initial collision and the rebounding should produce an effect equal to that which caused the collision; thus, the final relative speed of the two bodies should equal the initial relative speed, albeit with opposite direction. *Hence, the final relative speed of two elastic bodies after the collision should be the same as the initial relative speed.*

Let us now seek the laws that govern the distribution of motion among colliding bodies, whether they be elastic or inelastic.

We will derive these laws from only one principle and, from the same principle, we will derive the laws of mechanical equilibrium.

General Principle

When a change occurs in Nature, the quantity of action necessary for that change is as small as possible.

The **quantity of action** is the product of the mass of the bodies times their speed and the distance they travel. When a body is transported from one place to another, the action is proportional to the mass of the body, to its speed and to the distance over which it is transported.

Problem I: Laws of Motion for Inelastic Bodies

Let there be two inelastic bodies of masses m_1 and m_2 moving in the same direction with speeds v_1 and v_2, respectively. Let the first mass move more quickly, so that it overtakes the second mass and collides with it. After the collision, let the common velocity of the two bodies be v_f such that $v_2 < v_f < v_1$. The change in the universe is that, whereas the first mass was moving at a speed v_1 and was covering a distance v_1 per unit time, it now moves only at a speed v_f and covers only a distance v_f per unit time; and whereas the second mass was moving only at a speed v_2 and was covering only a distance v_2 per unit time, it now moves at speed v_f and covers a distance v_f per unit time.

This change would be the same if

- while the first body was moving at speed v_1 and was covering a distance v_1 per unit time, it were being transported backwards by an invisible, massless plane moving at speed $v_1 - v_f$ and covering a distance $v_1 - v_f$ per unit time; and
- while the second body was moving at speed v_2 and was covering a distance v_2 per unit time) it were being transported forwards by an invisible, massless plane moving at speed $v_f - v_2$ and covering a distance $v_f - v_2$ per unit time.

The motion of these immaterial planes conveying the masses m_1 and m_2 are the same, regardless of whether the masses are moving relative to these planes or are at rest. Hence, the quantities of action produced in Nature are $m_1(v_1 - v_f)^2$ and $m_2(v_f - v_2)^2$, the sum of which should be minimized. Thus, we have

$$m_1 v_1^2 - 2m_1 v_1 v_f + m_1 v_f^2 + m_2 v_f^2 - 2m_2 v_2 v_f + m_2 v_2^2 = Minimum$$

or, rather,

$$-2m_1 v_1 dv_f + 2m_1 v_f dv_f + 2m_2 v_f dv_f - 2m_2 v_2 dv_f = 0$$

from which one can derive the final speed

$$v_f = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2}$$

In this case where the two bodies are moving in the same direction, the quantity of momentum produced and destroyed is the same; the total momentum is constant, being the same after the impact as beforehand.

It is easy to extend the same reasoning to the case where the two bodies are moving towards each other, by making the second speed v_2 negative. In that case, the final speed is

$$v_f = \frac{m_1 v_1 - m_2 v_2}{m_1 + m_2}$$

If the second body is at rest before the impact, $v_2 = 0$, and the final speed is

$$v_f = \frac{m_1 v_1}{m_1 + m_2}$$

If the first body encounters an impassable barrier, one can consider that barrier as a body of infinite mass at rest; since m_2 is infinite, the final speed $v_f = 0$.

Now let us consider what happens to elastic bodies, by which I mean perfectly elastic bodies.

Problem II: Laws of Motion for Elastic Bodies

Let there be two elastic bodies of masses m_1 and m_2 moving in the same direction with speeds v_1 and v_2, respectively. Let the first mass move more quickly, so that it overtakes the second mass and collides with it. Let u_1 and u_2 represent the speeds of the two bodies after the collision; the sum or difference of these speeds is the same as that before the collision.

The change in the universe is that, whereas the first mass was moving at speed v_1 and was covering a distance v_1 per unit time, now it moves at speed u_1 and covers a distance u_1 per unit time; and whereas the second mass was moving at speed v_2 and was covering a distance v_2 per unit time, now it moves at speed u_2 and covers a distance u_2 per unit time.

This change would be the same if

- while the first body was moving at speed v_1 and was covering a distance v_1 per unit time, it were being transported backwards by an invisible, massless plane moving at speed $v_1 - u_1$ and covering a distance $v_1 - u_1$ per unit time; and
- while the second body was moving at speed v_2 and was covering a distance v_2 per unit time, it were being transported forwards by an invisible, massless plane moving at speed $u_2 - v_2$ and covering a distance $u_2 - v_2$ per unit time.

The motion of these immaterial planes conveying the masses m_1 and m_2 are the same, regardless of whether the masses are moving relative to these planes or are at rest. Hence, the quantities of action produced in Nature are $m_1 (v_1 - u_1)^2$ and $m_2 (u_2 - v_2)^2$, the sum of which should be minimized. Thus, we have

$$m_1 v_1^2 - 2m_1 v_1 u_1 + m_1 u_1^2 + m_2 u_2^2 - 2m_2 v_2 u_2 + m_2 v_2^2 = Minimum$$

or, rather,

$$-2m_1 v_1 du_1 + 2m_1 u_1 du_1 + 2m_2 u_2 du_2 - 2m_2 v_2 du_2 = 0$$

For elastic bodies, the relative speed after the impact should equal the relative speed before the impact; hence, we have $u_2 - u_1 = v_1 - v_2$ or, rather, $u_2 = u_1 + v_1 - v_2$ and, thus, $du_2 = du_1$. Substitution into the preceding equation yields the final speeds

$$u_1 = \frac{m_1 v_1 - m_2 v_1 + 2m_2 v_2}{m_1 + m_2}$$

and

$$u_2 = \frac{2m_1 v_1 - m_1 v_2 + m_2 v_2}{m_1 + m_2}$$

It is easy to extend the same reasoning to the case where the two bodies are moving towards each other, by making the second speed v_2 negative. In that case, the final speeds are

$$u_1 = \frac{m_1 v_1 - m_2 v_1 - 2m_2 v_2}{m_1 + m_2}$$

and

$$u_2 = \frac{2m_1 v_1 + m_1 v_2 - m_2 v_2}{m_1 + m_2}$$

If the second body is at rest before the impact, $v_2 = 0$, and the final speeds are

$$u_1 = \frac{m_1 v_1 - m_2 v_1}{m_1 + m_2}$$

and

$$u_2 = \frac{2m_1 v_1}{m_1 + m_2}$$

If the first body encounters an impassable barrier, one can consider that barrier as a body of infinite mass at rest. In that case, the final speed $u_1 = -v_1$, i.e., the first mass rebounds at the same speed with which it struck the barrier.

If one takes the sum of the kinetic energies, one sees that they are the same after the impact as before; thus,

$$m_1 v_1^2 + m_2 v_2^2 = m_1 u_1^2 + m_2 u_2^2$$

the sum of the kinetic energies is conserved after the impact. However, this conservation applied only to elastic bodies, and not to inelastic bodies. The general principle that applies to both types of bodies is that *the quantity of action required to cause a change in Nature is as small as possible*.

This principle is so universal and so fruitful that one can also derive the law of mechanical equilibrium from it. At equilibrium, there is no difference between elastic and inelastic bodies.

Problem III: Law of Mechanical Equilibrium for Bodies

I now suppose that two bodies are attached to a lever, and I seek the point about which they remain in equilibrium. Thus, I seek the point about which, if the lever moves slightly, the quantity of action is as small as possible.

Let L be the length of the lever (which I suppose to be massless), and let two masses m_1 and m_2 be placed at either end. If z represents the distance from the first mass m_1 to the equilibrium point being sought, then $L - z$ represents the corresponding distance to the second mass m_2. Obviously, if the lever rotates slightly about a point, the two masses describe geometrically similar arcs, whose size is proportional to their respective distances from the point of rotation. Thus, these arcs are the distances traveled by the bodies and also represent their speeds per unit time. Hence, the quantity of action is proportional to the product of the mass of each body multiplied by the square of its arc length; or, equivalently (since the two arcs are geometrically similar), the quantity of action is proportional to the product of the mass of each body multiplied by the square of its distance to the point of rotation, i.e., $m_1 z^2$ and $m_2 (L - z)^2$. The sum of these two terms should be minimized, giving the equation

$$m_1 z^2 + m_2 L^2 - 2m_2 Lz + m_2 z^2 = Minimum$$

or, rather,

$$2m_1 Ldz - 2m_2 Ldz + 2m_2 zdz = 0$$

from which the equilibrium position may be derived

$$z = \frac{m_2 L}{m_1 + m_2}$$

This is the basic law of mechanical equilibrium.