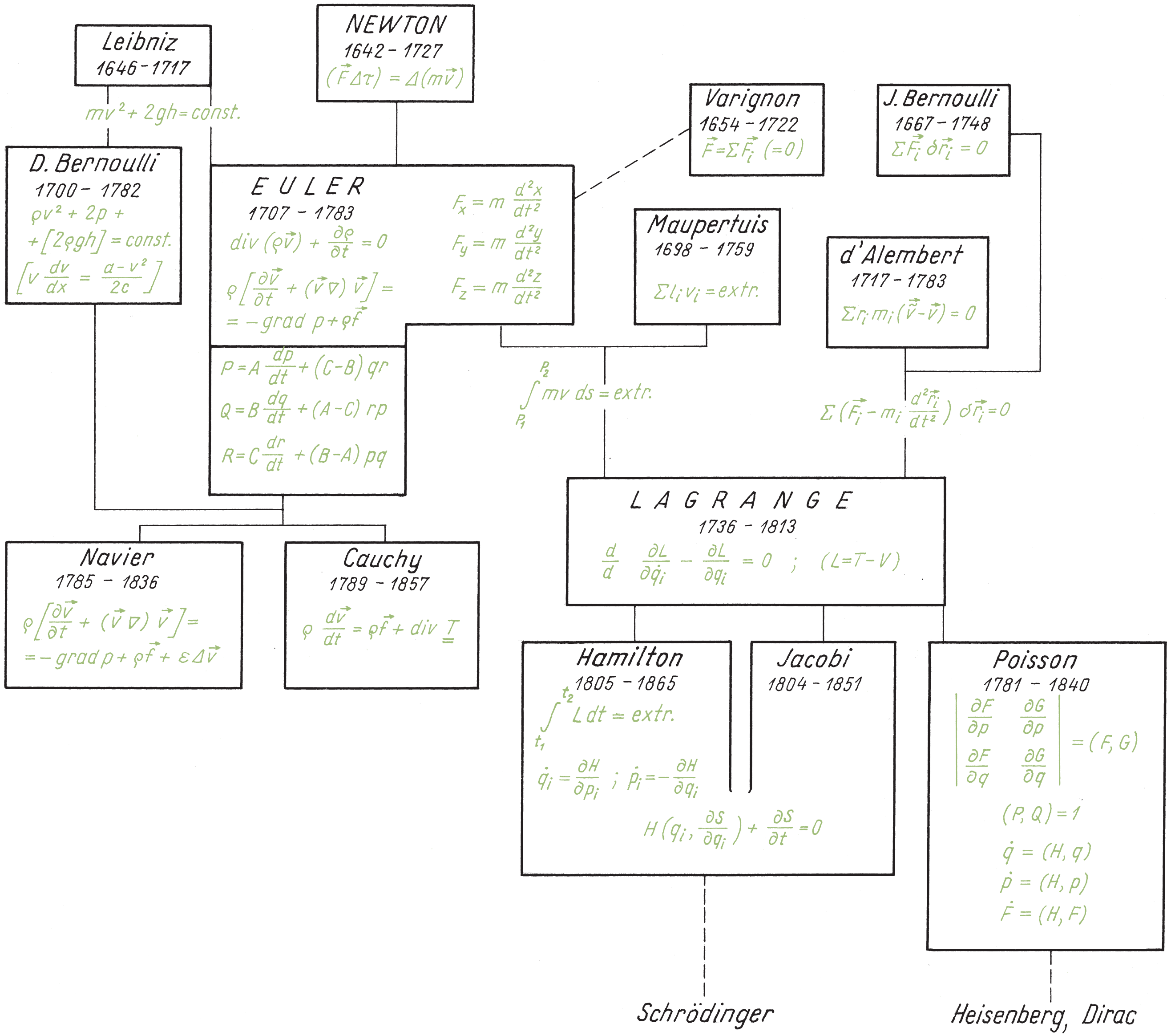
and, ultimately,

$$y = \frac{C}{g} \sqrt{2g(a+x) - C^2} + C',$$

Principle of least action (Euler, 1744 – App. 2)

16. Therefore, this principle is broadly applicable, except to the case of motion in a resistant medium. The reason for this exception is easy to see, because the speed of the particle at the endpoints will depend on the path taken. Hence, neglecting any resistance to the particle's motion, the momentum integrated along the path should be a minimum. Moreover, this minimum law is true not only for the motion of single particles, but also for systems of particles bound together. No matter what their reciprocal interactions are, the path integral of their momenta is always minimal. Compared to traditional mechanics methods, the motion may be more difficult to calculate using our new method; however, it seems easier to grasp from first principles. Because of their inertia, bodies are reluctant to move, and obey applied forces as though unwillingly; hence, external forces generate the smallest possible motion consistent with the endpoints. A rigorous proof for this principle is lacking, I realize. Nevertheless, it agrees with experiment and I do not doubt that it will be verified by stronger proofs that use the principles of a complete Metaphysics. But such proofs I leave to the professors of Metaphysics.

Mechanics’ Formalism over 2 centuries



Source: Simonyi (2012)

Questions for discussion

- Why does Maupertuis involve a massless invisible plane in his examples of the change occurring in other ways? Where does this idea come from?
- Why choose the least action? Which metaphysical considerations were made in the choosing of this principle?